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Extremal Problems in Some Classes of Measures (III) Functions of Positive Real Part

Problemy ekstremalne w pewnych klasach miar (III) Funkcje o dodatniej części rzeczywistej

Abstract. In this paper, being a continuation of [15,16], we consider the sets of extreme and support points for compact convex classes of holomorphic functions with ranges in a given strip or else with the fixed part of their Taylor expansions. It appears that these extremal sets can be dense subsets. By means of suitable affine homeomorphisms we reduce the extremal problems to some sets of Borel measures.

1. Introduction. This paper is a continuation of our previous works [15,16]. Let $\Delta_r = \{z : |z| < r\}$, $\Delta = \Delta_1$, and let $H(\Delta)$ denote the class of all complex functions holomorphic in Δ. Next let $a_j(f) = f^{(j)}(0)/j!$ for $f \in H(\Delta)$, $j = 0, 1, \ldots$. Endowed with the topology of uniform convergence on compacta, the linear space $H(\Delta)$ is metrizable locally convex [4,9,20] and its dual

$$H(\Delta)^* = \left\{ \sum_{j=0}^{\infty} b_j a_j(\cdot) : \overline{\lim} \sqrt[j]{|b_j|} < 1 \right\} = H(\overline{\Delta}) , \tag{1.1}$$

see the Teoplitz theorem [4, 9, 20].

In the present paper we shall be working within the class

$$\mathcal{P} = \{ f \in H(\Delta) : f(0) \geq 0 , \ \operatorname{Re} f(z) \geq 0 \text{ for } z \in \Delta \} \tag{1.2}$$

of all Carathéodory functions. The class (1.2) has been of interest to a number of mathematicians and its basic properties are well known [4,7,8,9,18,20,23]. We recall only the useful Riesz–Herglotz integral representation. Namely, let $\mathcal{B}$ consist of all Borel subsets of the unit circle $\partial\Delta$ and let M be the family of all finite nonnegative measures on the σ-algebra $\mathcal{B}$. Then

$$\mathcal{P} = \{ f_\nu : \nu \in M \} , \tag{1.3}$$

where

$$(1.4)\qquad f_\nu(z) \equiv \int_{\partial\Delta} q(z,x)\,d\nu(x) \quad \text{and} \quad q(z,x) \equiv (1+xz)/(1-xz)\,.$$

Furthermore, assuming the weak–star (metrizable) topology in M, the map $M \ni \nu \mapsto f_\nu$ is an affine homeomorphism from M onto $\mathcal{P}$, see [1,9]. Hence for each $f \in \mathcal{P}$ the equation $f_\nu = f$ has the unique solution $\nu = \nu_f \in M$. Moreover, any measure ν_f is the weak–star limit of a sequence (ν_{f_n}), whenever $f, f_1, f_2, \ldots \in \mathcal{P}$ and $f_n \to f$ uniformly on compact subsets of Δ. For instance, if $f \in \mathcal{P}$, $f_n(z) \equiv f((1-n^{-1})z)$, then

$$(1.5)\qquad \nu_{f_n}(A) = (2\pi)^{-1} \int_A \operatorname{Re} f((1-n^{-1})x)\,d\arg x\,, \quad A \in \mathcal{B}\,,$$

and $\nu_{f_n} \xrightarrow{w^*} \nu_f$ as $n \to \infty$.

In the paper we shall consider such compact convex subclasses of $\mathcal{P}$ to which the methods from [15,16] are especially successful and complete. Namely, let $0 \le c \le L$ and let

$$(1.6)\quad \mathcal{P}(L) = \{f \in H(\Delta) : f(0) \ge 0\,,\ 0 \le \operatorname{Re} f(z) \le L \text{ for } z \in \Delta\}\,,\ L > 0\,,$$

$$(1.7)\quad \mathcal{P}(L,c) = \{f \in \mathcal{P}(L) : f(0) = c\}\,.$$

Next consider

$$(1.8)\qquad \mathcal{P}(n;g) = \{f \in \mathcal{P} : a_{jn}(f) = a_{jn}(g) \text{ for } j = 0,1,2,\ldots\}$$

$$(1.9)\qquad \mathcal{P}[n;g] = \{f \in \mathcal{P} : \operatorname{Re} a_{jn}(f-g) = 0 \text{ for } j = 0,1,2,\ldots\}$$

for an arbitrarily chosen positive integer n and $g \in \mathcal{P}$.

Clearly, $\mathcal{P}(L,c) = \{f \in H(\Delta) : f \prec F_c \text{ in } \Delta\}$, $\mathcal{P}(L) = \bigcup_{0\le c\le L} \mathcal{P}(L,c)$, where

$$(1.10)\qquad F_c(z) \equiv c + \frac{ic}{\alpha}\log\frac{1-e^{i\alpha}z}{1-e^{-i\alpha}z} \equiv c + 2c\sum_{j=1}^{\infty}\frac{\sin j\alpha}{j\alpha}z^j\,, \quad \alpha = \frac{\pi c}{L}\,.$$

However, it seems to the authors that the classical arguments on subordination are not always useful in solving extremal problems for the classes (1.6), (1.7).

Let n be a positive integer and let

$$f_{(n)}(z) \equiv \sum_{j=0}^{\infty} a_{jn}(f) z^{jn} \equiv \sum_{k=0}^{n-1} f(\varepsilon^k z)/n\,, \quad \varepsilon = \exp(2\pi i/n)\,,$$

$$f_{[n]}(z) \equiv \sum_{j=0}^{\infty} (\operatorname{Re} a_{jn}(f)) z^{jn} \equiv \big(f_{(n)}(z) + \overline{f_{(n)}(\bar z)}\big)/2\,,$$

$$\mathcal{P}_{(n)} = \{z \mapsto f(z^n) : f \in \mathcal{P}\}\,, \quad \mathcal{P}_{[n]} = \{z \mapsto \big(f(z^n) + \overline{f(\bar z^n)}\big)/2 : f \in \mathcal{P}\}\,.$$

Then the mapping $f \mapsto f_{(n)}$ (resp. $f \mapsto f_{[n]}$) is a projection of $\mathcal{P}$ onto $\mathcal{P}_{(n)}$ (resp. of $\mathcal{P}$ onto $\mathcal{P}_{[n]}$). The equivalence relation:

$$f \sim g \text{ if and only if } f_{(n)} = g_{(n)} \quad (\text{resp. iff } f_{[n]} = g_{[n]})$$

decomposes $\mathcal{P}$ into compact convex equivalence classes (1.8) (resp. (1.9)) with g ranging over $\mathcal{P}$. This way

$$\mathcal{P} = \bigcup_{g\in\mathcal{P}_{(n)}} \mathcal{P}(n;g) = \bigcup_{g\in\mathcal{P}_{[n]}} \mathcal{P}[n;g]\,.$$

The classes defined in (1.6)–(1.9) have many interesting properties, some of them are curious enough. Namely, every considered class has extreme points of a convenient form and hence maxima of convex continuous functionals over such a class can be found. Furthermore, the classes (1.6), (1.7) and many of (1.8), (1.9) are strongly convex ($= H(\Delta)^*$–rotund), and hence their extreme points form dense subsets.

In a few places we shall use the symbols $\mathcal{S}, \mathcal{S}^*, \mathcal{K}, \mathcal{Y}$ and $\mathcal{T}$ for the subsets of $H(\Delta)$ consisting of these usual normalized functions that are univalent, starlike, convex, circulary symmetric and typically real, respectively. We recall only that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$ and

1° $\mathcal{Y} = \{f \in H(\Delta) : f(0) = f'(0) - 1 = 0,\ \ \operatorname{Im}(zf'(z)\overline{f(z)}) \operatorname{Im} z \geq 0$ for $z \in \Delta\}$,

2° for $f \in \mathcal{Y}$ and $0 < r < 1$ the function $t \mapsto |f(re^{it})|$ decreases in $[0,\pi]$ and increases in $[\pi, 2\pi]$ (strictly if $f(z) \not\equiv z$),

3° for $f \in \mathcal{Y}$ we have $\max\{|f(z)| : |z| \leq r\} = f(r)$ and the function $f|(0,1)$ strictly increases,

4° $\mathcal{Y} \cap \mathcal{S} = \mathcal{Y} \cap \mathcal{T}$,

see [13]. Moreover,

5° $f \in \mathcal{Y}$, $f'(z) \neq 0$ for all $z \in \Delta$ if and only if $f \in H(\Delta)$, $f'(0) = 1$ and the function $z \mapsto (z/(1+z)^2)(zf'(z)/f(z))$ belongs to the $\mathcal{T}$,

6° $f \in \mathcal{Y} \cap \mathcal{S}^*$ if and only if $f \in H(\Delta)$, $f'(0) = 1$ and the function $z \mapsto (z/(1+z)^2)(zf'(z)/f(z))^2$ belongs to the $\mathcal{T}$, see [21].

2. Simple results.

Proposition 2.1. *Let $f \in \mathcal{P}(L,c)$ and $|z| = r < 1$. Then*
(i) $0 \leq F_c(-r) \leq \operatorname{Re} f(z) \leq F_c(r) \leq L$, *see* (1.10)

(ii) $|\operatorname{Im} f(z)| \leq (L/\pi)\log\left[\left(\sqrt{(1-r^2)^2 + 4r^2\sin^2\alpha} + 2r\sin\alpha\right)/(1-r^2)\right]$
$\leq (L/\pi)\log((1+r)/(1-r)) = \operatorname{Im} F_{L/2}(ir)$,

where $\alpha = \pi c/L$.

In particular, for $c = L/2$ we have the following sharp estimation:

$$|\operatorname{Re} f(z) - L/2| \leq (2L/\pi)\arctan r\,.$$

Proposition 2.2.

(i) $\max\{|a_j(f)| : f \in \mathcal{P}(L,c)\} = a_1(F_c) = (2L/\pi)\sin(\pi c/L) \leq 2L/\pi$, $j = 1, 2, \ldots$ *and*

(ii) $\sum_{j=1}^{n} |a_j(f)|^2 \leq \sum_{j=1}^{n} |a_j(F_c)|^2$ *for all $f \in \mathcal{P}(L,c)$, $n = 1, 2, \ldots$.*

Both propositions follow from the well known properties of subordinate functions. We let add that for $0 \le c \le L$, $n = 1, 2, \ldots$, we have

$$\sum_{j=1}^{n} |a_j(F_c)|^2 \le \sum_{j=1}^{\infty} |a_j(F_c)|^2 = 2c(L-c) \le L^2/2$$

and

$$\sum_{j=1}^{n} |a_j(F_c|^2 \le \sum_{j=1}^{n} |a_j(F_{L/2})|^2 = (4L^2/\pi^2) \sum_{2j \le n+1} (2j-1)^{-2}.$$

The last inequality follows from the Fejér inequality

$$(2.1) \qquad \sum_{j=1}^{n} \frac{\sin jt}{j} \ge 0 \quad \text{for} \quad 0 \le t \le \pi \quad \text{and} \quad n = 1, 2, \ldots,$$

stated first as a conjecture by L. Fejér (1910) and proved first by Dunham–Jackson (1911) and by T.H. Gronwall (1912), see [22]. Another way of proving (2.1) depends on the fact that the polynomial $p(z) \equiv z + z^2/2 + \ldots + z^n/n$ is close-to-convex in Δ: Re $((1-z)p'(z)) \ge 0$ in Δ. Thus p is univalent and, consequently, it is typically real.

Now we use the notation from [15,16]. By (1.4), (1.5) we get

Proposition 2.3.

$$\mathcal{P}(L) = \{f_\nu : \nu, \mu - \nu \in M\} = \{f_\nu : \nu \in M^{id}(\partial\Delta, \mathcal{B}, \mu)\}$$

and

$$\mathcal{P}(L, c) = \{f_\nu : \nu \in M^{id}(\partial\Delta, \mathcal{B}, \mu, c)\},$$

where $2\pi\mu(A)/L$ denotes the linear Lebesgue measure of the set $\widetilde{A} = \{t \in [0, 2\pi) : e^{it} \in A\}$, i.e. $\mu(A) = L(2\pi)^{-1}|\widetilde{A}|$ for all $A \in \mathcal{B}$.

For (1.8), (1.9) we have

Proposition 2.4. *Let $\varepsilon = \exp(2\pi i/n)$.*

(i) $$\mathcal{P}(n; g) = \{f_\nu : \nu \in \widehat{M}^h(X, \mathcal{B}, \nu_{g_{(n)}})\},$$

where $X = \partial\Delta$ and $h(x) = \varepsilon x$. Moreover, $X = \widetilde{X}_n$, $\operatorname{orb}(x) = \{x, \varepsilon x, \ldots, \varepsilon^{n-1}x\}$ and $A^{[h]} = \bigcup_{j=0}^{n-1} h^j(A)$.

(ii) $$\mathcal{P}[n; g] = \{f_\nu : \nu \in \widehat{M}^h(X, \mathcal{B}, \nu_{g_{[n]}})\},$$

where $h(x) \equiv \overline{x}\varepsilon^{1+\mathrm{Ent}(n \arg x/\pi)}$ and $X = \partial\Delta = X_{2n} = \widetilde{X}_n \cup \widetilde{X}_{2n}$. Furthermore,

1° $\operatorname{orb}(x) = \{x, \varepsilon x, \ldots, \varepsilon^{n-1}x, \overline{x}, \varepsilon\overline{x}, \ldots, \varepsilon^{n-1}\overline{x}\}$ for all $x \in X$, $\widetilde{X}_n = \operatorname{orb}(e^{\pi i/n}) \cup \operatorname{orb}(e^{2\pi i/n}) = \{e^{k\pi i/n} : k = 0, 1, \ldots, 2n-1\}$ and $A^{[h]} = \bigcup_{j=0}^{2n-1} h^j(A) = \bigcup_{j=0}^{n-1} h^j(A \cup \widehat{A})$ for $A \in \mathcal{B}$, where $\widehat{A} = \{x \in \partial\Delta : \overline{x} \in A\}$, 2° $h(x) = \varepsilon x$ *for all* $x \in \widetilde{X}_n$,

3° *for* $x \in \widetilde{X}_{2n}$ *the point* $h(x)$ *is symmetric to* x *about the halfline:* $\arg z = \pi(1 + Ent(n \arg x/\pi))/n$.

Proof. (i). Observe first that $f \in \mathcal{P}(n; g)$ if and only if $f \in \mathcal{P}$ and $ng_{(n)}(z) \equiv nf_{(n)}(z) \equiv \sum_{k=0}^{n-1} f(\varepsilon^k z)$. Hence $n \int_{\partial\Delta} q(z,x)\, d\nu_{g_{(n)}}(x) \equiv \sum_{k=0}^{n-1} \int_{\partial\Delta} q(\varepsilon^k z, x)\, d\nu_f(x) \equiv \sum_{k=0}^{n-1} \int_{\partial\Delta} q(z, \varepsilon^k x)\, d\nu_f(x) \equiv \int_{\partial\Delta} q(z,x)\, d\Big(\sum_{k=0}^{n-1} \nu_f \circ h^{-k}\Big)(x)$, so that $n\nu_{g_{(n)}} = \nu_f + \nu_f \circ h + \ldots + \nu_f \circ h^{n-1}$. By [15, Proposition 7.1] the proof is complete.

(ii). Similarly we deduce that $f \in \mathcal{P}[n; g]$ if and only if $f \in \mathcal{P}$ and $2ng_{[n]}(z) \equiv 2nf_{[n]}(z) \equiv n(f_{(n)}(z) + \overline{f_{(n)}(\overline{z})})$. Thus $2n\nu_{g_{[n]}}(A) = \sum_{k=0}^{n-1}(\nu_f(\varepsilon^k A) + \nu_f(\varepsilon^k \widehat{A})) = \sum_{k=0}^{2n-1} \nu_f \circ h^k(A)$ and $2n \int_{\partial\Delta} q(z,x)\, d\nu_{g_{[n]}}(x) \equiv \int_{\partial\Delta} q(z,x)\, d\Big(\sum_{k=0}^{2n-1} \nu_f \circ h^k\Big)(x)$ for all $A \in \mathcal{B}$. Finally, the desired result follows from [15, Proposition 7.1].

3. Strong convexity. Let $\mathcal{A}$ be a nonempty compact convex subset of $H(\Delta)$ or, more general, of a locally convex Hausdorff space. By $\mathcal{E}\mathcal{A}$ we denote the set of all extreme points of $\mathcal{A}$, i.e. $\mathcal{E}\mathcal{A} = \{f \in \mathcal{A} : \mathcal{A} \setminus \{f\}$ is convex $\}$. The symbol $\sigma\mathcal{A}$ will denote the set of all support points of $\mathcal{A}$, i.e. $f_0 \in \sigma\mathcal{A}$ if and only if $f_0 \in \mathcal{A}$ and $\operatorname{Re}\Phi(f_0) = \max\{\operatorname{Re}\Phi(f) : f \in \mathcal{A}\}$ for some $\Phi \in H(\Delta)^*$ with $\operatorname{Re}\Phi|\mathcal{A} \neq$ const. The following are well known :

1° $\mathcal{E}\mathcal{A} \subset \sigma\mathcal{A}$ if $\dim \mathcal{A} < \infty$,

2° $\overline{\sigma\mathcal{A}} = \mathcal{A} = \partial\mathcal{A}$ if $\dim \mathcal{A} = \infty$,

3° $\mathcal{A} = \overline{\text{conv}}\,(\mathcal{E}\mathcal{A} \cap \sigma\mathcal{A})$, a generalization of Krein–Milman's theorem [2,12].

Recall that $\mathcal{A}$ is said to be *strongly convex* or $H(\Delta)^*$*-rotund* if $\sigma\mathcal{A} \subset \mathcal{E}\mathcal{A}$. By 2° we obtain

Proposition 3.1. (Klee [14].) *For all infinite dimensional compact strongly convex sets* $\mathcal{A} \subset H(\Delta)$ *we have* $\mathcal{A} = \overline{\sigma\mathcal{A}} = \overline{\mathcal{E}\mathcal{A}} = \partial\mathcal{A}$

Note that in the case of Proposition 3.1 the property 3° is of no interest since then $\mathcal{A} = \overline{\sigma\mathcal{A}} \subset \overline{\text{conv}}\,\sigma\mathcal{A} \subset \overline{\text{conv}}\,\mathcal{E}\mathcal{A} \subset \overline{\mathcal{A}} = \mathcal{A}$

Example 3.2. (Poulsen [19]). Let $\mathcal{A} = \{(x_j) \in l^2 : \sum_{j=1}^{\infty} 4^j |x_j|^2 \le 1\}$. Then $\mathcal{A}$ is an infinite dimensional compact convex subset of l^2 and

$$\sigma\mathcal{A} = \{(x_j) \in l^2 : \sum_{j=1}^{\infty} 4^j |x_j|^2 = 1\} = \mathcal{E}\mathcal{A}.$$

By Proposition 3.1 we get $\mathcal{A} = \overline{\sigma\mathcal{A}} = \overline{\mathcal{E}\mathcal{A}} = \partial\mathcal{A}$

Example 3.3 (Arens, Buck, Carleson, Hoffman, Royden, see [3,9,10]). Let $\mathcal{A} = \{f \in H(\Delta) : |f(z)| \le 1$ for $z \in \Delta\}$. Then $\mathcal{A}$ is an infinite dimensional compact convex subset of $H(\Delta)$ and

$$\sigma\mathcal{A}=\Big\{z\mapsto\bigcap_{j=1}^{n}((z-a_j)/(1-\overline{a}_jz)) : |a_j|\le 1 \text{ for } j=1,\ldots,n,\ n=1,2,\ldots\Big\}\subset\mathcal{E}\mathcal{A}$$

$$=\Big\{f\in\mathcal{A}:\ \int_0^{2\pi}\log(1-|f(e^{it})|)\,dt=-\infty\Big\}.$$

Thus $\mathcal{A}=\overline{\sigma\mathcal{A}}=\overline{\mathcal{E}\mathcal{A}}=\partial\mathcal{A}$ by Proposition 3.1.

For interesting generalizations concerning extreme points of classes of bounded holomorphic functions see [9].

Theorem 3.4. *Let $\psi:\partial\Delta\to R$ be a Lebesgue integrable function on $\partial\Delta$ (with respect to the Lebesgue arc measure on $\partial\Delta$). Then for $0<c<L$*

(i) $$\max\Big\{\int_{\partial\Delta}\psi\,d\nu_f : f\in\mathcal{P}(L)\Big\}=L(2\pi)^{-1}\int_0^{2\pi}\psi^+(e^{it})\,dt\,,$$

(ii) $$\max\Big\{\int_{\partial\Delta}\psi\,d\nu_f : f\in\mathcal{P}(L,c)\Big\}=L(2\pi)^{-1}\int_{\widetilde{A}(\lambda_c)}(\psi(e^{it})-\lambda_c)\,dt+\lambda_c c\,,$$

where $\widetilde{\ }$, $A(\cdot)$ and λ_c are defined by the formulas: $\widetilde{A}=\{t\in[0,2\pi):e^{it}\in A\}$, $A(\lambda)=\{x\in\partial\Delta:\psi(x)\ge\lambda\}$ and $\lambda_c=\sup\{\lambda\in R:|\widetilde{A}(\lambda)|\ge 2\pi c/L\}$.

Moreover,

(iii) $$\sigma\mathcal{P}(L)=\Big\{f_A=L(2\pi)^{-1}\int_A q(\cdot,e^{it})\,dt : A\subset R \text{ is a finite union of intervals, diam } A\le 2\pi\Big\}$$
$$\subsetneq \mathcal{E}\mathcal{P}(L)=\{f_A : A\subset R \text{ is a Borel set, diam } A\le 2\pi\}\,,$$

(iv) $$\sigma\mathcal{P}(L,c)=\{f\in\sigma\mathcal{P}(L):f(0)=c\}\subsetneq\mathcal{E}\mathcal{P}(L,c)=\{f\in\mathcal{E}\mathcal{P}(L):f(0)=c\}\,,$$

(v) $$\mathcal{A}=\overline{\sigma\mathcal{A}}=\overline{\mathcal{E}\mathcal{A}}=\partial\mathcal{A} \text{ if } \mathcal{A}=\mathcal{P}(L) \text{ or } \mathcal{A}=\mathcal{P}(L,c)\,.$$

Proof. (i)–(iv). On account of [15, Th. 6.1,9.1], [16, Th. 3.1, 4.1–4.3] and Proposition 2.3, 3.1, we have (i), (ii) and (v), whereas concerning (iii), (iv) it is sufficient to check only the following inclusions:

$$\{f_A : A\subset R \text{ is a finite union of intervals, diam } A\le 2\pi\,,\ |A|=2\pi c/L\}\subset\sigma\mathcal{P}(L,c)$$

and $\sigma\mathcal{P}(L)\subset\bigcup_{0\le c\le L}\sigma\mathcal{P}(L,c)$. Let $t_0<t_1<\ldots<t_{2n-1}<t_{2n}<t_0+2\pi$, $A=\bigcup_{j=1}^{n}[t_{2j-1},t_{2j}]$, $|A|=2\pi c/L$. The function $w(t)\equiv-\prod_{j=1}^{2n}\sin((t-t_j)/2)$ is a

trigonometric polynomial of n^{th} degree and $A = \{t \in [t_0, t_0 + 2\pi] : w(t) \geq 0\}$. Observe next that there is an algebraic polynomial p such that $w(t) \equiv \operatorname{Re} p(e^{it})$. Thus if we set

$$\Phi(f) \equiv a_0(p)a_0(f) + \sum_{j=1}^{n} a_j(p)a_j(f)/2 ,$$

we obtain the functional $\Phi \in H(\Delta)^*$, see (1.1), such that

1° $\Phi(q(\cdot, x)) = p(x)$ for all $x \in \partial\Delta$

and

2° $\operatorname{Re} \Phi|\mathcal{EP}(L,c) \neq \text{const.}$

Since $\{e^{it} : t \in A\} = \{x \in \partial\Delta : \operatorname{Re} \Phi(q(\cdot, x)) \geq 0\} = A(0)$ and $|A| = 2\pi c/L$, it follows from 3.4(ii) that $\max(\operatorname{Re} \Phi)(\mathcal{P}(L,c)) = L(2\pi)^{-1} \operatorname{Re} \Phi(f_A)$, which proves the first inclusion. The latter is trivial.

Recall that a continuous functional $J : \mathcal{A} \to \mathbf{C}$ is *weakly differentiable relative to* $\mathcal{A}$ if for any $f \in \mathcal{A}$ there exists a complex functional J'_f continuous on $H(\Delta)$ and linear with respect to the field R such that to each variation $f + \varepsilon g + o(\varepsilon) \in \mathcal{A}$ as $\varepsilon \to 0^+$, we have $J(f + \varepsilon g + o(\varepsilon)) = J(f) + \varepsilon J'_f(g) + o(\varepsilon)$ as $\varepsilon \to 0^+$. The functional J'_f is called the *weak derivative of* J *at* f *relative to* $\mathcal{A}$. It is clear that then for each $f \in \mathcal{A}$ there are Φ_f, $\Psi_f \in H(\Delta)^*$ so that $J'_f = \Phi_f + \overline{\Psi}_f$, namely $2\Phi_f(g) \equiv J'_f(g) - iJ'_f(ig)$ and $2\Psi_f(g) \equiv \overline{J'_f(g) - iJ'(ig)}$. Moreover, every $\Phi \in H(\Delta)^*$ is weakly differentiable relative to $\mathcal{A}$ and $\Phi'_f = \Phi$ for all $f \in \mathcal{A}$.

Let $\nu \in M$ and $A \in \mathcal{B}$. Later on we shall use the notation ν_A for the new measure obtained by means of ν and A as follows: $\nu_A(B) = \nu(A \cap B)$, $B \in \mathcal{B}$.

Because of [16, Remark 3.2, Theorems 4.1; 4.2] we have

Theorem 3.5. *Let* $\mathcal{A} = \mathcal{P}(L)$ *or* $\mathcal{A} = \mathcal{P}(L,c)$ *and let* $\mathcal{A}_0$ *consist of all* $f_0 \in \mathcal{A}$ *for which there is a complex functional* J *weakly differentiable relative to* $\mathcal{A}$ *such that* $\operatorname{Re} J(f_0) = \max\{\operatorname{Re} J(f) : f \in \mathcal{A}\}$ *and* $\operatorname{Re} J'_f|\mathcal{A} \neq \text{const.}$ *Then* $\mathcal{A}_0 = \sigma\mathcal{A}$

Remarks 3.6. It is known that

(i) for any $f \in \mathcal{P}$ the limits $\lim_{r\to 1^-} f(re^{it}) \stackrel{df}{=} f(e^{it})$ exist almost everywhere on $[0, 2\pi)$, see [3,10],

(ii) $f \in \mathcal{EP}(L,c)$ if and only if

$$f \in \mathcal{P}(L,c) \quad \text{and} \quad \operatorname{Re} f(e^{it})(L - \operatorname{Re} f(e^{it})) = 0 \quad \text{a.e. on } [0, 2\pi) , \tag{3.1}$$

see [9,17].

We give the proof that (ii) follows easily from Theorem 3.4. Namely, by (1.4), (1.5) and by the Lebesgue dominated convergence theorem we get $\nu_f(A) = (2\pi)^{-1} \int_{\widetilde{A}} \operatorname{Re} f(e^{it})dt$ for all $f \in \mathcal{P}(L,c)$, $A \in \mathcal{B}$, where as previous $\widetilde{A} = \{t \in [0, 2\pi) : e^{it} \in A\}$. Take now any $f \in \mathcal{EP}(L,c)$. According to Theorem 3.4 there exists $B \in \mathcal{B}$ so that $\nu_f = \mu_B$, where $\mu(A) = L(2\pi)^{-1}|\widetilde{A}|$. Thus

$$0 = \nu_f(\partial\Delta \setminus B) = (2\pi)^{-1} \int_{[0,2\pi)\setminus\widetilde{B}} \operatorname{Re} f(e^{it})\,dt$$

and

$$0=\mu(B)-\nu_f(B)=(2\pi)^{-1}\int_{\widetilde{B}}(L-\operatorname{Re} f(e^{it}))\,dt\,,$$

whence (3.1) follows.

Suppose now (3.1) and consider the set $B=\{x\in\partial\Delta:\ \operatorname{Re} f(x)=L\}$. Then $\operatorname{Re} f(e^{it})=0$ a.e. on $[0,2\pi)\setminus\widetilde{B}$ and for all $A\in\mathcal{B}$ we have

$$\nu_f(A)=(2\pi)^{-1}\int_{\widetilde{A}}\operatorname{Re} f(e^{it})\,dt=(2\pi)^{-1}L|\widetilde{A}\cap\widetilde{B}|\,,$$

which means that $f=f_B\in\mathcal{EP}(L,c)$, the desired result.

Using [15, Theorem 8.1, 9.1, 11.1, 11.2] and Proposition 2.4 we obtain

Theorem 3.7. *Let n be a positive integer, $n\geq 2$, $\varepsilon=e^{2\pi i/n}$, and let $g\in\mathcal{P}$. Then we have*

(i) $\mathcal{EP}(n;g)=\Big\{nf_\nu:\nu=(\nu_{g_{(n)}})_A$ *and the sets $A,\varepsilon A,\ldots,\varepsilon^{n-1}A$ form a Borel decomposition of the circle* $\partial\Delta\Big\}$.

(ii) $\max\Big\{\int_{\partial\Delta}\psi\,d\nu_f:f\in\mathcal{P}(n;g)\Big\}=\int_{\partial\Delta}\psi^*\,d\nu_{g_{(n)}}=\int_{\partial\Delta}\psi^*\,d\nu_{\widehat{f}}$ *for all bounded Borel functions $\psi:\partial\Delta\to R$ and all $\widehat{f}\in\mathcal{P}(n;g)$, where* $\psi^*(x)\equiv\max\{\psi(x),\psi(\varepsilon x),\ldots,\psi(\varepsilon^{n-1}x)\}$.

(iii) $\mathcal{EP}(n;g)=\mathcal{P}(n;g)$ *if ν_g is nonatomic.*

Theorem 3.8. *Let n be a positive integer, $\varepsilon=e^{2\pi i/n}$, let $g\in\mathcal{P}$, and given A let $\widehat{A}=\{x\in\partial\Delta:\overline{x}\in A\}$. Then*

(i) $\mathcal{EP}[n;g]=\Big\{nf_{\nu_1}+2nf_{\nu_2}:\nu_j=(\nu_{g_{[n]}})_{A_j}$ *and the sets $A_j,\ldots,\varepsilon^{n-1}A_j,\widehat{A}_j,\ldots,\varepsilon^{n-1}\widehat{A}_j$ form a Borel decomposition of the set $\widetilde{X}_{jn}$, $j=1,2\Big\}$,*

where, we recall, $\widetilde{X}_n=\{e^{k\pi i/n}:k=0,1,\ldots,2n-1\}$, $\widetilde{X}_{2n}=\partial\Delta\setminus\widetilde{X}_n$ and $\widehat{A}_1=\{e^{(2k+1)\pi i/n},\ e^{2l\pi i/n}\}$ for some $k,l\in\{0,1,\ldots,n-1\}$,

(ii) $\max\Big\{\int_{\partial\Delta}\psi\,d\nu_f:f\in\mathcal{P}[n;g]\Big\}=\int_{\partial\Delta}\psi^*\,d\nu_{g_{(n)}}=\int_{\partial\Delta}\psi^*\,d\nu_{\widehat{f}}$ *for all bounded Borel functions $\psi:\partial\Delta\to R$ and all $\widehat{f}\in\mathcal{P}[n;g]$, where*

$$\psi^*(x)=\max\{\psi(x),\ldots,\psi(\varepsilon^{n-1}x),\psi(\overline{x}),\ldots,\psi(\varepsilon^{n-1}\overline{x})\}\,,$$

(iii) *in the case when ν_g is nonatomic, we have $\overline{\mathcal{EP}[n;g]}=\mathcal{P}[n;g]$ and $\mathcal{EP}[n;g]=\Big\{2nf_\nu:\nu=(\nu_{g_{[n]}})_A$, the sets $A,\ldots,\varepsilon^{n-1}A,\widehat{A},\ldots,\varepsilon^{n-1}\widehat{A}$ form a Borel decomposition of the circle $\partial\Delta\Big\}$.*

Theorem 3.9. *Suppose that $g\in\mathcal{P}$, $g(0)>0$, and that ν_g is nonatomic. Next let $n\geq 2$ be a prime number. Then the class $\mathcal{P}(n;g)$ is strongly convex so that $\sigma\mathcal{P}(n;g)=\mathcal{EP}(n;g)=\mathcal{P}(n;g)$. More precisely, the set $\sigma\mathcal{P}(n;g)$ consists of such functions from the set $\mathcal{EP}(n;g)$ for which in 3.7(i) the corresponding A is a finite union of arcs.*

Proof. Let $f_0 \in \sigma\mathcal{P}(n;g)$. Then $n\nu_{g_{(n)}} - \nu_{f_0} \in M$ and for some $\Phi \in H(\Delta)^*$ with $\operatorname{Re}\Phi|\mathcal{P}(n;g) \neq \text{const}$ we have $\operatorname{Re}\Phi(f_0) = \max\{\operatorname{Re}\Phi(f) : f \in \mathcal{P}(n;g)\}$. Put $\varphi(x) = \operatorname{Re}\Phi(q(\cdot,x))$, $\varepsilon = e^{2\pi i/n}$, $\varphi^*(x) = \max\varphi(\operatorname{orb} x)$. By Theorem 3.7 we get

$$\operatorname{Re}\Phi(f_0) = \int_{\partial\Delta}\varphi(x)\,d\nu_{f_0}(x) = \int_{\partial\Delta}\varphi^*(x)\,d\nu_{f_0}(x) = \int_{\partial\Delta}\varphi^*(x)\,d\nu_{g_{(n)}}(x)$$
$$= n\int_G \varphi^*(x)\,d\nu_{g_{(n)}}(x)\,,$$

where G is any measurable generator for $\langle\partial\Delta,\mathcal{B},h\rangle$, see [15] and Proposition 2.4. Consider the set $B = \{x \in \partial\Delta : \varphi(x) = \varphi^*(x)\}$. Then $\nu_{f_0} = (\nu_{f_0})_B$ and $B = B_0 \cup B_1$, where

$$B_0 = \bigcap_{j=1}^{n-1}\{x \in \partial\Delta : \varphi(x) > \varphi(\varepsilon^j x)\} \quad\text{and}\quad B_1 \subset \bigcup_{j=1}^{n-1}\{x \in \partial\Delta : \varphi(x) = \varphi(\varepsilon^j x)\}\,.$$

We shall show that B_1 is finite. In fact, assume that B_1 is infinite. Since n is a prime number, $\varepsilon^j \neq 1$ for all integers j indivisible by n. Because of [16, Lemma 1] we obtain that $\varphi(x) \equiv \varphi(\varepsilon^s x)$ for some $s \in \{1,\ldots,n-1\}$ and then $B = \partial\Delta$, $\operatorname{Re}\Phi|\mathcal{P}(n;g) = \text{const}$, a contradiction. Thus $\nu_{f_0}(B_1) \le n(\nu_{g_{(n)}})(B_1) = 0$, i.e. $\nu_{f_0} = (\nu_{f_0})_{B_0}$. By [15, Remark 2.1] there is a measurable generator G_0 for $\langle\partial\Delta,\mathcal{B},h\rangle$ such that $B_0 \subset G_0 \subset B$. Hence for all numbers $c \ge \|\varphi\|$ we have

$$0 \le \int_{G_0}(\varphi(x)+c)\,d(n\nu_{g_{(n)}} - \nu_{f_0})(x) = \int_{G_0}\varphi(x)\,d(n\nu_{g_{(n)}} - \nu_{f_0})(x)$$
$$= \operatorname{Re}\Phi(f_0) - \operatorname{Re}\Phi(f_0) = 0\,,$$

which means that $\nu_{f_0} = (n\nu_{g_{(n)}})_{G_0} = (n\nu_{g_{(n)}})_B$. An argument similar to that used in the proof of Theorem 3.4 shows that B is a finite union of arcs. This ends the proof.

Theorem 3.10. *Let $g \in \mathcal{P}$, $g(0) > 0$, $n \ge 2$, and suppose that ν_g is nonatomic. Then $\mathcal{P}(n;g)$ is strongly convex if and only if n is a prime number.*

Proof. By the previous theorem it is sufficient to check "only if". Let $\mathcal{P}(n;g)$ be strongly convex and let $n = kl$, where $k \ge 2$, $l \ge 2$ are integers. Then

$$\max\{\operatorname{Re} a_k(f) : f \in \mathcal{P}(n;g)\} = 2n\int_{-\pi/n}^{\pi/n}\cos kt\,d\nu_{g_{(n)}}(e^{it}) > 0$$

with extremal functions $f_1 = n\int_{-\pi/n}^{\pi/n} q(\cdot,e^{it})\,d\nu_{g_{(n)}}(e^{it})$ and $f_2(z) \equiv f_1(\varepsilon^l z)$, where $\varepsilon = e^{2\pi i/n}$. We shall show that

1° $f_1 \neq f_2$

and

2° $\operatorname{Re} a_k|\mathcal{P}(n;g) \neq \text{const}$.

To see 1° observe that $\operatorname{Re} a_1(f_1) = 2n\int_{-\pi/n}^{\pi/n}\cos t\,d\nu_{g_{(n)}}(e^{it}) > 0$, whence $a_1(f_1 - f_2) = (1-\varepsilon^l)a_1(f_1) \neq 0$. Next consider $\widehat{f_1}(z) = f_1(\varepsilon z)$. Since $\widehat{f_1} \in \mathcal{P}(n;g)$

and $\operatorname{Re} a_k(f_1 - \widehat{f_1}) = 4n \int_{-\pi/n}^{\pi/n} \sin(kt + \pi/l)\sin(\pi/l)\, d\nu_{g_{(n)}}(e^{it}) > 0$, the property 2° holds. Finally, we have found distinct functions $f_1, f_2, (f_1+f_2)/2 \in \sigma\mathcal{P}(n;g)$, so that the proof is complete.

Theorem 3.11. *If $g \in \mathcal{P}$, $g(0) > 0$ and ν_g has an atom, then all the classes $\mathcal{P}(n;g)$, $n \geq 3$, are not strongly convex.*

Proof. Let $\lambda = \nu_g(\{b\}) > 0$ for some $b \in \partial\Delta$ and consider the functional $\Phi(f) = \bar{b}e^{-i\pi/n}a_1(f)$. Then by Theorem 3.8 $\max\{\operatorname{Re}\Phi(f) : f \in \mathcal{P}(n;g)\} = n\int_{A_j} \varphi\, d\nu_{g_{(n)}} \geq \lambda\varphi(b) = \lambda\varphi(\varepsilon b) > 0$, $j = 0,1$, where $\varphi(x) = \operatorname{Re}\Phi(q(\cdot,x))$, $A_j = A \cup \{\varepsilon^j b\}$ and $A = \{x \in \partial\Delta : -\pi/n < \arg \Phi(q(\cdot,x)) < \pi/n\}$. Consider extremal functions $f_j = n\int_{A_j} q(\cdot,x)\, d\nu_{g_{(n)}}(x)$, $j = 0,1$. It is sufficient to check that $f_0 \neq f_1$ and that $\operatorname{Re}\Phi|\mathcal{P}(n;g) \neq \text{const}$. In fact, $f_0(z) - f_1(z) = n\nu_{g_{(n)}}(\{b\})(q(z,b) - q(z,\varepsilon b)) = [\nu_g(\{b\}) + \ldots + \nu_g(\{\varepsilon^{n-1}b\})]\cdot[q(z,b) - q(z,\varepsilon b)] \neq 0$ for all $z \in \Delta\setminus\{0\}$. Furthermore, $-\pi/n \leq \arg\Phi(f_0) < \pi/n$, the function $\widehat{f_0}(z) \equiv f_0(\bar{\varepsilon}z)$ belongs to $\mathcal{P}(n;g)$ and $-\pi/2 < \pi/2 - 2\pi/n \leq \arg\Phi(f_0) + \arg(1-\bar{\varepsilon}) = \arg\Phi(f_0 - \widehat{f_0}) < \pi/n + (\pi/2 - \pi/n) = \pi/2$. This completes the proof.

Remark 3.12. Let $x_0 \in \partial\Delta$ and consider the case $g = q(\cdot,x_0)$. Then $\nu_g = \delta_{x_0}$, the set $\mathcal{P}(2;g)$ is identical with the segment $\{(1-\lambda)q(\cdot,x_0) + \lambda q(\cdot,-x_0) : 0 \leq \lambda \leq 1\}$ and amongst the classes $\mathcal{P}(n;g)$, $n \geq 2$, only $\mathcal{P}(2;g)$ is strongly convex: $\sigma\mathcal{P}(2;g) = \{q(\cdot,x_0), q(\cdot,-x_0)\} = \mathcal{E}\mathcal{P}(2;g)$.

Theorem 3.13. *All the classes $\mathcal{P}[n;g]$, $g \in \mathcal{P}$, $g(0) > 0$, $n \geq 3$, are not strongly convex.*

Proof. By Theorem 3.11 we can assume that $g_{[n]}$ is not of the form $z \mapsto \lambda(1+z^n)/(1-z^n)$, $\lambda > 0$, since then $\mathcal{P}[n;g] = \mathcal{P}(n;g)$. From Theorem 3.8 it follows that $\max\{\operatorname{Re} a_1(f) : f \in \mathcal{P}[n;g]\} = \operatorname{Re} a_1(f_j)$, $j = 1,2$, where $f_j = 2n\int_{A_j} q(\cdot,x)\, d\nu_{g_{[n]}}(x) + n\int_{B_j} q(\cdot,x)\, d\nu_{g_{[n]}}(x) \in \mathcal{P}[n;g]$ and for $j = 1,2$ we have $A_j = \{\exp((-1)^j it) : 0 < t < \pi/n\}$, $B_j = \{1, \exp((-1)^j i\pi/n)\}$ for $j = 1,2$. Since $\nu_{f_1} \neq \nu_{f_2}$, it remains to verify that $\operatorname{Re} a_1|\mathcal{P}[n;g] \neq \text{const}$. Put $\widehat{f_1}(z) = f_1(\bar{\varepsilon}z)$. Then $\widehat{f_1} \in \mathcal{P}[n;g]$, $-\pi/n \leq \arg a_1(f_1) < 0$, and $-\pi/2 < \pi/2 - 2\pi/n \leq \arg a_1(f_1 - \widehat{f_1}) < \pi/2 - \pi/n < \pi/2$, so that $\operatorname{Re} a_1(f_1 - \widehat{f_1}) > 0$. This completes the proof.

Remarks 3.14.

(i) When $g \in \mathcal{P}$, $g(0) > 0$ and ν_g is nonatomic, then $\mathcal{P}[1;g]$ is strongly convex (the proof is similar to that in 3.9).

(ii) If $g_{[2]} = q(z^2, \pm 1)$, then the class $\mathcal{P}[2;g] = \mathcal{P}(2;g)$ is strongly convex, see Remark 3.12.

4. Auxiliary lemmas.

Let the symbol $(p;q)$ denote the greatest common divisor of positive integers p and q.

Lemma 4.1. *Suppose that $u : R \to R$ has periods $p\alpha$ and $q\alpha$. Then the number*

$(p;q)\alpha$ is a period of the function u.

Proof. It is sufficient to observe that there are positive integers j,k such that $jp-kq=(p;q)$.

Let us consider now the following integral

$$J(t)=J(p,q,t)=\int_0^\pi |\sin(px)\sin(qx-t)|\,dx\,, \tag{4.1}$$

where p,q are arbitrarily chosen positive integers and $t\in R$.

Lemma 4.2. *The function J is even and has period $\pi(p;q)/p$.*

Proof. Since the integrand is periodic with period π relative to both variables x,t, we get $J(-t)\equiv J(t)\equiv J(t+p(\pi/p))\equiv J(t+q(\pi/p))$. Thus the conclusion follows from Lemma 4.1.

Lemma 4.3. *For $|t|\le \pi(p;q)/p$ we have*
(i) $J(t)\equiv(\pi/2-|t|)\cos t+\sin|t|$ *if* $p=q$,
(ii) $J(t)\equiv w(At/2)$ *if* $p\neq q$,
where $w(x)\equiv 4(w_A(x)-w_B(x))/(A^2-B^2)$, $w_Y(x)\equiv Y\cos((\pi-2|x|)/Y)/\sin(\pi/Y)$ and $A=2p/(p;q)$, $B=2q/(p;q)$.

Proof. Observe first that

$$J(p,q,t)\equiv J(p/(p;q),q/(p;q),t) \tag{4.2}$$

since $J(p,q,t)\equiv(p;q)^{-1}\int_0^{(p;q)\pi}|\sin(p_1u)\sin(q_1u-t)|\,du\equiv$ $\int_0^\pi|\sin(p_1u)\sin(q_1u-t)|\,du\equiv J(p_1,q_1,t)$, where $p/p_1=q/q_1=(p;q)$. Then (i) is trivial and in proving (ii) we may assume that $(p;q)=1$. By Fourier's expansion of the function $R\ni x\mapsto|\sin x|$ we can find similar expansions for $R\ni x\mapsto|\sin(px)|$ and $R\ni x\mapsto|\sin(qx-t)|$. Integration in x of the product of these Fourier series leads to the following

$$J(t)\equiv 4\pi^{-1}\left[1+2\sum_{k=1}^{\infty}(4k^2p^2-1)^{-1}(4k^2q^2-1)^{-1}\cos(2kpt)\right]. \tag{4.3}$$

To calculate the sum (4.3) on the interveal $[-\pi/p,\pi/p]$ use Fourier's expansion of the function $t\mapsto w_Y(pt)$, $|t|\le\pi/p$, and verify that $(w_{2p}(pt)-w_{2q}(pt))/(p^2-q^2)=J(t)$ for $|t|\le\pi/p$. By (4.2) this is what the lemma asserts.

Lemma 4.4. *$J(\pi/A)\le J(t)\le J(0)$ for all real t, where A,B are defined in the previous lemma. Moreover, $J(\pi/A)=1$, $J(0)=\pi/2$ if $p=q$, and $(A^2-B^2)J(\pi/A)=4(A/\sin(\pi/A)-B/\sin(\pi/B))$, $(A^2-B^2)J(0)=4(A\cot(\pi/A)-B\cot(\pi/B))$.*

Proof. It is enough to consider the case $p\neq q$. The inequality: $J(t)\le J(0)$ for all $t\in R$ is trivial by (4.3). However, the proof below suits to both inequalities.

Namely, because of Lemma 4.2 we get the identity $J(t) \equiv J(2\pi/A \pm t)$. Thus it sufficies to check that $dJ/dt < 0$ for $0 < t < \pi/A$. Observe first that $\partial(y\tan(x/y))/\partial y = (\sin(2x/y) - 2x/y)/(2\cos^2(x/y)) < 0$ for $0 < x/y < \pi/2$, whence $(p\tan(x/p) - q\tan(x/q))/(p-q) < 0$ for $0 < x < \pi(p;q)/2$. Therefore the function $u(x) \equiv (\sin(x/q)/\sin(x/p))/(p-q)$ strictly decreases on the interval $[0, \pi(p;q)/2]$. Since $J'(t) = C(t)(u(\pi p/A) - u(\pi p/A - pt))$ for $0 < t < \pi/A$, where $C(t) = 4p(A + B)^{-1}\sin(\pi/A - t)/\sin(\pi/B)$, so indeed $J'(t) < 0$ for $0 < t < \pi/A$.

Lemma 4.5. *Fixed $z \in \Delta$ let us consider the integrals $I(f, A) = \int_A f(e^{it}z)\,dt$, where $f \in H(\Delta)$ and A is a Borel subset of R. If $k_\alpha(\zeta) \equiv \zeta/(1 - e^{i\alpha}\zeta)^2$, $\alpha \in R$, then, independently of α, the following sets: $K_1(\alpha) = \{I(k_\alpha, [a,b]) : a, b \in R,\ a \le b\}$, $K_2(\alpha) = \{I(k_\alpha, A) : \operatorname{diam} A \le 2\pi\}$ and $K_3(\alpha) = \{I(f, A) : f \in \overline{\operatorname{conv}}\, S^*,\ \operatorname{diam} A \le 2\pi\}$ are identical with the closed disc $K = \{w : |w| \le 2|z|/(1 - |z|^2)\}$.*

Proof. Let $r = |z|$. Since $K = \{w_1 - w_2 : |w_j - ie^{-i\alpha}/(1 - r^2)| = r/(1 - r^2),\ j = 1, 2\} = K_1(\alpha) \subset K_2(\alpha) \subset K_3(\alpha)$, it is enough to check the inclusion: $K_3(\alpha) \subset K_1(\alpha)$. To see this we find the numbers: $S(\varphi) = \max\{\operatorname{Re}(e^{-i\varphi}w) : w \in K_3(\alpha)\}$, $\varphi \in R$. Since $\mathcal{E}\overline{\operatorname{conv}}\ S^* = \{k_\alpha : 0 \le \alpha < 2\pi\}$, see [1,9,20], for each φ there is α that $S(\varphi) = \int_0^{2\pi} \operatorname{Re}^+(e^{-i\varphi}k_\alpha(e^{it}z))\,dt = \int_a^b \operatorname{Re}(e^{-i\varphi}k_\alpha(e^{it}z))\,dt$ and $\{e^{it} : a \le t \le b\} = \{\zeta \in \partial\Delta : \operatorname{Re}(e^{-i\varphi}k_\alpha(\zeta z)) \ge 0\}$. By identity $K_1(\alpha) = K$ we obtain that $S(\varphi) \le 2r/(1 - r^2)$ for all real φ, whence we conclude the desired inclusion.

Lemma 4.6. *Let k, n be fixed positive integers, $m = n/(k;n)$ and let $\varepsilon = \exp(2\pi i/n)$. For the function $\partial\Delta \ni x \mapsto d(x) = \max\{\operatorname{Re}(\varepsilon^{jk}x) : j = 0, 1, \dots, n-1\}$, we have $d(x) = \operatorname{Re} x$ if $|\arg x| \le \pi/m$ and*
$$d(x) \equiv (m/\pi)\sin(\pi/m)\left[1 - 2\operatorname{Re}\sum_{j=1}^{\infty}(-1)^j x^{jm}/(j^2m^2 - 1)\right] \text{ if } m > 1.$$

Proof. Observe first that $\{\varepsilon^{jk} : j = 0, 1, \dots, n-1\} = \{\varepsilon_1^s : s = 0, \dots, m-1\}$, where $\varepsilon_1 = \varepsilon^{(k;n)} = \exp(2\pi i/m)$. Hence $d(x) \equiv \max\{\operatorname{Re}(\varepsilon_1^s x) : s = 0, \dots, m-1\}$ and the first equality holds. Since the function

$$R \ni t \mapsto d(e^{it}) \tag{4.4}$$

has period $2\pi/m$, it remains to expand the periodic function (4.4) in the Fourier series.

Lemma 4.7. *Let $\zeta = e^{-it}$, $t \in R$, and consider the function*

$$\partial\Delta \ni x \mapsto D(\zeta, x) = \max\{d(\zeta x), d(\zeta\overline{x})\},$$

where d is defined in the previous lemma. Then

1° $D(\zeta, x) = \cos(|\arg x| - |t|)$ *for* $|\arg x| \le \pi/m$, $|t| \le \pi/m$,

2° $D(\zeta, x) \equiv \operatorname{Re} x\cos t + 2\pi^{-1}(1 - 2\sum_{j=1}^{\infty}(4j^2 - 1)^{-1}\operatorname{Re} x^{2j})|\sin t|$ *if* $m = 1$ *and*

3° $D(\zeta, x) \equiv A_0(t)/2 + \sum_{j=1}^{\infty} A_j(t) \operatorname{Re} x^{jm}$ *if* $m > 1$,

where the A_j, $j = 0, 1, \ldots$, *are periodic functions on* R *with period* $2\pi/m$ *such that* $A_j(t) = 2\pi^{-1}\left[\sin|t| + (-1)^j \sin(\pi/m - |t|)\right] m/(1 - j^2m^2)$ *for* $|t| \le \pi/m$ *and* $j = 0, 1, \ldots$.

Proof. It is easy to see that $D(\zeta, x) = \operatorname{Re}(\zeta x)$ for $-\pi/m \le \arg x \le 0$ and $-\pi/m \le t \le 0$. Since $D(\zeta, x) = D(\overline{\zeta}, x) = D(\zeta, \overline{x}) = D(\varepsilon_1\zeta, x) = D(\zeta, \varepsilon_1 x)$ for all $\zeta, x \in \partial\Delta$, where $\varepsilon_1 = \exp(2\pi i/m)$, we obtain 1° and then 2°, 3°.

Lemma 4.8. *The support function* $S(\varphi) = \max\{\operatorname{Re}(e^{-i\varphi}w) : w \in D\}$, $\varphi \in R$, *of any compact convex subset* D *of the complex plane* $\mathbf{C}$ *has the following properties:*

(i) *S satisfies a Lipschitz condition,*

(ii) *for any real φ there exist the one-sided derivatives $S'_+(\varphi)$ and $S'_-(\varphi)$,*

(iii) *S' exists on R except a countable subset of R,*

(iv) *for the set $\mathcal{E}D$ of all extreme points of D we have the following identities:* $\mathcal{E}D = \mathcal{E}_+ \cup \mathcal{E}_- = \overline{\mathcal{E}}_+ = \overline{\mathcal{E}}_- = \overline{\mathcal{E}_+ \cap \mathcal{E}_-}$, *where*

$$\mathcal{E}_+ = \left\{\left[S(\varphi) + iS'_+(\varphi)\right]e^{i\varphi} : 0 \le \varphi < 2\pi\right\},\ \mathcal{E}_- = \left\{\left[S(\varphi) + iS'_-(\varphi)\right]e^{i\varphi} : 0 \le \varphi < 2\pi\right\},$$

(v) $D = \operatorname{conv}(\mathcal{E}D)$.

Proof. (i). Let $L = \max\{|w| : w \in D\}$. Then for any $w \in D$, $\varphi \in R$ and $-\pi/2 \le t \le \pi/2$ we have $\operatorname{Re}(e^{-i(\varphi+t)}w) \le S(\varphi)\cos t + L\sin|t|$, whence $|S(\varphi + t) - S(\varphi)| \le L(1 - \cos t + \sin|t|) \le L\sqrt{2}|t|$.

(ii). Observe first that $\{w \in D : \operatorname{Re}(e^{-i\varphi}w) = S(\varphi)\} = \operatorname{conv}\{u(\varphi), w(\varphi)\}$ for any $\varphi \in R$, where, to avoid an ambiguity, we assume that

$$\text{(4.5)} \qquad \operatorname{Im}(e^{-i\varphi}u(\varphi)) \le \operatorname{Im}(e^{-i\varphi}w(\varphi)) \quad \text{for every } \varphi \in R\,.$$

Next, for each $\varphi \in R$ and $t \in (-\pi, 0) \cup (0, \pi)$ the system

$$\text{(4.6)} \qquad \operatorname{Re}(ze^{-i\varphi}) = S(\varphi) \quad , \quad \operatorname{Re}(ze^{-i(\varphi+t)}) = S(\varphi + t)$$

has the unique solution

$$\text{(4.7)} \qquad z = z_{\varphi,t} = e^{i\varphi}(S(\varphi)e^{it} - S(\varphi + t))/(i\sin t)\,.$$

It is easy to check that for all real φ

$$1^\circ \quad u(\varphi^+) = w(\varphi^+) = w(\varphi) \quad , \quad 2^\circ \quad u(\varphi^-) = w(\varphi^-) = u(\varphi) \quad ,$$
$$3^\circ \quad \lim_{t \to 0^+} z_{\varphi,t} = w(\varphi) \quad , \quad 4^\circ \quad \lim_{t \to 0^-} z_{\varphi,t} = u(\varphi)$$

Indeed, take any $t_n \to 0^+$. Since D is compact, there is a subsequence (t_{k_n}) of (t_n) such that $w(\varphi + t_{k_n}) \to w_0 \in D$ when $n \to \infty$. By continuity of S, see (i), we obtain $S(\varphi) = \lim_{n\to\infty} S(\varphi + t_{k_n}) = \operatorname{Re} e^{-i\varphi}w_0$ and hence $w_0 \in \operatorname{conv}\{u(\varphi), w(\varphi)\}$.

1) If $u(\varphi)=w(\varphi)$, then $w_0=w(\varphi)$, i.e. $w(\varphi)$ is the unique cluster point of the sequence $(w(\varphi+t_n))$ and then $w(\varphi^+)=w(\varphi)$.

2) In the case $u(\varphi)\neq w(\varphi)$ we argue as follows. A simple calculation gives

$$(4.8)\ |w(\varphi+t)-u(\varphi)|^2\geq|w(\varphi+t)-z_{\varphi,t}|^2+|w(\varphi)-u(\varphi)|^2$$
$$\geq|u(\varphi+t)-z_{\varphi,t}|^2+|w(\varphi)-u(\varphi)|^2\geq|w(\varphi)-u(\varphi)|^2 \text{ for } 0<t<\pi/2\,.$$

Indeed, in view of (4.5), (4.6) we have

$$|w(\varphi+t)-u(\varphi)|^2-|w(\varphi+t)-z_{\varphi,t}|^2-|z_{\varphi,t}-u(\varphi)|^2$$
$$=2\,\mathrm{Im}\,[e^{-i(\varphi+t)}(w(\varphi+t)-z_{\varphi,t})]\,\mathrm{Im}\,[e^{-i\varphi}(z_{\varphi,t}-u(\varphi))]\cos t\geq 0 \text{ for } 0<t<\pi/2,$$

whence (4.8) follows. Put now $w_0=(1-\lambda)u(\varphi)+\lambda w(\varphi)$ for some $0\leq\lambda\leq 1$. Passing in (4.8) to the limit as $t\to 0^+$ we get the inequality $\lambda|w(\varphi)-u(\varphi)|^2\geq|w(\varphi)-u(\varphi)|^2$, whence $\lambda=1$ and $w_0=w(\varphi)$, i.e. $w(\varphi)$ is the unique cluster point of the sequence $(w(\varphi+t_n))$.

Since the equality $w(\varphi^+)=w(\varphi)$ has been proved for all real φ, we may use (4.8) once again when $t\to 0^+$. We thus obtain $\lim_{t\to 0^+}|w(\varphi+t)-z_{\varphi,t}|^2=0=\lim_{t\to 0^+}|u(\varphi+t)-z_{\varphi,t}|^2$, whence $w(\varphi)=w(\varphi^+)=\lim_{t\to 0^+}z_{\varphi,t}=u(\varphi^+)$ for $\varphi\in R$.

Similarly we prove the remainder 2° and 4°. Finally, by (4.7) we have $(S(\varphi+t)-S(\varphi))/t=S(\varphi)(e^{it}-1)/t-ie^{-i\varphi}z_{\varphi,t}\sin t/t$ for all $\varphi\in R$, $0<|t|<\pi$, and hence, by 3° − 4°, we obtain that

$$(4.9)\qquad w(\varphi)\equiv e^{i\varphi}(S(\varphi)+iS'_+(\varphi))\,,$$
$$(4.10)\qquad u(\varphi)\equiv e^{i\varphi}(S(\varphi)+iS'_-(\varphi))\,.$$

(iii). The sum $s=\sum_{0\leq\varphi<2\pi}|w(\varphi-u(\varphi)|$ is finite since s is not greater than the perimeter of D (if D is a segment with ends a,b, then $s=2|a-b|$). So the set $\{\varphi\in[0,2\pi):u(\varphi)\neq w(\varphi)\}$ is countable. In view of (4.9), (4.10) the proof is complete.

(iv) follows immediately from (4.9), (4.10) since

$$\mathcal{E}D=\{u(\varphi):0\leq\varphi<2\pi\}\cup\{w(\varphi):0\leq\varphi<2\pi\}\,.$$

(v) is an immediate consequence of the Minkowski–Carathéodory theorem, see [12].

5. Selected estimations.

Theorem 5.1. *Let m,n be distinct positive integers, let $p=|m-n|$, $q=m+n$ and let*

$$D(\alpha)=\{a_m(f)-e^{i\alpha}a_n(f):f\in H(\Delta)\,,\ 0\leq\ \mathrm{Re}\,f\leq L\}\,,$$
$$S(\varphi,\alpha)=\max\{\,\mathrm{Re}\,(e^{-i\varphi}w):w\in D(\alpha)\}\,.$$

Then

(i) *for all real* α *we have* $D(\alpha) = e^{i\beta}D(\alpha) = e^{i\gamma(\alpha)}D(0)$, *where* $\beta = \pi(p;q)/p$, $\gamma(\alpha) = m\alpha/(m-n)$,

(ii) *for any* $\alpha, \varphi \in R$ *we have* $S(\varphi,\alpha) = (2L/\pi)J(p,q,\varphi-\gamma(\alpha))$, *see* (4.1) *and Lemmas* 4.2–4.4,

(iii) $\max\{|w| : w \in D(\alpha)\} = \max\{|a_m(f)|+|a_n(f)| : f \in H(\Delta),\ 0 \le \operatorname{Re} f \le L\} = S(0,0)$ *for any* $\alpha \in R$,

(iv) $\max\{\operatorname{Im}(a_m(f) - a_n(f)) : f \in H(\Delta),\ 0 \le \operatorname{Re} f \le L\} = S(\pi/2, 0)$
$= \begin{cases} S(0,0) & \text{if } p/(p;q) \text{ is even}, \\ S(\pi(p;q)/(2p), 0) & \text{if otherwise}, \end{cases}$

(v) *the set* $\{w : |w| \le S(\pi(p;q)/(2p), 0)\}$ *is the largest disc contained in each* $D(\alpha)$,

(vi) *the boundary of* $D(0)$ *has the equation* $[0,2\pi) \ni \varphi \mapsto 8L\pi^{-2}\sum_{k=-\infty}^{\infty}(1+kA)^{-1}(1-k^2B^2)^{-1}e^{i(1+kA)\varphi} = e^{i\varphi}(J(\varphi)+iJ'(\varphi))$, *see* (4.1) *and Lemmas* 4.2–4.4.

Proof. (i). Let k, l be integers satisfying the condition: $kp - lq = (p;q)$. Together with $f \in \mathcal{P}(L)$ consider the functions

$$f_1(z) \equiv f(e^{-2\pi i l/p}z)\quad,\quad f_2(z) \equiv L - f_1(z) \quad\text{and}\quad f_3(z) \equiv f(e^{i\gamma(\alpha)/m}z)\,.$$

Obviously, $f_1, f_2, f_3 \in \mathcal{P}(L)$ and $e^{i\beta}(a_m(f) - e^{i\alpha}a_n(f)) = \pm(a_m(f_1) - e^{i\alpha}a_n(f_1)) = a_m(f_j) - e^{i\alpha}a_n(f_j)$ for a suitable $j = 1$ or 2; $e^{i\gamma(\alpha)}(a_m(f) - a_n(f)) = a_m(f_3) - e^{i\alpha}a_n(f_3)$.

(ii). Let $\psi(t) \equiv \cos(mt - \varphi) - \cos(nt - \varphi + \alpha)$. By Theorem 3.4(i) we find

$$S(\varphi,\alpha) = (L/\pi)\int_0^{2\pi}\psi^+(t)\,dt = L(2\pi)^{-1}\int_0^{2\pi}(|\psi(t)|+\psi(t))\,dt = L(2\pi)^{-1}\int_0^{2\pi}|\psi(t)|\,dt$$
$$= (L/\pi)\int_{\alpha/(2m-2n)}^{\pi+\alpha/(2m-2n)}|\psi(2t)|\,dt = (2L/\pi)J(p,q,\varphi-\gamma(\alpha))\,.$$

(iii). Because of Lemma 4.4 and just proved (ii) we have $S(\varphi,\alpha) \le S(0,0)$, whence $\max\{|w| : w \in D(\alpha)\} \le S(0,0) = S(-m\alpha/(n-m),\alpha) \le \max\{|w| : w \in D(\alpha)\} \le d \overset{df}{=} \max\{|a_m(f)|+|a_n(f)| : f \in \mathcal{P}(L)\}$. Let $|a_m(f_0)|+|a_n(f_0)| = d$ for some f_0 from $\mathcal{P}(L)$. Then there is an $\hat{\alpha} \in R$ such that $d = |a_m(f_0) - e^{i\hat{\alpha}}a_n(f_0)| \le S(0,0)$.

(iv). By (ii) and Lemma 4.2 it is sufficient to observe that

$$\pi/2 = [p/(2(p;q))]\pi(p;q)/p = [p/(2(p;q)) - 1/2]\pi(p;q)/p + \pi(p;q)/(2p)\,.$$

(v). Apply (ii) and Lemma 4.4.

(vi) follows from (ii), (4.3) and Lemma 4.8.

Theorem 5.2. *For each fixed* $z \in \Delta$ *and* $L > 0$ *we have*

$$\{f'(z) : f \in \mathcal{P}(L)\} = \{w : |w| \le 2L/(\pi - \pi|z|^2)\}\,.$$

This way the set $\{f'(z) : f \in H(\Delta)\ ,\ f(\Delta)$ *is contained in a strip of width* $L\}$ *is identical with the closed disc* $\{w : |w| \le 2L/(\pi - \pi|z|^2)\}$.

Proof. In view of Theorem 3.4(iii) we have $\{f'(z) : f \in \mathcal{EP}(L)\} = \{(L/(\pi z))I(k_0, A) : \text{diam } A \le 2\pi\} = \{w : |w| \le (2L/\pi)/(1-|z|^2)\}$, see Lema 4.5. Consequently, the Krein–Milman theorem implies the first identity. The second is trivial by the previous one.

Corollary 5.3. *According to the well known characterization of* BMOA *functions* [6], *we have* $\text{BMOA} = \bigcup_{L,M>0} \mathcal{R}(L,M) = \bigcup_{L,M>0} \tilde{\mathcal{R}}(L,M)$, *where* $\mathcal{R}(L,M) = \{f + ig : f, g \in H(\Delta)$ *and* $|\text{Re } f| \le L\ ,\ |\text{Re } g| \le M\}$,
$\tilde{\mathcal{R}} = \{f + g : f, g \in H(\Delta)$ *and* $f(\Delta), g(\Delta)$ *are contained in some strips of width* $2L, 2M$, *respectively* $\}$.
As an easy consequence of Theorem 5.2 *we obtain* $\{f'(z) : f \in \mathcal{R}(L,M)\} = \{f'(z) : f \in \tilde{\mathcal{R}}(L,M)\} = \{w : |w| \le 4\pi^{-1}(L+M)/(1-|z|^2)\}$ *for* $z \in \Delta$.

Let us apply now Lemma 4.5 to classes of normalized univalent functions having the same bounds for the angular velocity of the radius–vector or of the tangent–vector. More precisely, given $L > 1$ consider

$$\mathcal{S}^*(L) = \{f \in H(\Delta) : f'(0) = 1\ ,\ 0 < \text{Re}\,(zf'/f) < L\} \subset \mathcal{S}^* = \mathcal{S}^*(\infty)$$

and

$$\mathcal{K}(L) = \{f \in H(\Delta) : f(0) = f'(0) - 1 = 0\ ,\ 0 < \text{Re}\,(1 + zf''/f') < L\} \subset \mathcal{K} = \mathcal{K}(\infty)\,.$$

Obviously, $\mathcal{S}^*(L) = \{zf' : f \in \mathcal{K}(L)\}$. Recall that for any $f \in \mathcal{S}^*$ and $g \in \mathcal{K}$ the functions f/z and g' are subordinate to k/z in Δ, where $k(z) \equiv z/(1-z)^2$, see [9]. A similar property holds in the classes $\mathcal{S}^*(L)$, $\mathcal{K}(L)$. But then, instead of the Koebe function k, we shall use the following

$$(5.1)\qquad H_L(z) \equiv z\exp\Big[-(L/\pi)\int_{-\pi/L}^{\pi/L} \log(1 - e^{it}z)\,dt\Big],\quad L > 1 \qquad (H_\infty = k)\,,$$

whose properties are stated in

Lemma 5.4. *Let* $L > 1$ *and* $h_L = \log(H_L/z)$. *Then*
(i) $H_L \in \mathcal{Y} \cap \mathcal{S}^*(L)$,
(ii) $h_L(z) \equiv (2L/\pi)\sum_{j=1}^{\infty} z^j \sin(j\pi/L)/j^2$ *and* $h_L/a_1(h_L) \in \mathcal{K}$,
(iii) H_L/z *is one–to–one.*

Proof. (i). Since $zH_L'/H_L = F_1$, see (1.10), and $(F_1 - 1)/a_1(F_1) \in \mathcal{K}$, $F_1((-1,1)) \subset R$, we obtain that $(F_1 - 1)/a_1(F_1) \in \mathcal{T}$ and hence $H_L \in \mathcal{S}^*(L) \cap \mathcal{Y}$.

(ii). Integration in t yields the desired expansion. The next conclusion follows from the inequality $\text{Re}\,(1 + zh_L''/h_L') = \text{Re}\,(z(F_1 - 1)'/(F_1 - 1)) > 0$ (even $> 1/2$).

(iii). Since $H_L/z \prec k/z$ in Δ, we obtain that $|\text{Im } h_L(z)| = |\arg\,(H_L(z)/z)| \le |\arg\,(k(z)/z)| < \pi$ for $z \in \Delta$. Thus H_L/z is a composition of two univalent functions: $\exp|\{w : |\text{Im } w| < \pi\}$ and h_L.

Theorem 5.5. *Let $L > 1$, $f \in \mathcal{S}^*(L)$, $g \in \mathcal{K}(L)$ and let $|z_0| \le r < 1$. Then*

(i) $f/z \prec H_L/z$, $g' \prec H_L/z$ *in* Δ,

(ii) $-H_L(-1) < -H_L(-r)/r \le |f(z_0)/z_0| \le H_L(r)/r < H_L(1)$,

(iii) $\int_0^1 (-H_L(-t)/t)\,dt < \int_0^r (-H_L(-t)/t)\,dt \le |g(z_0)| \le \int_0^r (H_L(t)/t)\,dt < \int_0^1 (H_L(t)/t)\,dt$,

(iv) $|\arg(f(z_0)/z_0)|$ *and* $|\arg g'(z_0)|$ *are less than or equal to* $\arg(H_L(a)/a)$, *where* $a = r^2\cos(\pi/L) + ir(1 - r^2\cos(\pi/L))^{1/2}$, *and* $\arg H_L(a)/a < \pi - \pi/L = \arg(H_L(e^{i\pi/L})/e^{i\pi L})$.

The functions $f_\varphi(z) \equiv e^{-i\varphi}H_L(e^{i\varphi}z)$, $g_\varphi(z) = \int_0^r (f_\varphi(tz/r)/t)\,dt$, $0 \le \varphi < 2\pi$, *show that equalities are possible in* (i)–(iv).

Proof. (i). Let us find the numbers

$$S(b,\varphi) = \max\{\operatorname{Re}[e^{-i\varphi}\log(f(b)/b)] : f \in \mathcal{S}^*(L)\}\ , \quad \text{where } b \in \Delta \text{ and } \varphi \in R\,.$$

Since the correspondence $\mathcal{S}^*(L) \ni f \leftrightarrow zf'/f = p \in \mathcal{P}(L,1)$ is a homeomorphism and $J(p) \overset{df}{=} \int_0^1 (p(bt) - 1)t^{-1}\,dt = \log(f(b)/b)$, we get that $S(b,\varphi) = \max\{\operatorname{Re}(e^{-i\varphi}J(p)) : p \in \mathcal{E}\mathcal{P}(L,1)\} = \max\{L(2\pi)^{-1}\int_A \operatorname{Re}[e^{-i\varphi}J(q(\cdot,e^{it}))]\,dt : A$ is a Borel subset od R, diam $A \le 2\pi$ and $|A| = 2\pi/LL\}$, see Theorem 3.4. But $J(q(\cdot,\zeta)) = -2\log(1-b\zeta)$ and the function $\zeta \mapsto \log(1-b\zeta)$ is convex in $\overline{\Delta}$. Therefore we have successively:

1° for every $\lambda \in R$ the set $\{\zeta \in \partial\Delta : \operatorname{Re}[e^{-i\varphi}J(q(\cdot,\zeta))] \ge \lambda\}$ is a closed connected subset of the circle $\partial\Delta$, i.e. it is a closed subarc of $\partial\Delta$ (including perhaps a one–element set or the empty set),

2° there exists the unique α depending on φ such that $S(b,\varphi) = L(2\pi)^{-1}\int_{\alpha-\pi/L}^{\alpha+\pi/L} \operatorname{Re}[e^{-i\varphi}J(q(\cdot,e^{it}))]\,dt = \operatorname{Re}[e^{-i\varphi}J(z \mapsto F_1(e^{i\alpha}z))] = \operatorname{Re}[e^{-i\varphi}h_L(e^{i\alpha}b)] \le S(\varphi) \overset{df}{=} \max\{\operatorname{Re}(e^{-i\varphi}h_L(z)) : |z| \le |b|\}$,

3° $J(\mathcal{P}(L,1)) = \bigcap_{0\le\varphi<2\pi}\{w \in \mathbf{C} : \operatorname{Re}(e^{-i\varphi}w) \le S(b,\varphi)\} \subset \bigcap_{0\le\varphi<2\pi}\{w \in \mathbf{C} : \operatorname{Re}(e^{-i\varphi}w) \le S(\varphi)\} = h_L(\{z \in \mathbf{C} : |z| \le |b|\})$,

4° $\log(f/z) \prec h_L = \log(H_L/z)$ in Δ by subordination principle and

5° $f/z \prec H_L/z$ in Δ.

(ii). Applying Lemma 5.4(i) or (ii) we obtain

$$(H_L/z)(\Delta_r) \subset \{w \in \mathbf{C} : -H_L(-r)/r < |w| < H_L(r)/r\} \quad \text{for} \quad 0 < r < 1\,,$$

whence, by (i), we get the desired conclusion.

(iii). The right inequalities follow trivially by integrating (ii). For the rest we argue as follows. Denote $m(r) = \min\{|g'(z)| : |z| = r\}$, $0 \le r < 1$. Clearly, m decreases on $[0,1)$ and for any $0 < r < 1$ there is $z(r)$, $|z(r)| = r$, such that $\min\{|g(z)| : |z| = r\} = |g(z(r))| > 0$. Fix r, set $z(t) = g^{-1}(tg(z(r))/r)$ for $0 \le t \le r$ and consider the set $\Gamma = \{z(t) : 0 \le t \le r\}$. Obviously, Γ is an analytic Jordan arc with endpoints 0, $z(r)$ and $|g(z(r))| = \int_0^r |g'(z(t))z'(t)|\,dt = \int_0^{|\Gamma|} |g'(z(\tau^{-1}(s)))|\,ds \ge \int_0^{|\Gamma|} m(|z(\tau^{-1}(s))|)\,ds \ge \int_0^{\min\{1,|\Gamma|\}} m(s)\,ds \ge \int_0^r m(s)\,ds$, where we have denoted $s = \tau(t) = \int_0^t |z'(x)|\,dx$ for $0 \le t \le r$ ($|z(t)| \le \tau(t)$ if $0 \le t \le r$). However by (ii) we have $m(s) \geq -H_L(-s)/s$ for all $s \in (0,r)$ and integration in s gives what the left inequality asserts.

(iv). Observe that if for some $F \in H(\Delta)$ we have $\max\{\operatorname{Im} F(z) : |z| = r\} = \operatorname{Im} F(a)$, $|a| = r$, then $\operatorname{Re}(aF'(a)) = 0$. Putting $F = \log(H_L/z)$ we obtain $\operatorname{Re} F_1(a) = 1$, whence $\operatorname{Re} a = r^2 \cos(\pi/L)$, see (1.10).

Theorem 5.6. *For any $f \in \mathcal{P}$, $\varphi \in R$ and positive integers k, m, $m \geq 2$, we have*

$$\operatorname{Re}(e^{-i\varphi} a_k(f)) \leq S_k(\varphi, f), \tag{5.2}$$

where $S_k(\varphi, f) = (2m/\pi)\sin(\pi/m) \operatorname{Re} \sum_{j=0}^{\infty} (-1)^{j+1}(j^2m^2-1)^{-1} e^{-ijm\varphi} a_{jkm}(f)$. The above estimation is sharp in the following sense: for each $g \in \mathcal{P}$ there is $f \in \mathcal{P}$ with equality in (5.2) such that $a_{jkm}(f) = a_{jkm}(g)$ for $j = 0, 1, \ldots$. Equivalently, $\max\{\operatorname{Re}(e^{-i\varphi}a_k(f)) : f \in \mathcal{P}(n; g)\} = S(\varphi, g)$, whenever k is not divisible by n and $m = n/(k; n)$.

Proof. Use Theorem 3.7 and Lemma 4.6.

Corollaries 5.7. (i) *For any $f \in \mathcal{P}$ we have the following sharp inequalities*

$$|\operatorname{Re} a_k(f)| \leq (4/\pi) \sum_{j=0}^{\infty} (-1)^{j+1}(4j^2-1)^{-1} \operatorname{Re} a_{2jk}(f),$$

$$|\operatorname{Im} a_k(f)| \leq (4/\pi) \sum_{j=0}^{\infty} (1-4j^2)^{-1} \operatorname{Re} a_{2jk}(f), \quad k = 1, 2, \ldots.$$

(ii) *Let $D_{k,n}(c) = \{a_k(f) : f \in \mathcal{P},\ f(0) = 1,\ a_{jn}(f) = c \text{ for } j = 1, 2, \ldots\}$, where $0 \leq c \leq 2$ and k is a positive integer indivisible by n. Then $D_{k,n}(c) = \operatorname{conv} \bigcup_{j=0}^{m-1} \varepsilon^j \Gamma$, where $m = n/(k; n)$, $\varepsilon = \exp(2\pi i/m)$ and $\Gamma = \{c + (2-c)(m/\pi)\sin(\pi/m)e^{i\varphi} : -\pi/m \leq \varphi \leq \pi/m\}$.*

In the limit cases we obtain:

$$D_{k,n}(0) = \{w : |w| \leq 2(m/\pi)\sin(\pi/m)\} \text{ and } D_{k,n}(2) = \operatorname{conv}\{2\varepsilon^j : j = 0, 1, \ldots, m-1\}.$$

Proof. (i). Apply Theorem 5.6 to $m = 2$ and $\varphi = 0, \pi$ or $\varphi = \pm\pi/2$.

(ii). Use Theorem 5.6 in the case $2g(z) \equiv 2 - c + c(1+z)/(1-z)$ and apply Lemma 4.8. The support function of $D_{k,n}(c)$ has the form $S(\varphi) = c\cos\varphi + (2-c)(m/\pi)\sin(\pi/m)$.

Theorem 5.8. *For all $f \in \mathcal{P}$, $\varphi \in R$ and positive integers k, m we have*

$$\operatorname{Re}(e^{-i\varphi} a_k(f)) \leq \widehat{S}(\varphi, f), \tag{5.3}$$

where

$$\widehat{S}(\varphi, f) = (2m/\pi)\sum_{j=0}^{\infty}(1-j^2m^2)^{-1}[\sin|\varphi| + (-1)^j\sin(\pi/m - |\varphi|)]\,\mathrm{Re}\,a_{jmk}(f)$$

if $m > 1$ *and* $|\varphi| \le \pi/m$,

$$\widehat{S}(\varphi, f) = \mathrm{Re}\,a_k(f)\cos\varphi + (2/\pi)\sin|\varphi|\sum_{j=0}^{\infty}(1-4j^2)^{-1}\,\mathrm{Re}\,a_{2jk}(f)$$

if $m = 1$ *and* $|\varphi| \le \pi$,

$\widehat{S}(\varphi, f) = \widehat{S}(\varphi + 2\pi/m, f)$ *for all real* φ.

The estimation (5.3) is sharp in the following sense: for each $g \in \mathcal{P}$ *there is* $f \in \mathcal{P}$ *with equality in (5.3) such that* $\mathrm{Re}\,a_{jkm}(f) = \mathrm{Re}\,a_{jkm}(g)$ *for* $j = 0, 1, \ldots$. *Equivalently,* $\max\{\mathrm{Re}\,(e^{-i\varphi}a_k(f)) : f \in \mathcal{P}[n; g]\} = \widehat{S}(\varphi, g)$, *where* $m = n/(k; n)$.

Proof. Use Theorem 5.6 and Lemmas 4.7, 4.8.

Corollaries 5.9. (i) *For any* $g \in \mathcal{P}$ *and all positive integers* k, n *we have*

$$\{a_k(f) : f \in \mathcal{P}[n; g]\} = \mathrm{conv}\{a, \overline{a}, \varepsilon a, \varepsilon\overline{a}, \ldots, \varepsilon^{m-1}a, \varepsilon^{m-1}\overline{a}\},$$

where $m = n/(k; n)$, $\varepsilon = \exp(2\pi i/m)$ *and*

$$a = (2mi/\pi)\sum_{j=0}^{\infty}(1-j^2m^2)^{-1}(1-(-1)^j e^{\pi i/m})\,\mathrm{Re}\,a_{jmk}(g) \text{ if } m > 1,$$

$$a = \mathrm{Re}\,a_k(g) - (4i/\pi)\sum_{j=0}^{\infty}(4j^2-1)^{-1}\,\mathrm{Re}\,a_{2jk}(g) \text{ if } m = 1.$$

In particular, $a = c + (2-c)(2m/\pi)\sin(\pi/(2m))\exp(\pi i/(2m))$ *for* $a_0(g) = 1$, $\mathrm{Re}\,a_n(g) = \mathrm{Re}\,a_{2n}(g) = \ldots = c$, $0 \le c \le 2$, *so that*

1° *the set* $\{a_k(f) : f \in \mathcal{P},\ f(0) = 1,\ \mathrm{Re}\,a_{jn}(f) = 0$ *for* $j = 1, 2, \ldots\}$ *is identical with the regular polygon*

$$\mathrm{conv}\{(4m/\pi)\sin(\pi/(2m))e^{\pi i/(2m)}\eta^j : j = 0, 1, \ldots, 2m-1\},$$

where $m = n/(k; n)$ *and* $\eta = \exp(\pi i/m)$,

2° $\{a_k(f) : f \in \mathcal{P},\ f(0) = 1\ \ \mathrm{Re}\,a_{jn}(f) = 2$ *for* $j = 1, 2, \ldots\} = D_{k,n}(2)$, *see Corollaries 5.7(ii).*

(ii) *For any* $f \in \mathcal{P}$ *and all positive integers* k, m, $m \ge 2$, *we have sharp inequalities*

$$|\mathrm{Re}\,a_k(f)| + |\mathrm{Im}\,a_k(f)| \le (8/\pi)\sum_{j=0}^{\infty}(1-16j^2)^{-1}\,\mathrm{Re}\,a_{4jk}(f)$$

$$(= \mathrm{Re}\,a + \mathrm{Im}\,a \text{ if } n = 2k, 4k)$$

and

$$|a_k(f)|^2 \le \frac{16m^2}{\pi^2}\sin^2\frac{\pi}{2m}\Big(\sum_{j=0}^{\infty}\frac{\operatorname{Re} a_{2jmk}(f)}{1-4j^2m^2}\Big)^2 + \frac{16m^2}{\pi^2}\cos^2\frac{\pi}{2m}\Big(\sum_{j=1}^{\infty}\frac{\operatorname{Re} a_{(2j-1)mk}(f)}{(2j-1)^2m^2-1}\Big)^2 .$$

Proof. (i). Use Theorem 5.8 and Lemma 4.8. (ii) follows from (i). We let add that the inequality before last is a direct consequence of Corollaries 5.7(i). However, its sharpness follows from (i) since $\{a_k(f) : f \in \mathcal{P}[2k;g]\} = \operatorname{conv}\{a, \overline{a}, -a, -\overline{a}\}$.

Theorem 5.10. *For any positive integer n and $f \in \mathcal{P}$ with $f(0) = 1$ we have sharp inequalities*

$$2 + \sum_{j=1}^{\infty} |a_j(f)|^2 \le n\Big(2 + \sum_{j=1}^{\infty} |a_{jn}(f)|^2\Big) , \tag{5.4}$$

$$2 + \sum_{j=1}^{\infty} |a_j(f)|^2 \le 2n\Big(2 + \sum_{j=1}^{\infty} \operatorname{Re}^2 a_{jn}(f)\Big) . \tag{5.5}$$

Moreover, assuming for a Carathéodory function f to be in the Hardy second class H^2, we obtain that ν_f is nonatomic and

1° *equality in* (5.4) *is equivalent to the condition: $f \in \mathcal{EP}(n; f_{(n)})$,*

2° *equality in* (5.5) *is equivalent to the condition: $f \in \mathcal{EP}[n; f_{[n]}]$.*

Proof. Recall some known facts from the theory of H^p spaces. Namely, $H^p \subset H^q$ for $p \ge q > 0$ (trivial), for any $p > 0$ all functions $f \in H^p$ have the nontangential limits $f(e^{it})$ almost everywhere and if for some $f \in H^p$ with $p > 0$ the equality $f(e^{it}) = 0$ holds on a set of positive Lebesgue measure, then $f(z) \equiv 0$, see [3]. Moreover, $d\nu_f(e^{it}) = (2\pi)^{-1}\operatorname{Re} f(e^{it})\,dt$ for each $f \in \mathcal{P} \cap H^1$. To verify the last statement, denote $f_n(z) \equiv f((1-n^{-1})z)$ and $d\nu(e^{it}) = (2\pi)^{-1}\operatorname{Re} f(e^{it})\,dt$. Then for all real functions u continuous on $[0, 2\pi)$ we have

$$\Big|\int_0^{2\pi} u(t)\,d[\nu_{f_n}(e^{it}) - \nu(e^{it})]\Big| \le (2\pi)^{-1}\|u\| \int_0^{2\pi} |f_n(e^{it}) - f(e^{it})|\,dt \to 0 \text{ as } n \to \infty ,$$

see [3], whence ν is the weak-star limit of the sequence (ν_{f_n}), i.e. $\nu = \nu_f$.

If the right sides of (5.4) and (5.5) are infinite then the inequalities holds. So we can assume that $f_{(n)}$ in (5.4) and $f_{[n]}$ in (5.5) belong to H^2.

Let $\widehat{f} \in \mathcal{P}$, $\widehat{f}(0) = 1$, $\widehat{f}_{(n)} \in H^2$. Then, by $\operatorname{Re}(n\widehat{f}_{(n)} - \widehat{f}) \ge 0$,

$$\int_0^{2\pi} |\widehat{f}(re^{it})|^2\,dt < 2\int_0^{2\pi} \operatorname{Re}^2 \widehat{f}(re^{it})\,dt \le 2n^2 \int_0^{2\pi} |\widehat{f}_{(n)}(e^{it})|^2\,dt ,$$

which means that $\widehat{f} \in H^2$. Therefore $2+\sum_{j=1}^{\infty} |a_j(\widehat{f})|^2 = \pi^{-1}\int_0^{2\pi} \text{Re}^2 \widehat{f}(e^{it})\,dt \le \max\{\pi^{-1}\int_0^{2\pi} \text{Re}^2 f(e^{it})\,dt : f \in \mathcal{EP}(n;\widehat{f}_{(n)})\} = \max\{n^2\pi^{-1}\int_G \text{Re}^2 \widehat{f}_{(n)}(x)\,d\arg x : G, \varepsilon G, \dots, \varepsilon^{n-1}G$ form a Borel decomposition of $\partial\Delta\}$, where $\varepsilon = \exp(2\pi i/n)$, cf. Theorem 3.7. But $\int_{\varepsilon^j G} \text{Re}^2 \widehat{f}_{(n)}(x) d\arg x = \int_G \text{Re}^2 \widehat{f}_{(n)}(x) d\arg x$, so $2+\sum_{j=1}^{\infty} |a_j(\widehat{f})|^2 \le n\pi^{-1}\int_{\partial\Delta} \text{Re}^2 \widehat{f}_{(n)}(x) d\arg x = n(2+\sum_{j=1}^{\infty} |a_{jn}(\widehat{f})|^2)$. Following the above considerations, we remark that the functional $f \mapsto 2+\sum_{j=1}^{\infty} |a_j(f)|^2$ is constant on the set $\mathcal{EP}(n;\widehat{f}_{(n)})$. So it remains to show that the conditions: $f \in \mathcal{P}\cap H^2$, equality in (5.4) imply: $f \in \mathcal{EP}(n; f_{(n)})$. If not, we have $f=(1-\lambda)f_1+\lambda f_2$ with $f_1, f_2 \in \mathcal{P}(n; f_{(n)})$, $f_1 \ne f_2$, $0<\lambda<1$, and $n\int_{\partial\Delta} \text{Re}^2 f_{(n)}(x) d\arg x = A = \int_{\partial\Delta} \text{Re}^2 f(x) d\arg x = (1-\lambda)^2\int_{\partial\Delta} \text{Re}^2 f_1(x) d\arg x + 2(1-\lambda)\lambda\int_{\partial\Delta} \text{Re} f_1(x)\,\text{Re} f_2(x) d\arg x + \lambda^2\int_{\partial\Delta} \text{Re}^2 f_2(x) d\arg x \le (1-\lambda)^2 A + 2(1-\lambda)\lambda\sqrt{A}\sqrt{A} + \lambda^2 A = A$. Hence there is $t_0 \ge 0$ such that $\text{Re} f_1(x) = t_0\,\text{Re} f_2(x)$ almost everywhere on $\partial\Delta$. Thus $t_0 = 1$ and $f_1 = f_2$, a contradiction.

The proof of (5.5) and 2° proceeds similarly by Theorem 3.8(iii).

An open problem 5.11. What is the sharp upper bound for the integral

$$I(p) = (2\pi)^{-1}\int_0^{2\pi} |p(e^{it})|^2\,dt$$

over the class $\mathcal{P}_n$ of all Carathéodory polynomials p of at most n^{th} degree with $p(0)=1$? From (5.4) it follows that

$$I(p) < 2n+1 \quad \text{for any } p \in \mathcal{P}_n\,, \tag{5.6}$$

since $\mathcal{P}_n \subset \mathcal{P}(n+1; z\mapsto 1)$. The inequality (5.6) one can also get from the following sharp estimations

$$|a_j(p)| + |a_{n-j+1}(p)| \le 2 \quad \text{for } j=1,\dots,n \text{ and } p \in \mathcal{P}_n\,, \tag{5.7}$$

due to Egerváry, Szász [5].

We let add that the Holland result:

$$|p(z)| \le n+1 \quad \text{for } p \in \mathcal{P}_n\,,\ z \in \Delta\,, \text{ see [11]}\,,$$

is a simple consequence of (5.7):

$$2|p(z)| \le 2 + \sum_{j=1}^{n}(|a_j(p)| + |a_{n-j+1}(p)|) \le 2(n+1) \quad \text{for any } z \in \Delta \text{ and } p \in \mathcal{P}_n\,.$$

Let ν be a complex Radon measure on $\partial\Delta$, i.e. $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$, where $\nu_j \in M$ for $j=1,2,3,4$. If $\int_{\partial\Delta} x^n\,d\nu(x) = 0$ for $n=1,2,\dots$, then the measure ν is absolutely continuous with respect to the Lebesgue arc measure on $\partial\Delta$ (the theorem of F. and M. Riesz, see [3,8]). This result is trivial for real Radon measures, since then ν is a multiple of the Lebesgue arc measure. Indeed, if $\nu = \nu_1 - \nu_2$, $\nu_1, \nu_2 \in M$, then $f_{\nu_1} - f_{\nu_2} =$ const, so that $f_{\nu_1} = f_{\nu_2+\mu}$, where μ is a multiple of the Lebesgue arc measure. Thus $\nu = \nu_1 - \nu_2 = \mu$.

Since for any $f \in \mathcal{P}(n; z \mapsto c)$, where $c > 0$, we have: $\nu_f(\partial\Delta) = f(0) = c$ and $\nu_f(A) \leq nc(2\pi)^{-1}|\{t \in [0, 2\pi) : e^{it} \in A\}|$ for $A \in \mathcal{B}$, we obtain

Proposition 5.12. *Let n be a positive integer and let $\nu \in M$. If*

$$\int_{\partial\Delta} x^{jn}\, d\nu(x) = 0 \quad \text{for } j = 1, 2, \ldots,$$

then for all $A \in \mathcal{B}$ we have $\nu(A) \leq n\nu(\partial\Delta)|\{t \in [0, 2\pi) : e^{it} \in A\}|/(2\pi)$, whence it follows that ν is absolutely continuous with respect to the Lebesgue arc measure on $\partial\Delta$.

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STRESZCZENIE

W pracy, będącej kontynuacją artykułów [15,16], rozważamy zbiory punktów ekstremalnych i podpierających dla zwartych wypukłych klas funkcji holomorficznych, których wartości są w zadanym pasie, bądź których część rozwinięcia Taylora jest ustalona. Okazuje się, że te zbiory ekstremalne mogą być gęstymi podzbiorami. Za pomocą odpowiednich homeomorfizmów afinicznych redukujemy problemy ekstremalne do pewnych zbiorów miar borelowskich.