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Extremal Problems in Some Classes of Measures (IV) Typically Real Functions

Problemy ekstremalne w pewnych klasach miar (IV) Funkcje typowo rzeczywiste

Abstract. This paper is a conclusion of [8–10] and deals with compact convex classes of typically real functions whose ranges are in a given horizontal strip or else whose all odd coefficients are fixed. Like in [10] extreme and support points can form dense subsets and hence every extremal continuous problem over such class reduces to the extremal problem over its extreme (support) points. Some applications concern with the class of all typically real functions bounded in modulus by a common constant.

1. Introduction. Let $H(\Delta)$ be the linear space of all complex functions holomorphic in the open unit disc Δ, endowed with the topology of uniform convergence on compacta. In this paper, being a conclusion of [8–10], we shall be interested in subsets of the class

$$\mathcal{T} = \{f \in H(\Delta) : f(0) = 0\ ,\ \operatorname{Im} f(z) \operatorname{Im} z \geq 0 \text{ for } z \in \Delta\}\ , \tag{1.1}$$

parallel to those considered in the previous part [10]. Since $\mathcal{T}$ is the smallest convex cone in $H(\Delta)$ that contains the known class of all normalized typically real functions, we have the Rogosinski representation (1932) :

$$\mathcal{T} = \{z \mapsto zf(z)/(1-z^2) : f \in \mathcal{P}_R\}\ , \tag{1.2}$$

where

$$\mathcal{P}_R = \{f \in H(\Delta) : f(z) + \overline{f(\bar{z})} \geq 0 \text{ for } z \in \Delta\}. \tag{1.3}$$

This is equivalent to the Robertson integral representation (1935) :

$$\mathcal{T} = \{f_\nu : \nu \in M\}. \tag{1.4}$$

where

$$(1.5) \qquad f_\nu(z) \equiv \int_0^\pi q(z,\cos x)d\nu(x) \quad , \quad q(z,t) \equiv z/(1-2tz+z^2)\,,$$

and M is the family of all finite nonnegative Borel measures on the interval $[0,\pi]$. For details see [1, 3–5, 11].

According to [8–10] we suppose that $\mathcal{B}$ consists of all Borel subsets of $[0,\pi]$ and that M is endowed with the weak–star topology. Then the map $\nu \mapsto f_\nu$ is an affine homeomorphism from M onto $\mathcal{T}$ [1], so we get that

(I) the equation $f_\nu = f$ with $f \in \mathcal{T}$ has the unique solution $\nu = \nu_f \in M$,

(II) ν_f is the weak–star limit of a sequence (ν_{f_n}) whenever $f, f_1, f_2, \ldots \in \mathcal{T}$ and $f_n \to f$ uniformly on compacta.

For instance, if $f \in \mathcal{T}$ and $f_n(z) \equiv f((1-1/n)z)$, then

$$(1.6) \qquad d\nu_{f_n}/dx = (2/\pi)\mathrm{Im}\, f((1-1/n)e^{ix})\sin x \;\text{ on }\; [0,\pi] \;\text{ and }\; \nu_{f_n} \xrightarrow{w^*} \nu_f$$

(recover the function $g_n(z) \equiv (1-z^2)f_n(z)/z$ from its boundary function $g_n|\partial\Delta$ by means of the Poisson integral and use the property : $2f_n(z) \equiv f_n(z) + \overline{f_n(\overline{z})}$).

Most of the paper is concerned with the compact convex sets :

$$(1.7) \qquad \mathcal{T}(L) = \{f \in \mathcal{T} : |\mathrm{Im}\, f(z)| \le \pi L \;\text{ for }\; z \in \Delta\} \quad , \quad L > 0\,,$$

$$(1.8) \qquad \mathcal{T}(L,c) = \{f \in \mathcal{T}(L) : f'(0) = c\} \quad , \quad 0 \le c \le 4L\,,$$

and

$$(1.9) \qquad \mathcal{T}[g] = \{f \in \mathcal{T} : a_{2m-1}(f) = a_{2m-1}(g) \text{ for } m = 1,2,\ldots\}\,,$$

where $g \in \mathcal{T}$ and $a_j(f) = f^{(j)}(0)/j!$ for $j = 0,1,2,\ldots$. Obviously, by subordination principle, $\mathcal{T}(L) = \bigcup_{0\le c\le 4L} \mathcal{T}(L,c)$.

Moreover, for any $0 < r < 1$ the real functional

$$(1.10) \qquad \mathcal{T} \ni f \mapsto \sum_{j=1}^{\infty} a_j^2(f) r^{2j} = (2/\pi)\int_0^\pi \mathrm{Im}^2 f(re^{ix})dx$$

is continuous convex and hence

$$(1.11) \qquad \mathcal{T}(L) \subset H^2.$$

We let add that $\mathcal{T} \subset H^p$ for $0 < p < 1/2$. From the theory of H^p spaces [2], there follows the existence of nontangential boundary limits $f(e^{ix})$ a.e. on $[0,\pi]$ for all $f \in H^p$ with $0 < p \le \infty$. Thus

$$(1.12) \qquad \sum_{j=1}^{\infty} a_j^2(f) = (2/\pi)\int_0^\pi \mathrm{Im}^2 f(e^{ix})dx \quad \text{for} \quad f \in \mathcal{T} \cap H^2$$

and

$$(1.13) \qquad d\nu_f/dx = (2/\pi)\mathrm{Im}\, f(e^{ix})\sin x \text{ a.e. on } [0,\pi] \;\text{ for }\; f \in \mathcal{T} \cap H^1\,,$$

see (I), (II) and (1.6), see also the proof of [10, Th.5.10].

Observe now that the map $f \mapsto \widetilde{f}$, where $\widetilde{f}(z) \equiv (f(z) - f(-z))/2$, is a projection of $\mathcal{T}$ onto the class $\widetilde{\mathcal{T}}$ of all odd functions from $\mathcal{T}$. Thus the equivalence relation

$$f \sim g \quad \text{if and only if} \quad \widetilde{f} = \widetilde{g}$$

decomposes $\mathcal{T}$ into equivalence classes (1.9) with g ranging over $\widetilde{\mathcal{T}}$. This way

$$\mathcal{T} = \bigcup_{g \in \widetilde{\mathcal{T}}} \mathcal{T}[g] \quad , \quad \text{c.f. [6,7]} .$$

Just as in [10], the classes (1.7), (1.8) and many of (1.9) are strongly convex, so their extreme points form dense subsets. Some applications will concern the class of all typically real functions that are bounded in modulus by a common constant.

To comply with the previous notation, let $\mathcal{E}\mathcal{A}$ (resp. $\sigma\mathcal{A}$) denote the set of all extreme (resp. support) points of $\mathcal{A}$. Moreover, let $\nu_A(B) = \nu(A \cap B)$ for all $\nu \in M$ and $A, B \in \mathcal{B}$, and let $h(x) = \pi - x$ for $0 \le x \le \pi$. The support of $\nu \in M$ will be denoted by supp ν.

2. Basic results. Using (I), (II), (1.6) and the notation from [8] we get

Proposition 2.1. *$\mathcal{T}(L) = \{f_\nu : \nu, \mu - \nu \in M\} = \{f_\nu : \nu \in M^{id}([0,\pi], \mathcal{B}, \mu)\}$ and $\mathcal{T}(L, c) = \{f_\nu : \nu \in M^{id}([0,\pi], \mathcal{B}, \mu, c)\}$, where $d\mu/dt = 2L \sin t$, $0 \le t \le \pi$.*

For the classes (1.9) we have

Proposition 2.2. *Let $g \in \mathcal{T}$. Then $\mathcal{T}[g] = \{f_\nu : \nu \in \widehat{M}^h(X, \mathcal{B}, \nu_{\widetilde{g}})\}$, where* orb$(x) \equiv \{x, \pi - x\}$, $X = [0,\pi] = X_2 = X_1 \cup \widetilde{X}_2$, $X_1 = \{\pi/2\}$ *and* $\widetilde{X}_2 = X \setminus \{\pi/2\}$, *see* [8].

Proof. Note first that $f \in \mathcal{T}[g]$ if and only if $f \in \mathcal{T}$ and $2\widetilde{g}(z) \equiv f(z) - f(-z)$. Since $h = h^{-1}$, we have $2\nu_{\widetilde{g}}(A) = \nu_f(A) + \nu_f(h(A))$ for all $A \in \mathcal{B}$, and the desired result follows from [8, Proposition 7.1].

The classes $\mathcal{T}(L)$ and $\mathcal{T}(L, c)$, $0 < c < 4L$, are strongly convex and the following properties hold.

Theorem 2.3. *Let $\psi : [0,\pi] \to R$ be a Lebesgue integrable function on $[0,\pi]$ and $0 < c < 4L$. Then*

(i) $$\max\left\{\int_0^\pi \psi \, d\nu_f : f \in \mathcal{T}(L)\right\} = L \int_0^\pi (\psi(x) + |\psi(x)|) \sin x \, dx ,$$

(ii) $$\max\left\{\int_0^\pi \psi \, d\nu_f : f \in \mathcal{T}(L, c)\right\} = 2L \int_{A(\lambda_c)} (\psi(x) - \lambda_c) \sin x \, dx + \lambda_c c ,$$

where $A(\lambda) = \{x \in [0,\pi] : \psi(x) \ge \lambda\}$ and $\lambda_c = \sup\{\lambda \in R : 2L \int_{A(\lambda)} \sin x \, dx \ge c\}$. Furthermore,

(iii) $\sigma\mathcal{T}(L) = \{f_A = 2L\int_A q(\cdot,\cos x)\sin x\,dx : A \subset [0,\pi]$ *is a finite union of intervals* $\} \subsetneq \mathcal{E}\mathcal{T}(L) = \{f_A : A \in \mathcal{B}\} = \{2L\int_B q(\cdot,t)\,dt : B$ *is a Borel subset of* $[-1,1]\}$
and
(iv) $\sigma\mathcal{T}(L,c) = \{f \in \sigma\mathcal{T}(L) : f'(0) = c\} \subsetneq \mathcal{E}\mathcal{T}(L,c) = \{f \in \mathcal{E}\mathcal{T}(L) : f'(0) = c\}$.
Thus for $\mathcal{A} = \mathcal{T}(L)$ *or* $\mathcal{A} = \mathcal{T}(L,c)$ *we have*
(v) $\overline{\sigma\mathcal{A}} = \overline{\mathcal{E}\mathcal{A}} = \mathcal{A}$.
Moreover,
(vi) $f \in \mathcal{E}\mathcal{A}$ *iff* $f \in \mathcal{A}$ *and* $(\pi L - \operatorname{Im} f(e^{ix}))\operatorname{Im} f(e^{ix}) = 0$ *a.e. on* $[0,\pi]$.

Proof. In contrast to the proofs of [10, Th. 3.4, Remarks 3.6] it is sufficient to observe that for $0 \le x_1 < x_2 < \cdots < x_{2n-1} < x_{2n} \le \pi$ we have $A \overset{\text{df}}{=} \bigcup_{j=1}^{n}[x_{2j-1},x_{2j}] = \{x \in [0,\pi] : w(x) \ge 0\}$, where $w(x) \equiv -\prod_{j=1}^{2n}(\cos x_j - \cos x)$. Moreover, if $\Phi(f) = \sum_{j=1}^{2n+1} d_j a_j(f)$, where $w(x)\sin x \equiv \sum_{j=1}^{2n+1} d_j \sin jx$, then $\Phi \in H(\Delta)^*$ and $\Phi(q(\cdot,\cos x)) \equiv w(x)$. Taking A such that $2L\int_A \sin x\,dx = c$ we have $f_A \in \mathcal{T}(L,c)$ and $\max\Phi(\mathcal{T}(L,c)) = \Phi(f_A)$. In the proof of (vi) we use (I), (1.11) and (1.13).

Let now $\widetilde{\mathcal{T}}(L) \overset{\text{df}}{=} \mathcal{T}(L) \cap \widetilde{\mathcal{T}}$ for $L > 0$, $\widetilde{\mathcal{T}}(L,c) \overset{\text{df}}{=} \mathcal{T}(L,c) \cap \widetilde{\mathcal{T}}$ for $0 < c < 4L$, and let ψ be a real Lebesgue integrable function on $[0,\pi]$. Clearly,

$$\widetilde{\mathcal{T}}(L) = \{\widetilde{f} : f \in \mathcal{T}(L)\} = \{f \in \mathcal{T}(L) : \nu_f = \nu_f \circ h\}$$

and

$$\widetilde{\mathcal{T}}(L,c) = \{\widetilde{f} : f \in \mathcal{T}(L,c)\} = \{f \in \mathcal{T}(L,c) : \nu_f = \nu_f \circ h\}\,.$$

Analogously to the previous theorem we deduce

Theorem 2.4.

(i) $$\max\left\{\int_0^\pi \psi\,d\nu_f : f \in \widetilde{\mathcal{T}}(L)\right\} = L\int_0^{\pi/2}(\psi(x)+\psi(\pi-x)+|\psi(x)+\psi(\pi-x)|)\sin x\,dx,$$

(ii) $$\max\left\{\int_0^\pi \psi\,d\nu_f : f \in \widetilde{\mathcal{T}}(L,c)\right\} = 2L\int_{A(\lambda_c)}(\psi(x)+\psi(\pi-x)-\lambda_c)\sin x\,dx + \lambda_c c/2,$$

where

$$A(\lambda) = \{x \in [0,\pi/2] : \psi(x)+\psi(\pi-x) \ge \lambda\}$$

and

$$\lambda_c = \sup\left\{\lambda \in R : 2L\int_{A(\lambda)} \sin x\,dx \ge c/2\right\}.$$

Furthermore,

(iii) $\sigma\widetilde{\mathcal{T}}(L) = \widetilde{\mathcal{T}} \cap \sigma\mathcal{T}(L) \subsetneq \{2(\widetilde{f}_A) : A$ *is a Borel subset of* $[0,\pi/2]\} =$

$$= \left\{2L\int_B (q(\cdot,t)+q(\cdot,-t))\,dt : B \text{ is a Borel subset of } [0,1]\right\} = \mathcal{E}\widetilde{\mathcal{T}}(L) = \widetilde{\mathcal{T}} \cap \mathcal{E}\mathcal{T}(L)$$

and

(iv) $\sigma\widetilde{\mathcal{T}}(L,c)=\widetilde{\mathcal{T}}\cap\sigma\mathcal{T}(L,c)\subsetneq\mathcal{E}\widetilde{\mathcal{T}}(L,c)=\widetilde{\mathcal{T}}\cap\mathcal{E}\mathcal{T}(L,c)$, c.f Th. 2.3 .

Moreover, the classes $\widetilde{\mathcal{T}}(L)$, $\widetilde{\mathcal{T}}(L,c)$ *are strongly convex so that* (v) *and* (vi) *of Theorem* 2.3 *with* $\mathcal{A}=\widetilde{\mathcal{T}}(L)$ *or* $\mathcal{A}=\widetilde{\mathcal{T}}(L,c)$ *holds.*

The proof is very similar. Observe only that

$$\widetilde{\mathcal{T}}(L)=\{f_{\nu_{[0,\pi/2]}+\nu_{[0,\pi/2]}\circ h}:\nu\in M\ ,\ \ d\nu/dx\le 2L\sin x\ \text{ a.e. on } [0,\pi/2]\}$$

and

$$\widetilde{\mathcal{T}}(L,c)=\{f_\nu\in\widetilde{\mathcal{T}}(L):\nu([0,\pi/2])=c/2\}\ .$$

If now $0\le x_1<x_2<\cdots<x_{2n-1}<x_{2n}\le\pi/2$, $-\sin x\prod_{j=1}^{2n}(\cos 2x,-\cos 2x)\equiv\sum_{j=1}^{2n+1}d_{2j-1}\sin(2j-1)x$, $\Phi(f)\equiv\sum_{j=1}^{2n+1}d_{2j-1}a_{2j-1}(f)$ and $A=\bigcup_{j=1}^{n}[x_{2j-1},x_{2j}]$, then $\Phi\in H(\Delta)^*$, $\Phi(q(\cdot,t))\equiv\Phi(q(\cdot,-t))$, $A=\{x\in[0,\pi/2]:\Phi(q(\cdot,\cos x))\ge 0\}$ and $(\widetilde{f}_A)\in\widetilde{\mathcal{T}}(L)$.

Remarks 2.5.

(i) $\sigma\widetilde{\mathcal{T}}(L)\subsetneq\{\widetilde{f}:f\in\sigma\mathcal{T}(L)\}$, $\mathcal{E}\widetilde{\mathcal{T}}(L)\subsetneq\{\widetilde{f}:f\in\mathcal{E}\mathcal{T}(L)\}$,

(ii) $\sigma\widetilde{\mathcal{T}}(L,c)\subsetneq\{\widetilde{f}:f\in\sigma\mathcal{T}(L,c)\}$, $\mathcal{E}\widetilde{\mathcal{T}}(L,c)\subsetneq\{\widetilde{f}:f\in\mathcal{E}\mathcal{T}(L,c)\}$.

(iii) Let $g(z)\equiv 2L\log((1+z)/(1-z))$. Then $f\in\mathcal{T}(L,c)$ (resp. $f\in\widetilde{\mathcal{T}}(L,c)$) if and only if $g-f\in\mathcal{T}(L,4L-c)$ (resp. $g-f\in\widetilde{\mathcal{T}}(L,4L-c)$).

Proof. (i) – (ii). Take any $A\in\mathcal{B}$ with $|A\cap h(A)|>0$ and let $d\mu/dx=2L\sin x$ a.e. on $[0,\pi]$. Then $f_A\in\mathcal{E}\mathcal{T}(L)$, see Theorem 2.3(iii). If B is a measurable subset of $A\cap[0,\pi/2]$ or of $A\cap[\pi/2,\pi]$ with $\mu(B)=\mu(A)/2$, then $(\widetilde{f_A})=(f_A+f_{h(A)})/2=(f_{B\cup h(B)}+f_{A\setminus B\cup h(A\setminus B)})/2\notin\mathcal{E}\widetilde{\mathcal{T}}(L)$.

In proving (iii) observe that for all $f\in\mathcal{T}(L,c)$ we have $g'(0)-f'(0)=4L-c$ and $d(\nu_g-\nu_f)=2(L-\operatorname{Im}f(e^{ix})/\pi)\sin x\,dx$ a.e. on $[0,\pi]$.

By [9, Remark 3.2, Theorems 4.1, 4.2] we get

Theorem 2.6. *Let* $0<c<4L$ *and let* $\mathcal{A}$ *be one of the following sets :* $\mathcal{T}(L)$, $\mathcal{T}(L,c)$, $\widetilde{\mathcal{T}}(L)$ *or* $\widetilde{\mathcal{T}}(L,c)$. *If* $\mathcal{A}_0$ *consists of all* $f_0\in\mathcal{A}$ *for which there is a complex functional* J *weakly differentiable relative to* $\mathcal{A}$ *such that* $\operatorname{Re}J(f_0)=\max(\operatorname{Re}J)(\mathcal{A})$ *and* $\operatorname{Re}J'_{f_0}|\mathcal{A}\ne\text{const}$, *then* $\mathcal{A}_0=\sigma\mathcal{A}$.

Using [8, Theorems 8.1, 9.1, 11.2] and Proposition 2.2 we obtain

Theorem 2.7. *Let $g \in \mathcal{T}$ and $\mu = \nu_{\widetilde{g}}$. Then*

(i) $\mathcal{E}\mathcal{T}[g] = \{f_\nu : \nu = \mu_{\{\pi/2\}} + 2\mu_D$ *and the sets* D, $h(D)$, $\{\pi/2\}$ *form a Borel decomposition of the interval* $[0,\pi]\}$,

(ii) $$\max\{\int_0^\pi \psi\, d\nu_f : f \in \mathcal{T}[g]\} = \int_0^\pi \max\{\psi(x), \psi(\pi - x)\}\, d\nu_F(x)$$
$$= \int_0^\pi \psi\, d\mu + (1/2)\int_0^\pi |\psi(x) - \psi(\pi - x)|\, d\mu(x)$$

for all bounded Borel functions $\psi : [0,\pi] \to R$ and all $F \in \mathcal{T}[g]$, see [6–7]. *Moreover, f realizes the maximum if and only if $f \in \mathcal{T}[g]$ and $\nu_f(\{x \in [0,\pi] : \psi(\pi - x) > \psi(x)\}) = 0$.*

Corollary 2.8. [6,7]. *$\mathcal{T}(g) = \{g\}$ if and only if $g(z) \equiv \lambda z/(1+z^2) = f_{\lambda\delta_{\pi/2}}$ for some nonnegative number λ.*

Proof. The original proof has been found by means of 2.7(ii) (consider all continuous functions $\psi : [0,\pi] \to R$). An alternative proof of the theorem depends on 2.7(i). If $\mathcal{T}[g] = \{g\}$, then $g = \widetilde{g}$ and $\nu_g = (\nu_g)_{\{\pi/2\}}$.
Conversely, putting $\nu = \lambda\delta_{\pi/2}$, $\lambda \ge 0$, $g = f_\nu$, we obtain that $\mathcal{E}\mathcal{T}[g] = \{g\}$, that is $\mathcal{T}[g] = \{g\}$.

The class $\mathcal{T}[g]$ can be strongly convex. Namely,

Theorem 2.9. *Let $g \in \mathcal{T}$ and $\mu = \nu_{\widetilde{g}}$. The class $\mathcal{T}[g]$ is strongly convex if and only if either*
1° $\mu - \mu_{\{\pi/2\}}$ *is nonzero and nonatomic*
or
2° $\operatorname{supp}\mu \setminus \{\pi/2\}$ *consists of 2 elements.*
In the case 1° we have
(i) $\overline{\sigma\mathcal{T}[g]} = \overline{\mathcal{E}\mathcal{T}[g]} = \mathcal{T}[g]$
and
(ii) $\sigma\mathcal{T}[g] = \{f_\nu \in \mathcal{E}\mathcal{T}[g] : \operatorname{supp}\nu$ *is the finite union of subintervals of* $[0,\pi]\}$.

Proof. Let $\alpha = \mu - \mu_{\{\pi/2\}}$.
"if". If 1° holds, the proof is similar to that found in [10, Th. 3.9]. Namely, without loss of generality we can assume that μ is nonzero and nonatomic. The truth is that $\mathcal{T}[g] = \mu(\{\pi/2\})q(\cdot,\pi/2) + \mathcal{T}[f_\alpha]$. Next replace n, $\mathcal{P}(n;g)$, $g_{(n)}$, $\partial\Delta$ and $h(x) \equiv \varepsilon x$ by 2, $\mathcal{T}[g]$, $\widetilde{g}$, $[0,\pi]$ and $h(x) \equiv \pi - x$, respectively. In the case 2° the class $\mathcal{T}[g]$ is a segment in $H(\Delta)$ and, hence, it is strongly convex.
"only if". Suppose that $\mathcal{T}[g]$ is a strongly convex set different from a segment. Obviously, the measure α is nonzero, and if b is an atom of α, then also $\pi - b$ is an atom of α. We can assume that $0 \le b < \pi/2$. Consider now the functional $\Phi(f) \equiv 2a_2(f)\cos 2b - a_4(f)$. By 2.7(ii) we get $\max\Phi(\mathcal{T}[g]) = \int_0^\pi |\psi(x)|\, d\mu(x) = 2\int_0^{\pi/2} |\psi(x)|\, d\mu(x) = \Phi(f_1) = \Phi(f_2)$, where $\psi(x) \equiv 4\cos x(\cos 2b - \cos 2x)$, $\nu_{f_1} = \mu_{\{\pi/2\}} + 2\mu_{[b,\pi/2)\cup(\pi-b,\pi]}$ and $\nu_{f_2} = \mu_{\{\pi/2\}} + 2\mu_{(b,\pi/2)\cup[\pi-b,\pi]}$. Clearly, $f_1, f_2 \in \mathcal{E}\mathcal{T}[g]$

and $f_1 \neq f_2$, see (I). Since $\mathcal{T}[g]$ is not a segment, there is $A \in \mathcal{B}$, $A \subset [0, \pi/2) \setminus \{b\}$ with $\mu(A) > 0$, and then $\Phi(f_1) \geq 2\int_A |\psi(x)|\, d\mu(x) > 0 = \Phi(g)$. Finally, $(f_1 + f_2)/2 \in \sigma\mathcal{T}[g] \setminus \mathcal{ET}[g]$, from which it follows that α has to be nonatomic.

3. Functions with range in a strip.

Theorem 3.1. *For any real numbers r, s and positive integers m, n we have*

(i) $\max\{r a_m(f) + s a_n(f) : f \in \mathcal{T}(L)\} =$

$$= L\Big[\int_0^{\pi} |r\sin(mx) + s\sin(nx)|\, dx + r(1-(-1)^m)/m + s(1-(-1)^n)/n\Big].$$

In particular, for $f \in \mathcal{T}(L)$ we obtain the following sharp inequalities:

(ii) $|a_n(f) - L(1-(-1)^n)/n| \leq 2L \quad , \quad n = 1, 2, \ldots \ ,$

(iii) $|a_m(f) \pm a_n(f) - L(1-(-1)^m)/m \mp L(1-(-1)^n)/n| \leq$

$$\leq \begin{cases} 8L(A\cot(\pi/A) - B\cot(\pi/B))/(A^2 - B^2) & \text{if } A/2 \text{ is even}, \\ 8L(A/\sin(\pi/A) - B/\sin(\pi/B))/(A^2 - B^2) & \text{if } A/2 \text{ is odd}, \end{cases}$$

where $A = 2|m \pm n|/d$, $B = 2|m \mp n|/d$, $m \neq n$, and d is the greatest common divisor of $m+n$ and $|m-n|$.

Proof. Apply 2.3(i) to $\psi(x) \equiv r\sin(mx) + s\sin(nx)$. To calculate the integrals in (i) for $r, s = \pm 1$ use [10, the formula (4.1), Lemmas 4.2–4.4 and the proof of Th. 5.1(iv)]. Then

$$\int_0^{\pi} |\sin mx \pm \sin nx|\, dx = \int_0^{\pi} |\sin 2mx \pm \sin 2nx|\, dx = 2J(|m \pm n|, |m \mp n|, \pi/2).$$

Corollary 3.2. *Suppose that $f \in \mathcal{T}$, $f(z) \equiv z + a_2 z^2 + \cdots + a_n z^n + \cdots$ and $|\mathrm{Im}\, f(z)| \leq \pi/2$ for $z \in \Delta$. Then we have*

$$|a_n - (1-(-1)^n))/(2n)| \leq 1 \quad \textit{for } n = 1, 2, \ldots .$$

This result is sharp.

Proof. The extremal functions realizing equality in 3.1(ii) belong all to the class $\mathcal{T}(L, 2L)$. Indeed, each of such functions belongs to $\mathcal{T}(L, c)$ with $c = 2L\int_{\{x \in [0,\pi] : \sin nx \geq 0\}} \sin x\, dx = 2L$. For $L = 1/2$ we get the corollary.

Theorem 3.3. *For all $0 \leq c \leq 4L$, $\max\{\sum_{j=1}^{\infty} a_j^2(f) : f \in \mathcal{T}(L, c)\}$ $4\pi L^2 \arccos(1 - c/(4L))$.*

Proof. The classical arguments on subordination [3–5] lead to the inequalities:

$$\sum_{j=1}^{n} a_j^2(f) \leq 16L^2 \sum_{1 \leq j \leq (n+1)/2} (2j-1)^{-2} < 2\pi^2 L^2 \quad \text{for} \quad f \in \mathcal{T}(L) \quad , \quad n = 1, 2, \ldots .$$

str 9 koniec that are sharp only in the class $\mathcal{T}(L,4L)=\{z\mapsto 2L\log((1+z)/(1-z))\}$. For the remainder we shall use the Krein–Milman theorem and Theorem 2.3(iv). Since for any $0<r<1$ the real functional (1.10) is convex continuous on $\mathcal{T}(L,c)$, we get that

$$\max\Big\{\sum_{j=1}^{\infty}a_j^2(f)r^{2j}:f\in\mathcal{T}(L,c)\Big\}=\max\Big\{\sum_{j=1}^{\infty}a_j^2(f)r^{2j}:f\in\mathcal{ET}(L,c)\Big\}$$

$$<\sup\Big\{\sum_{j=1}^{\infty}a_j^2(f):f\in\mathcal{ET}(L,c)\Big\}=\sup\Big\{(2/\pi)\int_0^{\pi}\operatorname{Im}^2f(e^{ix})\,dx:f\in\mathcal{ET}(L,c)\Big\},$$

see (1.12). By Theorem 2.3(iii) and by formula (1.13) we obtain that

$$\sup\Big\{\sum_{j=1}^{\infty}a_j^2(f):f\in\mathcal{ET}(L,c)\Big\}=\sup\Big\{2\pi L^2|A|:A\in\mathcal{B}\,,\ 2L\int_A\sin dx=c\Big\}\,.$$

Consider now the set $D=\{x\in[0,\pi]:\sin x\le\lambda\}$, where $\lambda=\sqrt{c}\sqrt{8L-c}/(4L)$. It is trivial to check that $|D|=2\arcsin\lambda$ and $2L\int_D\sin x\,dx=c$. If $A\in\mathcal{B}$ satisfies equality $2L\int_A\sin x\,dx=c$, then $|A|=|A\setminus D|+|A\cap D|\le\lambda^{-1}\int_{A\setminus D}\sin x\,dx+|A\cap D|=\lambda^{-1}\int_{D\setminus A}\sin x\,dx+|A\cap D|\le|D\setminus A|+|A\cap D|=|D|$. Thus

$$\max\Big\{\sum_{j=1}^{\infty}a_j^2(f):f\in\mathcal{ET}(L,c)\Big\}=2\pi L^2|D|\le\sup\Big\{\sum_{j=1}^{\infty}a_j^2(f):f\in\mathcal{T}(L,c)\Big\}$$

$$\le 2\pi L^2|D|=4\pi L^2\arccos(1-c/(4L)),$$

the desired result.

Remark 3.4. Since the extremal function giving equality in the last theorem is odd, we obtain also that

$$\max\Big\{\sum_{j=1}^{\infty}a_{2j-1}^2(f):f\in\mathcal{T}(L,c)\Big\}=4\pi L^2\arccos(1-c/(4L))\quad\text{for all } 0\le c\le 4L\,.$$

Theorem 3.5. *Let $f\in\mathcal{T}(L,c)$, $n\ge 5$ and $\varphi(n,x,y)\equiv(4x/n)\sin^2(n\arcsin\sqrt{y/(4x)}\,)$. Then the following sharp inequalities hold :*

(i) $|a_n(f)|\le\varphi(n,L,c)$ *if* $c\le 4L\sin^2(3\pi/(8n))$ *and n is even,*
(ii) $|a_n(f)|\le\varphi(n,L,4L-c)$ *if* $c\ge 4L\cos^2(3\pi/(8n))$ *and n is even,*
(iii) $a_n(f)\le\varphi(n,2L,c)$ *if* $c\le 8L\sin^2(3\pi/(8n))$ *and n is odd,*
(iv) $a_n(f)\ge 4L/n-\varphi(n,2L,4L-c)$ *if* $c\ge 4L\cos(3\pi/(4n))$ *and n is odd.*
The extremal functions for (i), (ii), (iv) are univalent.

Proof. Consider $w(x)=\sin(nx)/\sin x$ for $0\le x\le\pi$, $n\ge 5$, and put $p(t)\equiv w(\arccos t)$. Since $p(t)=2^{n-1}\prod_{j=1}^{n-1}(t-\cos(j\pi/n))$ for $-1\le t\le 1$, we obtain that w strictly decreases on $[0,x_0]$ and strictly increases on $[x_0,2\pi/n]$, where x_0 is the unique solution of the equation : $w'(x)=0$, $5\pi/(4n) < x < 3\pi/(2n)$. To this end observe that $w'(5\pi/(4n)) < 0$ and $w'(3\pi/(2n)) > 0$. Moreover, $w(\pi - x) \equiv (-1)^{n-1} w(x)$ and

$$w(x) \leq \begin{cases} 1/\sin(5\pi/(4n)) & \text{if } 5\pi/(4n) \leq x \leq \pi - 5\pi/(4n)\,, \\ w(3\pi/(4n)) & \text{if } 3\pi/(4n) \leq x \leq 5\pi/(4n) \\ & \text{or } \pi - 5\pi/(4n) \leq x \leq \pi - 3\pi/(4n)\,. \end{cases}$$

Thus $w(x) \leq w(3\pi/(4n))$ for $3\pi/(4n) \leq x \leq \pi - 3\pi/(4n)$. But Theorem 2.3(ii) implies that $\max a_n(\mathcal{T}(L,c)) = 2L\int_A \sin(nx)\,dx$, where $A = \{x \in [0,\pi] : w(x) \geq \lambda\}$ and $2L\int_A \sin x\,dx = c$, and sometimes A looks very simply.

If n is even and $w(3\pi/(4n)) \leq w(\alpha)$, then $A = [0,\alpha]$, $c = 4L\sin^2(\alpha/2) \leq 4L\sin^2(3\pi/(8n))$ and $2L\int_A \sin(nx)\,dx = \varphi(n,L,c)$.

If now n is odd and $w(3\pi/(4n)) \leq \lambda \leq n$, then $w^{-1}(\lambda) = \{\alpha, \pi-\alpha\}$ with $0 < \alpha \leq 3\pi/(4n)$, $A = [0,\alpha] \cup [\pi-\alpha,\pi]$, $c = 8L\sin^2(\alpha/2) \leq 8L\sin^2(3\pi/(8n))$ and $2L\int_A \sin(nx)\,dx = \varphi(n,2L,c)$.

This is what the theorem asserts. Since $f \in \mathcal{T}(L,c)$ iff $\{z \mapsto -f(-z)\} \subset \mathcal{T}(L,c)$, and by Remarks 2.5(iii), the proof is complete. In the cases (i), (ii), (iv) all the extremal functions are close–to–convex. For (iii) all of them are not locally univalent.

Remark 3.6.. Applying 2.3(ii) we obtain easily the sharp bounds for the initial coefficients in the class $\mathcal{T}(L,c)$: $\max\{|a_2(f)| : f \in \mathcal{T}(L,c)\} = c(4L-c)/(2L)$, $\min a_3(\mathcal{T}(L,c)) = c(c^2/(12L^2)-1$, $\max a_3(\mathcal{T}(L,c)) = c(6L-c)^2/(12L^2)$ and

$$\max\{|a_4(f)| : f \in \mathcal{T}(L,c)\} = \begin{cases} c(4L-c)(c-2L)^2/(4L^3) \text{ if } c \geq 0 \text{ and } |c-2L| \geq 2L\sqrt{6}/3, \\ c(4L-c)/(2L) + (c-2L)^4/(32L^3) \text{ if } |c-2L| \leq 2L\sqrt{6}/3, \end{cases}$$

see [12] for another proof. Now we find a global bound for the even coefficients in the class $\mathcal{T}(L,c)$.

Theorem 3.7. $\max\{|a_{2n}(f)| : f \in \mathcal{T}(L,c)\} \leq \sqrt{c(4L-c)} \leq 2L$, cf. Th. 3.1(ii), *and strict inequality holds for* $0 < c < 2L$ *and* $2L < c < 4L$.

Before passing to the proof, let us verify the following

Lemma 3.8. *Assume that* $0 < \lambda \leq 1$, $E = \{x \in [0,\pi] : \sin 2nx \geq 0\}$ *and let* $S = \sup\{|A| : A \in \mathcal{B}\,,\ A \subset E\,,\ \int_A \sin x\,dx = \lambda\}$. *Then* $S = \arccos(1-\lambda)$.

Proof. Take any $A \in \mathcal{B}$, $A \subset E$. Then $A_1 \stackrel{df}{=} A \cap [0,\pi/2] \cup h(A \cap [\pi/2,\pi]) \in \mathcal{B}$, $A_1 \subset [0,\pi/2]$, $|A_1| = |A \cap [0,\pi/2]| + |h(A \cap [\pi/2,\pi])| = |A \cap [0,\pi/2]| + |A \cap [\pi/2,\pi]| = |A|$ and $\int_{A_1} \sin x\,dx = \int_A \sin x\,dx$. Thus $S = \sup\{|A| : A \in \mathcal{B}\,,\ A \subset [0,\pi/2]\,,\ \int_A \sin x\,dx = \lambda\}$. Next apply the argument used in the proof of Theorem 3.3.

Proof of theorem 3.7. Because of 2.5(iii) it is sufficient to consider the case $0 < c \leq 2L$. By 2.3(iii) there is $B \in \mathcal{B}$ with $2L\int_B \sin x\,dx = c$ such that $\max a_{2n}(\mathcal{T}(L,c)) = 2L\int_B \sin 2nx\,dx$. Since $2L\int_E \sin x\,dx = 2L = 2L\int_0^{\pi/2} \sin x\,dx \geq$

c, we can find $C \in \mathcal{B}$, $C \subset E \setminus B$, for which $2L \int_{B \cap E \cup C} \sin x\, dx = c$. But $2L \int_{B \cap E \cup C} \sin 2nx\, dx \geq 2L \int_{B \cap E} \sin 2nx\, dx \geq \max a_{2n}(\mathcal{T}(L, c))$, that is $|C| = |B \setminus E| = 0$. Hence and by 2.3(iii) we obtain that $\max a_{2n}(\mathcal{T}(L,c)) = \max\{2L \int_A \sin 2nx\, dx : A \in \mathcal{B}, A \subset E, 2L \int_A \sin x\, dx = c\}$. Finally, it is less than or equal to $\sup\{2L \int_A \sin 2nx\, dx : A \in \mathcal{B}, A \subset E, |A| \leq S\} = 2nL \int_{(\pi-2S)/(4n)}^{(\pi+2S)/(4n)} \sin 2nx\, dx = 2L \sin S = 2L\sqrt{\lambda(2-\lambda)}$, see Lemma 3.8 with $\lambda = c/(2L)$. Also, $\min a_{2n}(\mathcal{T}(L,c)) = -\max a_{2n}(\mathcal{T}(L,c)) \geq -\sqrt{c(4L-c)}$, and the proof is complete.

4. **Bounded functions.** Let us consider the following classes

$$\mathcal{T}_L = \{f \in \mathcal{T} : |f(z)| \leq L \text{ for } z \in \Delta\},$$
$$\mathcal{T}_L(c) = \{f \in \mathcal{T}_L : f'(0) = c\}, \text{ where } 0 \leq c \leq L,$$
$$\mathcal{R} = \{f \in \mathcal{P}_R : f(0) = 1,\ \operatorname{Im} f(z) \operatorname{Im} z \geq 0 \text{ for } z \in \Delta\},$$

and let P_X denote the set of all probability measures on X. Clearly,

(4.1) $f \in \mathcal{T}_L$ if and only if $(1/2)\log((L+f)/(L-f)) \in \mathcal{T}(1/4)$,

(4.2) $f \in \mathcal{T}_L(c)$ if and only if $(1/2)\log((L+f)/(L-f)) \in \mathcal{T}(1/4, c/L)$,

and

(4.3) $f \in \mathcal{R}$ if and only if $(1/2)\log f \in \mathcal{T}(1/4)$.

In [12], as the basic result, it was established that the both classes $\{f^2 : f \in \mathcal{R}\}$ and $\{f/k : f \in \mathcal{T},\ f'(0) = 1\}$, where $k(z) \equiv z/(1+z)^2$, are identical. In particular, putting $Q(z,t) \equiv (1+z)^2/(1-2tz+z^2)$ we have

$$\mathcal{T}(L) = \left\{L \log \int_{-1}^{1} Q(\cdot, t)\, d\nu(t) : \nu \in P_{[-1,1]}\right\} \text{ and } \mathcal{T}_L = \left\{L(f-1)/(f+1) : f \in \mathcal{R}\right\}.$$

Also, in [12] it has been proved that

$$\left\{f^2 : f \in \mathcal{R},\ f'(0) = 2\tau\right\} = \left\{\int_{A(\tau)} \frac{Q(\cdot, x)Q(\cdot, y)}{Q(\cdot, x+y+1-2\tau)}\, d\nu(x,y) : \nu \in P_{A(\tau)}\right\},$$

where $A(\tau)$ is the rectangle $\{(x,y) \in R^2 : -1 \leq x \leq 2\tau - 1 \leq y \leq 1\}$, $0 \leq \tau \leq 1$. Thus

$$\mathcal{T}(L,c) = \{L \log f^2 : f \in \mathcal{R},\ f'(0) = c/(2L)\}, \quad 0 \leq c \leq 4L,$$

and

$$\mathcal{T}_L(c) = \{L(f-1)/(f+1) : f \in \mathcal{R},\ f'(0) = 2c/L\}, \quad 0 \leq c \leq L.$$

By Remark 2.5(iii) we have another interesting property:

$$f \in \mathcal{T}_L(c) \text{ if and only if } L(Lz - f)/(L - zf) \in \mathcal{T}_L(L-c).$$

Applications given in [12] concern mainly the variablity regions of the initial four coefficients in the mentioned classes. By means of the method described in Theorems 2.3–2.4 we can find easily the sharp estimations for the initial five coefficients of bounded typically real functions.

Theorem 4.1. *Let $f \in \mathcal{T}_1(c)$. Then we have*

(i) $|a_2(f)| \le 2c(1-c)$,

(ii) $|a_3(f) - c(1-c)^2| \le 2c(1-c)$,

(iii) $|a_4(f)| \le 2c(1-c)(2-8c+7c^2)$ *if* $0 \le c \le 1/11$,

(iv) $|a_4(f)| \le (1-c)(1+3c)(1+6c-3c^2)/8$ if $1/11 \le c \le 1$, *see* [12],

(v) $a_5(f) \le w(c) \equiv c(1-c)(5-15c+10c^2-2c^3)$ *if* $0 \le c \le (4-\sqrt{13})/3$,

(vi) $a_5(f) \le w(c) + (1-c)(1-8c+3c^2)^2/4$ *if* $(4-\sqrt{13})/3 \le c \le 1$, *and*

(vii) $a_5(f) \ge -c(1-c^2)(5-c^2)/4$.

Proof. Let $g = (1/2)\log((1+f)/(1-f))$ and $f \in \mathcal{T}_1(c)$. By (4.2) we obtain that $g \in \mathcal{T}(1/4, c)$, $a_2(f) = a_2(g)$, $a_3(f) = a_3(g) - c^3/3$, $a_4(f) = a_4(g) - c^2 a_2(g)$ and $a_5(f) = a_5(g) - c^2 a_3(g) - c a_2^2(g) + 2c^5/15$. Because of [12] we shall show only (v), (vi) and (vii). Since $\widetilde{\mathcal{T}}(1/4,c) = \widetilde{\mathcal{T}} \cap \mathcal{T}(1/4,c)$, we can assume that

$$g \in \widetilde{\mathcal{T}}(1/4,c) \quad \text{and} \quad a_5(f) = a_5(g) - c^2 a_3(g) + 2c^5/15\,.$$

In view of Theorem 2.4 we have $\max(\min) a_5(\mathcal{T}_1(c)) = \int_A (\sin 5x - c^2 \sin 3x + (2c^5/15)\sin x)\,dx$, where $\int_A \sin x\,dx = c$ and $A = \{x \in [0,\pi/2] : (\sin 5x - c^2 \sin 3x)/\sin x \ge \lambda\ (\le \lambda)\}$. Notice that $(\sin 5x - c^2\sin 3x)/\sin x \equiv w(\cos x)$, where $w(t) \equiv 16t^4 - 4(3+c^2)t^2 + 1 + c^2$, and that $\max(\min) a_5(\mathcal{T}_1(c)) = \int_B w(t)\,dt$, where $B = \{t \in [0,1] : w(t) \ge \lambda\ (\le \lambda)$ and $|B| = c$.

Since w strictly decreases on $[0, \sqrt{(3+c^2)/8}]$ and strictly increases on $[\sqrt{(3+c^2)/8}, 1]$, and because of $w(\sqrt{3+c^2}/2) = w(0)$, we have

1° $\max a_5(\mathcal{T}_1(c)) = \int_{1-c}^1 w(t)\,dt + 2c^5/15$ if $1 - c \ge \sqrt{3+c^2}/2$,

2° $\max a_5(\mathcal{T}_1(c)) = \int_0^{t_1} w(t)\,dt + \int_{t_2}^1 w(t)\,dt + 2c^5/15$ if $(4-\sqrt{13})/5 \le c \le 1$, where $0 \le t_1 < t_2$, $t_1 + (1-t_2) = c$ and $w(t_1) = w(t_2)$,

3° $\min a_5(\mathcal{T}_1(c)) = \int_{t_1}^{t_2} w(t)\,dt + 2c^5/15$, where $0 \le t_1 < t_2$, $t_2 - t_1 = c$ and $w(t_1) = w(t_2)$ (observe that $\sqrt{3+c^2}/2 \ge c$ for $0 \le c \le 1$).

Corollaries 4.2.

(i) $\max\{|a_2 f)| : f \in \mathcal{T}_1\} = 1/2$,

(ii) $\min a_3(\mathcal{T}_1) = \min a_3(\mathcal{T}_1(1/\sqrt{3})) = -2\sqrt{3}/9$,

(iii) $\max a_3(\mathcal{T}_1) = \max a_3(\mathcal{T}_1((4-\sqrt{7})/3)) = (14\sqrt{7}-20)/27$,

(iv) $\max\{|a_4(f)| : f \in \mathcal{T}_1\} = \max\{|a_4(f)| : f \in \mathcal{T}_1(c_0)\} = 0.508\ldots$, *where* $c_0 = 0.515\ldots$ *is the only zero of the polynomial* $c \mapsto 2 + 3c - 18c^2 + 9c^3$ *in* $[0,1]$, *see* [12],

(v) $\min a_5(\mathcal{T}_1) = \min a_5(\mathcal{T}_1(\sqrt{7/5} - \sqrt{2/5})) = (14\sqrt{35} - 44\sqrt{10})/125$, *and*

(vi) $\max a_5(\mathcal{T}_1) = \max a_5(\mathcal{T}_1(c_1)) = 0.571\ldots$, where $c_1 = 0.4819\ldots$ is the only zero of the polynomial $c \mapsto 3 + 12c - 54c^2 + 36c^3 - 5c^4$ *in* $[0,1]$.

Remarks 4.3.

(i) Each extremal function f in 4.1, 4.2 satisfies the equation :

$$\log((1+f)/(1-f)) = \int_B q(\cdot,t)\,dt \tag{4.4}$$

where the set B is one of the following : $[1-2c,1]$, $[-1,-1+2c]$, $[-1,-1+c]\cup[1-c,1]$, $[-c,c]$, $[t_1,c-1/2]\cup[t_2,1]$, $[-1,-t_2]\cup[1/2-c,-t_1]$, $[-1,-\tau_2]\cup[-\tau_1,\tau_1]\cup[\tau_2,1]$ and $[-s_2,-s_1]\cup[s_1,s_2]$ with

$$t_1+t_2 = 1/2-c\ ,\ t_1t_2 = (-1-4c+3c^2)/4\ ,\ t_1<t_2\ ,\ \tau_2-\tau_1 = 1-c\ ,$$
$$\tau_1\tau_2 = (-1+8c-3c^2)/8\ ,\ s_2-s_1 = c\ ,\ s_1s_2 = 3(1-c^2)/8\ .$$

(ii) The function f satisfying (4.4) is univalent if and only if B is a subinterval of $[-1,1]$ (up to a set of measure zero).

5. Functions having a given part of their Taylor's expansions. As an application of Theorem 2.7(ii) we obtain

Theorem 5.1 [6,7]. *For any positive integer n and $f\in T$ we have*

$$|a_{2n}(f)| \le \sum_{j=1}^{\infty} b_j a_{2j-1}(f) \tag{5.1}$$

and

$$|a_{2n+2}(f)-a_{2n}(f)| \le \sum_{j=0}^{\infty} c_j\left(a_{2j(2n+1)+1}(f)-a_{2j(2n+1)-1}(f)\right)\ , \tag{5.2}$$

where $b_j = 8\pi^{-1}n(4n^2-(2j-1)^2)^{-1}\cot((2j-1)\pi/(4n))$, $c_j = 4\pi^{-1}(-1)^{j+1}(4j^2-1)^{-1}$ and $a_{-1}(f)=0$.

These inequalities are sharp in any class $T[g]$, i.e. for each $g\in T$ there is $f\in T$ with equality in (5.1) or in (5.2) such that $f(z)-f(-z)\equiv g(z)-g(-z)$.

Let us add that the proof needs the following Fourier's expansions :

$$|\sin 2nt| = \sum_{j=1}^{\infty} b_j \sin(2j-1)t \text{ for } 0\le t\le\pi$$

and

$$|\cos(2n+1)t| = c_0/2 + \sum_{j=1}^{\infty} c_j\cos 2(2n+1)jt \text{ for } t\in R\ .$$

Theorem 5.2 [7]. *For $f\in T$ we have*

$$\sum_{j=0}^{\infty}(a_{j+2}(f)+a_j(f))^2 \le 2\sum_{j=1}^{\infty}(a_{2j+1}(f)+a_{2j-1}(f))^2\ , \tag{5.3}$$

$$\sum_{j=0}^{\infty}(a_{j+2}(f)-a_j(f))^2 \le 2\sum_{j=0}^{\infty}(a_{2j+1}(f)-a_{2j-1}(f))^2\ ,\quad a_{-1}(f)=0\ , \tag{5.4}$$

and

$$(5.5) \qquad \sum_{j=1}^{\infty} a_j^2(f) \le 2 \sum_{j=1}^{\infty} a_{2j-1}^2(f).$$

These estimations are sharp in any class $\mathcal{T}[g]$, *see the previous theorem. Moreover, if* $f \in \mathcal{T} \cap H^2$, *then each one of equalities holding in* (5.3)–(5.5) *is equivalent to the following condition :* $f \in \mathcal{ET}[\widetilde{f}]$.

Let us add that the following facts were used in the proof :

$$\mathrm{Im}^2(f(z)+f(-z)) \le \mathrm{Im}^2(f(z)-f(-z)) \text{ for } f \in \mathcal{T},\ z \in \Delta\ ;$$

if $g \in \mathcal{T} \cap H^2$, then $f \in \mathcal{ET}[g]$ if and only if $f \in \mathcal{T}[g]$ and $\mathrm{Im}\, f(\zeta)\, \mathrm{Im}\, f(-\zeta) = 0$ a.e. on $\partial\Delta$.

Corollary 5.3. *Let* $f(z) \equiv z + \sum_{j=1}^{\infty} a_{2j} z^{2j}$ *be univalent in* Δ *and real on the real segment* $(-1,1)$. *Then*

$$|a_2| \le 8/(3\pi)$$

with equality only for the univalent functions : $\widehat{f},\ z \mapsto -\widehat{f}(-z)$, *where*

$$\widehat{f}(z) = z + 2\pi^{-1} - \pi^{-1}(iz+(iz)^{-1})\log((1+iz)/(1-iz)) = z - (8/\pi) \sum_{j=1}^{\infty} \frac{(-1)^j j}{4j^2-1} z^{2j}.$$

Moreover,

$$a_2^2 + \sum_{j=1}^{\infty}(a_{2j+2}+a_{2j})^2 \le 1\ , \quad a_2^2 + \sum_{j=1}^{\infty}(a_{2j+2}-a_{2j})^2 \le 3\ , \quad \sum_{j=1}^{\infty} a_{2j}^2 \le 1$$

with equality for $\widehat{f}$ *and* $z \mapsto -\widehat{f}(-z)$.

Proof. By (5.1) we have $|a_2(f)| \le 8/(3\pi)$ for all $f \in \mathcal{T}[g]$, where $g(z) \equiv z$. Consider the set $\mathcal{A} = \{f \in \mathcal{T}[g] : a_2(f) = 8/(3\pi)\}$. Since $d\nu_g = (2/\pi)\sin^2 x\, dx$, then by Theorem 2.7(ii) or by Theorem 2.9 we obtain that $\mathcal{A} = \{f_\nu\}$, where $\nu = 2(\nu_g)_{[0,\pi/2]}$. Thus it is enough to show that the function $f_\nu = \widehat{f}$ is univalent in Δ. Integration by parts leads to the identity $z\widehat{f}'(z) \equiv f_1(z)f_2(z)$, where $f_1(z) \equiv z/(1+z^2)$, $f_2(z) \equiv (4/\pi)\int_0^{\pi/2}(1-z^2)(1-2z\cos t+z^2)^{-1}\cos^2 t\, dt$. Because f_1 is univalent starlike in Δ and $f_2 \in \mathcal{P}_R$, we get that f is close-to-convex and so f is univalent.

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STRESZCZENIE

Praca jest zakończeniem cyklu [8–10] i dotyczy zwartych wypukłych klas funkcji typowo rzeczywistych, których wartości leżą w zadanym pasie poziomym, lub których wszystkie nieparzyste współczynniki są ustalone. Podobnie jak w [10] punkty ekstremalne i podpierające mogą tworzyć gęste podzbiory, więc każdy ciągły problem ekstremalny nad taką klasą redukuje się do problemu nad jej ekstremalnymi (podpierającymi) punktami. Niektóre zastosowania dotyczą klasy wszystkich funkcji typowo rzeczywistych ograniczonych przez wspólną stałą.