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Logarithmic Coefficients of Locally Univalent Functions

Współczynniki logarytmiczne funkcji lokalnie jednolistnych

Abstract. In this paper the authors obtain upper bounds of logarithmic coefficients of functions from a linearly invariant family of the order α.

1. Introduction. Let U_α^*, $\alpha \geq 1$ be the class of functions f analytic in the unit disk $\mathbf{D}$ such that

$$f'(z) = s'(z) \exp\left[-2 \int_0^{2\pi} \log \frac{1-\omega(z)e^{it}}{1-\omega(0)e^{it}}\, d\mu(t)\right],$$

where $s(z) = z + \cdots$ is a convex and univalent function, i.e. s maps $\mathbf{D}$ onto convex domain; ω is analytic in $\mathbf{D}$ and $|\omega(z)| < 1$, $z \in \mathbf{D}$; μ is a complex valued function with bounded variation on $[0, 2\pi]$ and satisfying the following conditions

$$\int_0^{2\pi} d\mu(t) = 0\,, \quad \int_0^{2\pi} |d\mu(t)| \leq \alpha - 1\,.$$

The class U_α^* is the linearly invariant family of the order α, [2], [3]. The class U_2^* contains the class of close-to-convex functions. Moreover, if $V_{2\alpha}$ is the class of functions of bounded boundary rotation, [2], then $V_{2\alpha} \subset U_\alpha^*$. As shown in [1], $f \in U_\alpha^*$ iff

$$f'(z) = s'(z) \exp\left[-2 \int_0^{2\pi} \log(1 - \omega_0(z)e^{it})\, d\mu(t)\right], \tag{1.1}$$

where s, μ are as above, and ω_0 is analytic in $\mathbf{D}$, $|\omega_0(z)| < 1$, $z \in \mathbf{D}$, $\omega_0(0) = 0$.

For a function $f \in U_\alpha^*$ its logarithmic coefficients γ_n, $n = 1, 2, \ldots$ are defined by the expansion

$$\log f'(z) = \sum_{n=1}^{\infty} \gamma_n z^n\,. \tag{1.2}$$

In this paper we obtain bounds for the coefficients γ_n.

2. The main result. By $\{h\}_n$ we will denote n-th coefficient in the series expansion of an analytic function h.

Theorem. *For $f \in U^*_\alpha$ and γ_n given by (1.2) we have*

$$|\gamma_n| \le 2\left(\alpha - \frac{n-1}{n}\right), \quad n = 1, 2, \dots .$$

Proof. Since U^*_α is rotationally invariant it suffices to consider Re γ_n. By (1.1) we have

$$\log f'(z) = \log s'(z) - 2 \int_0^{2\pi} \log(1 - \omega_0(z)e^{it})\, d\mu(t) . \tag{2.1}$$

It is known that for a convex function s there exists a function β of the total variation 1 on $[0, 2\pi]$ such that

$$\operatorname{Re}\, \{\log s'(z)\}_n = -2 \operatorname{Re} \int_0^{2\pi} \{\log(1 - ze^{it})\}_n\, d\beta(t) \le \frac{2}{n} . \tag{2.2}$$

The equality holds for

$$\beta(t) = \begin{cases} 0 & \text{for } t = 0 \\ 1 & \text{for } t \in (0, 2\pi]. \end{cases}$$

Now, we estimate coefficients of the second expression in (2.1). Let us introduce a new class U^+_α of functions f such that

$$f'(z) = s'(z) \exp\left[-2 \int_0^{2\pi} \log(1 - \omega(z,t))\, d\mu(t)\right] ,$$

where s, μ are as above and $\omega(z,t)$ is a function analytic with respect to z, $z \in \mathbf{D}$ and analytic with respect to t on an interval containing $[0, 2\pi]$. Moreover, $|\omega(z,t)| < 1$, $\omega(0,t) = 0$.

Observe that

$$U^*_\alpha \subset U^+_\alpha . \tag{2.3}$$

Let $f \in U^+_\alpha$ and

$$\log f'(z) = \sum_{n=1}^{\infty} \gamma_n z^n , \quad z \in \mathbf{D} .$$

Let Φ_α be a class of functions φ such that

$$\varphi(z) = -2 \int_0^{2\pi} \log(1 - \omega(z,t))\, d\mu(t) ,$$

where ω, μ are as above.

Let $\widehat{\omega}(z,t)$ be an extremal function for $|\gamma_n|$ with corresponding $\widehat{\mu}$ and let

$$\widehat{\varphi}(z) = -2\int_0^{2\pi} \log(1-\widehat{\omega}(z,t))\, d\widehat{\mu}(t) = \sum_{k=1}^{\infty} A_k z^k \in \Phi_\alpha .$$

Then for $\epsilon_n = e^{2\pi i/n}$ we have

$$\varphi_+(z) := \frac{1}{n}\sum_{k=1}^{n-1} \widehat{\varphi}(z\epsilon_n^k) = -2\int_0^{2\pi} \frac{1}{n}\sum_{k=0}^{n-1} \log(1-\widehat{\omega}(z\epsilon_n^k,t))\, d\widehat{\mu}(t) = \sum_{k=1}^{\infty} A_{kn} z^{kn} .$$

Now, we give

Lemma. *Let $\lambda_k \geq 0$, $\sum_{k=0}^{n-1} \lambda_k = 1$ and let $\omega_k(z,t)$, $k=0,1,\ldots,n-1$ be as in the definition of U_α^+. Then there exists the function $\omega_+(z,t)$ such as in the definition of U_α^+ and such that*

$$\sum_{k=0}^{n-1} \lambda_k \log(1-\omega_k(z,t)) = \log(1-\omega_+(z,t)) , \quad z \in \mathbf{D} .$$

The Lemma follows from the fact that the function $\log(1+\zeta)$ is convex in $\mathbf{D}$ and from properties of the functions ω_k. Thus from the Lemma we obtain that

$$\varphi_+(z) = -2\int_0^{2\pi} \log(1-\omega_+(z,t))\, d\widehat{\mu}(t) = A_n z^n + A_{2n} z^{2n} + \cdots ,$$

where $\omega_+(z,t) = \sum_{l=1}^{\infty} \delta_l z^{nl}$. We have that the function $\omega_\#(z,t) = \omega_+(z^{1/n},t)$ is such as in the definition of U_α^+ and therefore

$$\varphi_+(z^{1/n}) = -2\int_0^{2\pi} \log(1-\omega_\#(z,t))\, d\widehat{\mu}(t) = A_n z + A_{2n} z^2 + \cdots \in \Phi_\alpha .$$

Thus an estimation of the n-th coefficient in Φ_α reduces to an estimation of the first one.

Therefore, if $\varphi \in \Phi_\alpha$ then

$$\mathrm{Re}\,\{\varphi\}_1 = \mathrm{Re}\Big[\int_0^{2\pi} 2\{\omega(z,t)\}_1 d\mu(t)\Big] \leq 2\int_0^{2\pi} \Big|\frac{d}{dz}\Big|_{z=0}\, \omega(z,t)\Big|\, |d\mu(t)| \leq 2(\alpha-1).$$

Hence, by the inclusion (2.3) we obtain

$$\mathrm{Re}\Big[\Big\{-2\int_0^{2\pi} \log(1-\omega(z)e^{it})d\mu(t)\Big\}_n\Big] \leq 2(\alpha-1.$$

The equality holds for $\omega(z) = z^n$ and for μ with jumps : $\dfrac{\alpha-1}{2}$ for $t=0$ and $\dfrac{1-\alpha}{2}$ for $t=\pi$. Evidently the equality occurs for another μ.

Now, we deduce from this and (2.1), (2.2) that

$$\operatorname{Re} \gamma_n \le 2(\alpha - 1 + \frac{1}{n})$$

and this proves our Theorem.

3. Additional results. From our Theorem we have that for $n = 1, 2, \dots$

$$|\{\log f'(z)\}_n| \le \left| \left\{ 2(\alpha - 1)\frac{z}{1-z} - 2\log(1-z) \right\}_n \right| .$$

Hence

$$|\{f'(z)\}_n| \le \left| \left\{ \frac{1}{(1-z)^2} \exp \frac{2(\alpha-1)z}{1-z} \right\}_n \right| =$$
$$= |\{(1 + 2z + 3z^2 + \cdots)(1 + B_1 z + B_2 z^2 + \cdots)\}_n| =$$
$$= \sum_{k=0}^{n} (k+1) B_{n-k}, \quad B_0 = 1, \quad n = 1, 2, \dots .$$

Observe that

$$\left\{ \frac{z^k}{(1-z)^k} \right\}_n = \{(1-z)^{-k}\}_{n-k} = \frac{(n-1)!}{(k-1)!(n-k)!} := \binom{n-1}{k-1}$$

Therefore

$$B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{2^k (\alpha-1)^k}{k!}, \quad n = 1, 2, \dots$$
$$B_0 = 1 .$$

Thus we have

$$|\{f'\}_n| \le \sum_{k=0}^{n} \sum_{j=1}^{n-k} \binom{n-k-1}{j-1} \frac{2^j (\alpha-1)^j (k+1)}{j!}, \quad n = 1, 2, \dots .$$

From this we can obtain that

$$|\{f\}_n| \le \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=1}^{n-k-1} \binom{n-k-2}{j-1} \frac{2^j (\alpha-1)^j (k+1)}{j!}, \quad n = 2, 3, \dots$$

where $\sum_{j=1}^{0}$ by definition equals to $B_0 = 1$.

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STRESZCZENIE

W pracy autorzy otrzymali oszacowanie współczynników logarytmicznych funkcji z pewnej liniowo niezmienniczej rodziny rzędu α.