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On Radii of Univalence, Starlikeness and Bounded Turning

O promieniach jednolistności, gwiaździstości i ograniczonego obrotu

Abstract. This paper deals with a simple method which enables us to determine the largest disks on that every function from a given class is univalent, starlike or its turning is bounded.

1. Introduction. Let H be the class of all complex functions holomorphic on the open unit disk Δ. For brevity we use the notation: $\Delta_r = \{z : |z| < r\}$, $\Delta = \Delta_1$, $H_0 = \{f \in H : f(0) = f'(0) - 1 = 0\}$, $H_1 = \{f \in H_0 : f(z)/z \neq 0 \text{ for all } z \in \Delta\}$ and $L(A) = \{\log(f/z) : f \in A\}$ whenever $A \subset H_1$. In the last-defined set take $\log 1 = 0$. The convex hull of A and the closed convex hull of A we denote by conv A and $\overline{\text{conv}}\, A$, respectively.

Let us consider any $A \subset H_0$. In this paper we shall derive a simple method which enables us in many cases to determine the largest disks $\Delta_r \subset \Delta$ on that every function from A is univalent, starlike or its turning is bounded.

Strictly speaking, for classes $A \subset H_0$ that satisfy some geometric properties the following quantities will be examined:

$$r_A = \sup\{r \in (0,1) : \text{each } f \in A \text{ is univalent on } \Delta_r\},$$

i.e. the radius of univalence,

$$r^*_A = \sup\{r \in (0,1) : \text{Re}\,[zf'/f] > 0 \text{ on } \Delta_r \text{ for all } f \in A\},$$

i.e. the radius of starlikeness and

$$r'_A = \sup\{r \in (0,1) : \text{Re}\, f' > 0 \text{ on } \Delta_r \text{ for all } f \in A\},$$

i.e. the radius of bounded turning.

The class A is said to be

(i) convex if $(1-t)f + tg \in A$ whenever $f, g \in A$ and $0 \le t \le 1$,

(ii) conjugate invariant if for any $f \in A$ the function $z \mapsto \overline{f(\bar{z})}$ belongs to A,

(iii) rotation invariant if for all $f \in A$ and $|\eta| = 1$ the functions $z \mapsto \overline{\eta} f(\eta z)$ are in A.

The general results contained in Theorems 1-2 and Corollaries 1-2 concern just such classes and are useful in applications to the classes (2)-(6) or to their closed convex hulls.

2. Basic results.

Theorem 1. *If $A \subset H_0$ is nonempty convex and rotation invariant, then*

(1) $$r_A = \sup\{r \in (0,1) : f'(r) \neq 0 \ \text{ for all } f \in A\}.$$

In the proof we use

Lemma . *Suppose that $A \subset H_0$ is nonempty, convex and rotation invariant. Then for each $\zeta \in \Delta$ there is $f \in A$ such that $f'(\zeta) = 1$. If moreover A is compact, then A contains the identity mapping.*

Proof. Take any $f_0 \in A$ and fix $\zeta \in \Delta$. By the assumption the functions $z \mapsto \overline{\eta} f_0(\eta z)$, $|\eta| = 1$, are in A and

$$1 = (2\pi)^{-1} \int_0^{2\pi} f_0'(e^{it}\zeta)\, dt \in \overline{\operatorname{conv}}\{f_0'(\eta\zeta) : |\eta| = 1\}$$
$$= \operatorname{conv}\{f_0'(\eta\zeta) : |\eta| = 1\}$$

by the Minkowski theorem, see [1]. Thus there is a function $z \mapsto t_1\overline{\eta}_1 f_0(\eta_1 z) + \cdots + t_k\overline{\eta}_k f_0(\eta_k z)$, $t_j \geq 0$, $|\eta_j| = 1$, $t_1 + \cdots + t_k = 1$, having the desired property. We let add that in the function we can put $k = 2$, see [1], p.35.

If A is compact, then the function

$$z \mapsto (2\pi)^{-1} \int_0^{2\pi} e^{-it} f_0(e^{it} z)\, dt = z$$

belongs to $\overline{\operatorname{conv}}\{z \mapsto \overline{\eta} f_0(\eta z) : |\eta| = 1\} \subset A$.

Remark. The first part of Lemma follows also from the following facts. Namely, if $f_0 \in A$, $\zeta \in \Delta$ and $r = |\zeta|$, then $f_0'(\partial\Delta_r) \subset \{f'(\zeta) : f \in A\}$ and the last set is convex. By the maximum principle

$$1 = f_0'(0) \in f_0'(\Delta_r) \subset \{f'(\zeta) : f \in A\}.$$

Proof of Theorem 1. Denote the supremum in (1) by ρ. Obviously $\rho \geq r_A$. If $\rho = 0$, then $r_A = 0 = \rho$. Assuming that $\rho > 0$ fix an arbitrary point $\zeta \in \Delta$ and consider the functional $f \mapsto \Phi_\zeta(f) = f'(\zeta)$. Observe first that $\Phi_\zeta(A)$ is convex, $\Phi_\zeta(A) = \Phi_{|\zeta|}(A)$ and $0 \notin \Phi_\zeta(A)$. It follows by Lemma that $1 \in \Phi_\zeta(A)$ so there exists $t = t(|\zeta|) \in (-\pi/2, \pi/2)$ such that $\operatorname{Re}\left[e^{-it}\Phi_z(f)\right] > 0$ for all $f \in A$ and $|z| = |\zeta|$. By the maximum principle Re $[e^{-it}f'(z)] > 0$ for all $f \in A$ and $|z| \leq |\zeta|$ which means that each $f \in A$ is univalent on $\Delta_{|\zeta|}$. Since ζ was chosen arbitrarily, $\rho \leq r_A$. The theorem is proved.

There is a nice corollary to the proof. Namely, if we assume additionally that A is conjugate invariant, then for any $\zeta \in \Delta_\rho$ the set $\Phi_\zeta(A)$ is symmetric with respect to the real axis, i.e. there is $t(|\zeta|) = 0$ and we have

Corollary 1. *If $A \subset H_0$ is convex, rotation and conjugate invariant, then $r'_A = r_A$, where r_A is determined in* (1) *or, more precisely,*

$$r_A = \sup\{r \in (0,1) : \text{Re } f'(r) > 0 \text{ for all } f \in A\}.$$

A similar result is contained in

Theorem 2. *Let $A \subset H_1$ be nonempty and rotation invariant. If $L(A)$ is convex, then* (1) *holds.*

Proof. Following the previous proof denote the right side of (1) by ρ. Clearly $\rho \geq r_A$. Assuming that $\rho > 0$ take $\zeta \in \Delta_\rho$ and consider the functionals $f \mapsto \Phi_\zeta(f) = f'(\zeta)$, $g \mapsto \Psi_\zeta(g) = \zeta g'(\zeta) + 1$. Observe first that $\Psi_\zeta(L(A))$ is convex, $0 \notin \Phi_\zeta(A) = \Phi_{|\zeta|}(A)$ and $\Psi_\zeta(g) = \zeta f'(\zeta)/f(\zeta)$ for $g(z) \equiv \log[f(z)/z]$. Hence $0 \notin \Psi_\zeta(L(A)) = \Psi_{|\zeta|}(L(A))$ and a similar argument used in the proof of Lemma shows that there is a function $g \in L(A)$ for that $g'(\zeta) = 0$. Therefore $1 \in \Psi_\zeta(L(A))$ and there is $t = t(|\zeta|) \in (-\pi/2, \pi/2)$ such that Re $[e^{-it}zf'(z)/f(z)] > 0$ for all $f \in A$ and $|z| = |\zeta|$. By the maximum principle each $f \in A$ is t-spirallike on $\Delta_{|\zeta|}$ and, since this is true for all $|\zeta| < \rho$, we obtain $\rho \leq r_A$. The proof is complete.

If moreover in Theorem 2 we assume that A is conjugate invariant, then for each $\zeta \in \Delta$ the set $\Psi_\zeta(L(A))$ is symmetric with respect to the real axis, i.e. there is $t(|\zeta|) = 0$ and Theorem 2 has the following

Corollary 2. *Suppose that $A \subset H_1$ is rotation and conjugate invariant. If $L(A)$ is convex, then $r^*_A = r_A$, where r_A is determined in* (1).

3. Applications. For $0 \leq \alpha \leq 1$ let $P_\alpha = \{p \in H : \text{Re } p > \alpha \text{ on } \Delta,\ p(0) = 1\}$ and $P = P_0$. We shall solve some radius problems for the following classes or for their closed convex hulls:

(2) $\quad A(\alpha, \lambda) = \{zp^\lambda : p \in P_\alpha\}\ ,\quad 0 \leq \alpha \leq 1\ ,\quad \lambda \in \mathbf{R}\ ,$

(3) $\quad B(M) = \{f \in H_1 : |f| < M \text{ on } \Delta\}\ ,\quad M > 1\ ,$

(4) $\quad S^* = \{f \in H_0 : \text{Re}(zf'/f) > 0 \text{ on } \Delta\}\ ,$

(5) $\quad K(\beta) = \{f \in H_0 : \text{Re}[e^{i\beta}zf'/g] > 0 \text{ on } \Delta \text{ for some } g \in S^*\}\ ,$
$\quad -\pi/2 < \beta < \pi/2\ ,$

(6) $\quad S = \{f \in H_0 : f \text{ is univalent on } \Delta\}\ .$

As a first application we get

Theorem 3.

(i) $r'_{A(\alpha,\lambda)} = r_{A(\alpha,\lambda)}$ *if* $-1 \le \lambda \le 1$.

(ii) $r^*_{A(\alpha,\lambda)} = r_{A(\alpha,\lambda)}$ *if* $0 \le \alpha < 1$, $\lambda \in \mathbf{R}$.

(iii) $r_{A(\alpha,\lambda)}$ *is the unique positive solution* r *of the equation* $2\alpha - 1 + 2(1-\alpha)d(\lambda, r) = 0$, *where* $d(\lambda, r) = \min\{\mathrm{Re}[(1-\lambda)/(1-z) + \lambda/(1-z)^2] : |z| = r\}$.

Proof. (i). All the classes $A(\alpha,\lambda)$ with $0 \le \alpha \le 1$, $-1 \le \lambda \le 1$ are compact convex. Indeed, fix $0 \le \alpha < 1$, $-1 \le \lambda \le 1$, and consider the function $h(z) = \{[1+(1-2\alpha)z]/(1-z)\}^\lambda$ that is holomorphic and univalent on Δ. Since $zh'/h'(0) \in S^*$, the set $h(\Delta)$ is convex and we have the identity

$$A(\alpha,\lambda) = \{f \in H(\Delta) : f/z \prec h \text{ on } \Delta\}$$

which means the convexity of $A(\alpha,\lambda)$. Furthermore, $A(\alpha,\lambda)$ is conjugate and rotation invariant so we may use Corollary 1.

(ii). Fix $0 \le \alpha < 1$, $\lambda \in \mathbf{R}$ and consider the function $g = \log h$, where h has been defined in the proof of (i). The function g is univalent on Δ and the set $g(\Delta)$ is convex since $zg'/g'(0) \in S^*$. Thus

$$L(A(\alpha,\lambda)) = \{f \in H(\Delta) : f \prec g \text{ on } \Delta\},$$

whence the convexity of $L(A(\alpha,\lambda))$ follows. By Corollary 2 we get the desired conclusion.

(iii). For all $0 \le \alpha \le 1$, $\lambda \in \mathbf{R}$ the class $A(\alpha,\lambda)$ satisfies the hypotheses of Corollary 2. Therefore $r_{A(\alpha,\lambda)} = \sup\{r \in (0,1) : f'(r) \ne 0 \text{ for all } f \in A(\alpha,\lambda)\} = \sup\{r \in (0,1) : p(r) + \lambda r p'(r) \ne 0 \text{ for all } p \in P_\alpha\} = \sup\{r \in (0,1) : \mathrm{Re}\,[p(r) + \lambda r p'(r)] > 0 \text{ for all } p \in P_\alpha\}$. Since the set of all extreme points of the class P_α consists of the following functions $z \mapsto (1 + (1-2\alpha)\zeta z)/(1-\zeta z)$, $|\zeta| = 1$, we have hence

$$r_{A(\alpha,\lambda)} = \sup\{r \in (0,1) : 2\alpha - 1 + 2(1-\alpha)\, d(\lambda, r) \ge 0\}.$$

Corollary 3.

(i) $$r'_{A(\alpha,1)} = r^*_{A(\alpha,1)} = r_{A(\alpha,1)} = \begin{cases} \sqrt{2(1-\alpha)/(1-2\alpha)} - 1 & \text{for } 0 \le \alpha \le 1/10, \\ \sqrt{\alpha/(\alpha + \sqrt{\alpha - \alpha^2})} & \text{for } 1/10 \le \alpha \le 1, \end{cases}$$

see [2], v.II, pp.96, 98,

(ii) $$r^*_{A(0,\lambda)} = r_{A(0,\lambda)} = \sqrt{\lambda^2 + 1} - |\lambda|,$$

see the case $\lambda = 1$ in [2], v.I, p.129 (19) and v.II, p.98,

(iii) $$r^*_{A(1/2,\lambda)} = r_{A(1/2,\lambda)} = \begin{cases} \sqrt{1 + 2\sqrt{\lambda} - \lambda}/(1 + \sqrt{\lambda}) & \text{if } 0 \le \lambda \le 4, \\ 1/(\lambda - 1) & \text{if } \lambda \ge 4, \end{cases}$$

(iv) $$r^*_{A(\alpha,\lambda)} = r_{A(\alpha,\lambda)} = (\lambda - 1 + a + \sqrt{(1-\lambda)^2 + 2a\lambda})/a$$

for $0 \le \alpha < 1$, $\lambda \le 0$, where $a = (1-2\alpha)/(1-\alpha)$.

Proof. A quite elementary calculation shows us that

$$\begin{aligned} d(\lambda, r)(1-r^2)^2 &= \min\{\mathrm{Re}[(1-\lambda)(1-r^2)w + \lambda w^2] : |w-1| = r\} \\ &= \min\{2\lambda r^2 t^2 + r[1 - r^2 + \lambda(1+r^2)]t + 1 - r^2 : -1 \le t \le 1\}, \end{aligned}$$

whence it follows

1° $d(\lambda, r) = [-(1-\lambda)^2 r^4 + 2(1-4\lambda-\lambda^2 r^2 - \lambda^2 + 6\lambda - 1]/[8\lambda(1-r^2)^2]$ if $\lambda > 0$ and $(\lambda+1)/(2\lambda+\sqrt{3\lambda^2+1}) \le r < 1$,

2° $d(\lambda, r) = [1 + (1-\lambda)r]/(1+r)^2$ if $\lambda \ge 0$ and $0 \le r \le (\lambda+1)/(2\lambda+\sqrt{3\lambda^2+1})$ or else if $\lambda \le 0$ and $r^2 \le (1+\lambda)/(1-\lambda)$,

3° $d(\lambda, r) = [1 - (1-\lambda)r]/(1-r)^2$ if $\lambda < 0$ and $(1+\lambda)/(1-\lambda) \le r^2 < 1$.

The next step is to examine the equation stated in Theorem 3(iii) for suitable values of α and λ.

For bounded functions with the only zero at the origin we have the following Noshiro result.

Theorem 4.

$$r^*_{B(M)} = r_{B(M)} = 1 + \log M - \sqrt{(2 + \log M)\log M},$$

see [2], v.II, pp.95, 107.

Proof. Since $L(B(M)) = \log M - (\log M)P$, the class $B(M)$ satisfies the assumptions of Corollary 2. Thus

$$\begin{aligned} r^*_{B(M)} = r_{B(M)} &= \sup\{r \in (0,1) : 1 - rp'(r)\log M \ne 0 \text{ for all } p \in P\} \\ &= \sup\{r \in (0,1) : \mathrm{Re}[1 - rp'(r)\log M] > 0 \text{ for all } p \in P\}. \end{aligned}$$

Restricting our linear extremal problem to the extreme points of P we get

$$r^*_{B(M)} = r_{B(M)} = \max\{r \in (0,1) : \mathrm{Re}[2z/(1-z)^2] \le 1/\log M \text{ for } |z| = r\},$$

i.e. $r_{B(M)}$ satisfies the equation

$$2r/(1-r)^2 = 1/\log M.$$

This completes the proof.

The authors of [3] determined the radius of univalence for the class $\overline{\mathrm{conv}}\, S^*$ and proved that the same number is the radius of starlikeness. We shall find the radius of univalence in a different manner. Namely we have

Theorem 5. *$r'_{\overline{\text{conv}}\, S^*} = r_{\overline{\text{conv}}\, S^*} = \rho$, where $\rho = 0.403\ldots$ is the unique positive solution of the equation: $\rho^6 + 5\rho^4 + 79\rho^2 - 13 = 0$.*

Proof. The class $\overline{\text{conv}}\ S^*$ satisfies the assumptions of Corollary 1, so the radius of univalence and bounded turning is equal to $\sup\{r \in (0,1) : \text{Re}\ f'(r) > 0$ for all $f \in \overline{\text{conv}}\ S^*\} = \sup\{r \in (0,1) : \text{Re}[(1+z)/(1-z)^3] > 0$ for $|z| = r\}$ because the Koebe functions compose the set of all extreme points for $\overline{\text{conv}}\ S^*$. Thus the both radii are equal to $\max\{r \in (0,1) : p(r,t) \geq 0$ for all $-1 \leq t \leq 1\}$, where $p(r,t) \equiv 1 - 6r^2 + r^4 + (6r^3 - 2r)t + (6r^2 - 2r^4)t^2 - 4r^3t^3$. For $0 < r < (\sqrt{33}-5)/4$ and $-1 \leq t \leq 1$ we have $p(r,t) > 0$, since $\partial p/\partial t$ is negative at $t = -1$, $t = 1$, and $a_r > 1$, where $\partial^2 p(r, a_r)/\partial t^2 = 0$. If $(\sqrt{33}-5)/4 \leq r < 1$, then $p(r,t) \geq p(r,t_r)$, where $\partial p(r,t_r)/\partial t = 0$ with $-1 \leq t_r \leq 1$. The desired equation follows from the equation $p(\rho, t_\rho) = 0$ after removing all the irrationalities.

Theorem 6. *The radius $r_{\overline{\text{conv}}\, K(\beta)}$ is the least positive solution r of the equation*

$$4r^6 + 8r^4\cos 2\beta + 5r^2 - 1 = 0.$$

Proof. By Theorem 1 the considered radius is identical with $\sup\{r \in (0,1) : f'(r) \neq 0$ for $f \in \overline{\text{conv}}\ K(\beta)\} = \max\{r \in (0,1) : |\text{Im}\ \log[f'(z)/f'(\zeta)]| < \pi$ for all $f \in K(\beta)$, $|z| = |\zeta| = r\}$. The connection between $K(\beta)$ and the classes S^* and P gives

$$\begin{aligned} r_{\overline{\text{conv}}\, K(\beta)} &= \max\{r \in (0,1) : 2\arctan[2r\cos\beta/(1-r^2)] + 4\arcsin r \leq \pi\} \\ &= \max\{r \in (0,1) : \arctan[2r\cos\beta/(1-r^2)] \leq \arctan[(1-2r^2)/(2r\sqrt{1-r^2})]\} \\ &= \max\{r \in (0,1) : 4r^6 + 8r^4\cos 2\beta + 5r^2 - 1 \leq 0\}. \end{aligned}$$

Theorem 7. $r'_{\overline{\text{conv}}\, S} = r_{\overline{\text{conv}}\, S} = \sqrt{2-\sqrt{2}}/2 = 0.382\ldots$

Proof. By Corollary 1 we get that the both radii are equal to $\sup\{r \in (0,1) : \text{Re}\ f'(r) > 0$ for all $f \in S\} = \max\{r \in (0,1) : |\arg f'(r)| < \pi/2$ for all $f \in S\} = \max\{r \in (0,1) : \arcsin r \leq \pi/8\} = \sqrt{2-\sqrt{2}}/2$ because of the rotation theorem for the class S (see e.g. [2], v.I, p.66).

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STRESZCZENIE

W pracy przedstawiono prostą metodę, która pozwala wyznaczyć największe koła, na których każda funkcja z danej klasy jest jednolistna, gwiaździsta lub jej obrót jest ograniczony.