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Convex Mappings and the Lebesgue Measure of Omitted Values

Funkcje wypukłe i miara Lebesgue'a zbiorów niepokrytych

Мера Лебега множеств непокрытых значениями выпуклых однолистных функций

Let S denote the family of functions $f(z)$ analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$, normalized by: $f(0)=f'(0)-1=0$. According to the Koebe one quarter theorem the omitted values $f(z)$, $f \in S$, $z \in \Delta$ are distributed outside the disk $|w| < 1/4$. Let $B(f) \subset \Delta$ be any set of values not assumed by f in Δ. We propose to give an answer on the question: how "large" the set $B(f)$ could be?

Let $m(B)$ be the Lebesgue measure of B. If $B(f) = \Delta \setminus f(\Delta)$ and $m(B(f))$ is two-dimensional Lebesgue measure of $B(f)$ then A.W. Goodman [1] showed that

$$0.5\pi < \sup_{f \in S} m(B(f)) < 0.7728\pi .$$

Besides, James A. Jenkis [3] proved that if

$$B(f,r) = \{\Delta \setminus f(\Delta)\} \cap \{w : |w| = r\} ,$$

then

$$\sup_{f \in S} m(B(f,r)) = 2r \arccos(8\sqrt{r} - 8r - 1) \;, \quad 1/4 < r < 1 \;.$$

The latter problem was also solved within the subclass of S consisting of starlike functions w.r.t. the origin (see e.g. [4], [6]).

At present we take up the problem of omitted values within the class S^c consisting of all functions $f \in S$ for which $f(\Delta)$ is a convex domain in C. For any $f \in S^c$ the omitted values are located outside the disk $|z| < 1/2$, therefore it is natural question to find

(i) $A(r) = \sup m(A(f,r)) \;, \quad f \in S^c \;, \quad 1/2 < r < 1$

where $m(A(r,f))$ denote the Lebesgue measure of the set $A(r,f) = \Delta_r \setminus f(\Delta)$, $\Delta_r = \{w : |w| < r\}$, as well as

(ii) $L(r) = \sup m(B(f,r)) \;, \quad f \in S^c \;, \quad 1/2 < r < 1$.

The class S^c forms a compact family with respect to locally uniform convergence in Δ, hence for any admissible r, there exists in S^c the extremal function F say, such that

$$L(r) = m(B(F,r)) \;.$$

In order to determine the extremal function F we will recall that if $D = f(\Delta)$, $f \in S$, then the conformal inner radius of D at the point $w \in D$ which we will denote by $R(w;D)$ is equal to $(1 - |z|^2)|f'(z)|$ where $w = f(z)$. Moreover, $w \in D_1 \subset D_2$ implies $R(w;D_1) < R(w;D_2)$ (see e.g. [2] p. 80). Besides, if a convex domain D containing the origin is a proper subset of C then there exists on Δ a conformal mapping f such that $f(0) = 0$, $f'(0) > 0$. If in addition $R(0;D) = 1$ then $f \in S^c$.

We will also need theorem of Pólya and Szegö.

Theorem B. ([2] p. 61). Suppose that w_0 is a point of a domain D and that D^* is obtained from D by symmetrizing with respect to a line passing through w_0. Then $R(w_0;D) \leq R(w_0;D^*)$. Equality holds if D^* coincides with D.

In fact, we will use Theorem B in case D^* is a domain obtained from D by the Steiner symmetrization.

2. Auxiliary lemmas. In further considerations we will need the following lemmas:

Lemma 1. Let α_0 satisfy in $(0, \pi/2)$ the equation

$$\tan\alpha = 2\alpha \qquad (\alpha_0 = 1.165\ldots) . \tag{1}$$

Let

$$r_1 := \frac{1}{2\pi}\sqrt{4+\pi^2} \qquad (= 0.594\ldots)$$

and

$$r_2 := \pi/4\alpha_0 \qquad (= 0.673\ldots) .$$

Then for a given r, $r_1 < r < \pi/4$ the equation

$$\alpha^{-4}(\sin^2\alpha + \alpha^2 - \alpha\sin 2\alpha) = \left(\frac{4}{\pi}r\right)^2 \tag{2}$$

has a unique root $\alpha(r)$ in $(0;\pi/2)$. If moreover $r_1 < r < r_2$ then $\alpha(r) \in (\alpha_0;\pi/2)$, while $r_2 < r < \pi/4$ implies $\alpha(r) \in (0;\alpha_0)$.

Proof. Let $\varphi(\alpha)$ be the left hand side of (2). Then $\varphi(\alpha)$ decreases from 1 to $(2/\pi)^4 + (2/\pi)^2$ as α increases from 0 to $\pi/2$. This can be seen because $\varphi'(\alpha) < 0$ in the range $(0;\pi/2)$. In fact, inequality $\varphi'(\alpha) < 0$ is equivalent to

$$\alpha^2\sin^2\alpha - \alpha^2 - \sin^2\alpha + \alpha\sin 2\alpha < 0$$

or

$$\sin^2\alpha - 2\alpha\sin\alpha\cos\alpha + \alpha^2\cos^2\alpha > 0 .$$

Dividing both sides the latter inequality by $\cos^2\alpha$ we get $(\tan\alpha - \alpha)^2 > 0$ which is obviously true. Besides, the right hand side of (2) varies in the same range as the left hand side of (2) does. Thus the solution of (2) always exists and it is unique.

Suppose now, $r_1 < r < r_2$; $0 < \alpha < \alpha_0$. Then

$$\left(\frac{4}{\pi} r\right)^2 < \left(\frac{4}{\pi} r_2\right)^2 = \alpha_0^{-2} = \varphi(\alpha_0) < \varphi(\alpha) .$$

Hence, there is no solution of (2) in the range $0 < \alpha < \alpha_0$. Therefore $\alpha(r) \in (\alpha_0;\pi/2)$. Similarly, $r_2 < r < \pi/4$; $\alpha_0 < \alpha < \pi/2$ implies

$$\varphi(\alpha) < \varphi(\alpha_0) = \alpha_0^{-2} = \left(\frac{4}{\pi} r_2\right)^2 < \left(\frac{4}{\pi} r\right)^2$$

which gives the conclusion $\alpha(r) \in (0;\alpha_0)$.

Lemma 2. Suppose w_1 is a point of the complex plane C with $|w_1| = r$, $1/2 < r$, $\operatorname{re} w_1 = -d_1$, $0 < d_1 \leqslant 1/2$, $\operatorname{im} w_1 = h > 0$. Let $\mathcal{F}(w_1)$ denote the family of the sectors

$$S_\alpha = \{w \in C : |\arg (w-w_0)| < \alpha\} , \quad 0 < \alpha < \pi/2 , \quad -\infty < w_0 < 0$$

such that for any $S_\alpha \in \mathcal{F}(w_1)$, $w_1 \in \partial S_\alpha$. Then the conformal inner radius $R(0;S_\alpha)$ of the sector S_α at the origin is an increasing function of α in the range $0 < \alpha < \pi/2$ provided $1/2 < r \leqslant r_1$, $\operatorname{re} w_1 = -1/2$. (We recall that r_1 is defined in Lemma 1). If $r_1 < r < \pi/4$; $R(0;S_\alpha)$ increases in $0 < \alpha < \alpha(r,h)$ and it decreases in $\alpha(r,h) < \alpha < \pi/2$, where $\alpha(r,h)$ is a unique root of the equation

(3) $$h \cos\alpha + \sqrt{r^2 - h^2}\sin\alpha = h\frac{\alpha}{\sin\alpha} .$$

Proof. The function $f(z) = w_0\left\{\left(\frac{1+z}{1-z}\right)^{\frac{2\alpha}{\pi}} - 1\right\}$, $w_0 > 0$ maps conformally the unit disk Δ onto $S_\alpha \in \mathcal{F}(w_1)$ for some w_1. Hence $R(0;S_\alpha) = (4/\pi)\alpha w_0$. Let $w_0 = d_1 + d_2$. Since $d_1 = \sqrt{r^2 - h^2}$; $d_2 = h/\tan\alpha$ we see that

(4) $$R(0;S_\alpha) = \frac{4}{\pi}\frac{\alpha}{\sin\alpha}(h\cos\alpha + \sqrt{r^2 - h^2}\sin\alpha)$$

and

(5) $$\frac{d}{d\alpha}R(0;S_\alpha) = \frac{4}{\pi}\frac{1}{\sin\alpha}(h\cos\alpha + \sqrt{r^2-h^2}\sin\alpha - \frac{h\alpha}{\sin\alpha}).$$

Let $\varphi(\alpha) := h\cos\alpha + \sqrt{r^2 - h^2}\sin\alpha = r\sin(\alpha+\theta)$,
$\theta = \arc\sin\frac{h}{r}$; $\psi(\alpha) = h\frac{\alpha}{\sin\alpha}$.

An elementary computation shows that $\varphi(\alpha) > \psi(\alpha)$ in the range $0 < \alpha < \pi/2$ if $\varphi(\pi/2) > \psi(\pi/2)$. The latter inequality holds if $\sqrt{r^2 - h^2} > h\pi/2$ or equivalently

(6) $$h < r(1 + (\pi/2)^2)^{-1/2} .$$

If $S_\alpha \in \mathcal{F}(w_1)$, re $w_1 = -1/2$ then $h = (r^2 - 1/4)^{1/2}$ and condition (6) holds if $1/2 < r < r_1$. Thus

$$\frac{d}{d\alpha}R(0;S_\alpha) > 0 \quad \text{in} \quad (0;\pi/2) .$$

Suppose $r > r_1$. From the definition of r_1 we get then inequality

(7) $$(r^2 - 1/4)(1 + (\pi/2)^2) > r^2 .$$

Since $r^2-h^2 = d_1^2 \leqslant 1/4$ we obtain from (7) that
$h^2(1 + (\pi/2)^2) > r^2$ or equivalently $\sqrt{r^2 - h^2} < h\pi/2$

which means that $\varphi(\pi/2) < \psi(\pi/2)$. Moreover, $\varphi(\pi/2 - \theta) > \psi(\pi/2 - \theta)$. Therefore there exists the unique $\alpha = \alpha(r,h)$, $\pi/2 - \arcsin h/r < \alpha(r,h) < \pi/2$ such that $\frac{d}{d\alpha} R(0;S_{\alpha}) > 0$ in $(0;\alpha(r,h))$ and $\frac{d}{d\alpha} R(0;S_{\alpha}) < 0$ in $(\alpha(r,h);\pi/2)$ and hence the result.

Remark 1. A simple calculation shows that (3) is equivalent to

$$h^2(\sin^2\alpha + \alpha^2 - \alpha\sin 2\alpha) = r^2 \sin^4\alpha \quad . \tag{8}$$

Now, if we consider (8) and

$$h = \frac{\pi}{4} \left(\frac{\sin\alpha}{\alpha} \right)^2 \tag{9}$$

as a system of equations then the solution $\alpha(r)$ of (2) as well as

$$h(r) = \frac{\pi}{4} \left(\frac{\sin\alpha(r)}{\alpha(r)} \right)^2 \tag{10}$$

form the solution of the considered system. Besides,

$$r^2 - h^2(r) < 1/4 \quad . \tag{11}$$

That latter inequality can be seen as follows:
We substitute r and $h(r)$ into (11) from (2) and (10) resp. in order to get

$$\alpha^{-4}(\sin^2\alpha + \alpha^2 - \alpha\sin 2\alpha - \sin^4\alpha) < (2/\pi)^2 \quad . \tag{12}$$

The derivative of the left hand side of (12) is equal to $2(\tan\alpha - \alpha)(2\alpha - \sin 2\alpha)\alpha^{-5}\cos^{-2}\alpha$ and hence positive in $(0;\pi/2)$.

It follows from (4) and (3) that

$$R(0;S_{\alpha(r)}) = \frac{4h}{\pi} \left(\frac{\alpha(r)}{\sin\alpha(r)} \right)^2 \quad .$$

If we choose h to be given by (10) then $R(0;S_{\alpha(r)}) = 1$.

From Lemma 2 and Remark 1 we get

Corollary 1. If $r_1 < r < \pi/4$, $\alpha(r)$ is the root of (2) and $h = h(r)$ is given by (10) then for any $\beta \in (0;\pi/2)$, $\beta \neq \alpha(r)$ $R(0;S_\beta) < 1$ provided $S_\beta \in \mathcal{F}(w_1)$, $w_1 = -\sqrt{r^2 - h^2} + ih$.

3. Main result.

Theorem 1. Let $m(B(f,r))$ denote the linear measure of the set $\{\Delta \setminus f(\Delta)\} \cap \{w : |w| = r\}$. Let moreover $L(r) = \sup m(B(f,r))$, $f \in S^c$. Then

$$L(r) = \begin{cases} 2r \arcsin \dfrac{\sqrt{4r^2 - 1}}{2r}, & 1/2 < r \leq r_1 \\ 2r \arcsin \dfrac{h(r)}{r}, & r_1 < r < r_2 \end{cases}$$

where $h(r) = \frac{\pi}{4}\left(\frac{\sin\alpha(r)}{\alpha(r)}\right)^2$ and $\alpha(r)$ is the root of the equation (2).

The supremum $L(r)$ is attained by the function

$F(z) = \frac{\pi}{4\alpha}\left\{\left(\frac{1+z}{1-z}\right)^{\frac{2\alpha}{\pi}} - 1\right\}$ with $\alpha = \pi/2$ if $1/2 < r < r_1$ and $\alpha = \alpha(r)$ if $r_1 < r < r_2$. The numbers r_1, r_2 are defined in Lemma 1.

Remark 2. If $r_1 < r < r_2$ then the function F maps the unit disk Δ onto the sector

$$\left\{w : \left|\arg\left(w + \frac{\pi}{4\alpha}\right)\right| < \alpha\right\}$$

whose vertex $v = -\frac{\pi}{4\alpha}$ is located inside the disk $|w| < r$.

Remark 3. If $1/2 < r \leq r_1$ then the extremal function F

maps the unit disk Δ onto a half-plane. Therefore we obtain immediately

Theorem 2. Let $m(A(f,r))$ denote the area of the set $\Delta_r \setminus f(\Delta)$, $\Delta_r = \{w : |w| < r\}$, and $A(r) = \sup m(A(f,r))$, $f \in S^c$. Then

$$A(r) = r^2 \arccos \frac{1}{2r} - \frac{1}{4}\sqrt{4r^2 - 1}\,, \quad 1/2 < r \leq r_1 .$$

The supremum $A(r)$ is attained by the function $F(z) = \frac{z}{1-\eta z}$, $|\eta| = 1$.

4. Proof of Theorem 1. Let r be fixed in the range $1/2 < r < r_2$ and let $D = f(\Delta)$, $f \in S^c$. We observe that f can not be extremal if D can be expanded to a convex domain D' while the set of points on $|w| = r$ outside both domains D as well as D' remains the same. The reason is that if $D \subset D'$, then $R(0;D') > 1$. Now we may arrange such a variation of D' which provides a convex domain D'' with $R(0;D'') = 1$ and at the same time the Lebesgue measure of the set of points on $|w| = r$ outside D' will grow. Therefore if f is an extremal function then the boundary of D consists of the stright-lines segments or half-lines as well as the circular arcs located on $|w| = r$.

Suppose that there exists a function $f \in S^c$ such that $m(B(f,r)) > L(r)$. We will show then that the above assumption would led to a contradiction.

Let $D = f(\Delta)$. Then there is a supporting sector $\mathcal{S}$, $D \subset \mathcal{S}$ which has the following property:

Denote by v the vertex of $\mathcal{S}$ and $\partial\mathcal{S}$ consists of two half-lines l_1, l_2 say. Then the set $\partial D \cap \partial\mathcal{S} \cap \{w : |w| = r\}$ has common cut points z_1, z_2, z_3, z_4 such that

$z_1, z_2 \in l_1$, $z_3, z_4 \in l_2$ and

$|z_k - v| \leqslant |z_{k+1} - v|$, $k=1,3$ (if $|v| < r$ then $z_1 = z_2$; $z_3 = z_4$).

Now let $\mathcal{S}_\beta^*$ be the sector obtained from $\mathcal{S}$ by the Steiner symmetrization with respect to the line

$$w = (z_2 - z_4)it + \frac{1}{2}(z_2 + z_4) \quad , \quad t \in (-\infty ; +\infty) .$$

Hence $D \subset \mathcal{S}$ implies $R(0;D) < R(0;\mathcal{S})$ while $R(0;\mathcal{S}) \leqslant R(0;\mathcal{S}_\beta^*)$ according to Theorem B.

We may assume without loss of generality that $\mathcal{S}_\beta^* \in \mathcal{F}(w_1)$, where

$$w_1 = \sqrt{r^2 - h^2} + ih \quad , \quad h = \frac{1}{2}|z_2 - z_4| \quad .$$

If we exclude the trivial case $\mathcal{S}_\beta^* \subset F(\Delta)$ then the set $\partial \mathcal{S}_\beta^* \cap \partial F(\Delta)$ consists of two points ζ_1, ζ_2; $|\zeta_1| = |\zeta_2|$, $\zeta_2 = \overline{\zeta_1}$. There are two possibilities: $|\zeta_1| < r$ or $|\zeta_1| \geqslant r$.

Suppose $|\zeta_1| < r$. Then there exists a sector S_β (with the same β) such that $\mathcal{S}_\beta^* \subset S_\beta$ and

$$\partial S_\beta \cap \partial F(\Delta) = \partial F(\Delta) \cap \{w : |w| = r\} \quad .$$

Obviously, $\beta < \alpha(r)$. Therefore from Corollary 1 it follows that $R(0;S_\beta) < 1$.

Assume now $|\zeta_1| \geqslant r$. In this case there exists a sector S'_β (with the same β) such that $\mathcal{S}_\beta^* \subset S'_\beta$ whose boundary half-lines are tangent to $|w| = r$. Hence $R(0;S'_\beta) = (4/\pi) r (\beta/\sin\beta)$ and β does not exceed $\beta_0 = \arcsin \frac{1}{2r}$. Since $(\beta/\sin\beta)$ increases in $(0;\pi/2)$, $R(0;S'_\beta) \leqslant 8/\pi \; r^2 \arcsin 1/2r$. But $(8/\pi) r^2 \arcsin 1/2r < 1$ in $1/2 < r < \sqrt{2}/2$.

Finally, from both cases it follows that $R(0;D) = f'(0) < 1$

contrary to the assumption $D = f(\Delta)$, $f \in S^c$.

Thus the proof of Theorem 1 is complete.

REFERENCES

[1] Goodman, A.W., Note on regions omitted by univalent functions, Bull. Amer. Math. Soc. 55(1949), 363-369.

[2] Hayman, W.K., Multivalent functions, Cambridge Univ. Press. 1958.

[3] Jenkis, J.A., On values omitted by univalent functions, Amer. J. Math. 2(1953), 406-408.

[4] Lewandowski, Z., On circular symmetrization of starshaped domains, Ann. Univ. Mariae Curie-Skłodowska, Sectio A 17 (1963), 35-38.

[5] Pólya, G., Szegö, G., Isoperimetric inequalities in Mathematical Physics, Princeton 1951.

[6] Stankiewicz, J., On a family of starlike functions, Ann. Univ. Mariae Curie-Skłodowska, Sectio A 22/23/24 (1968/69/70) 175-181.

STRESZCZENIE

Niech $D_r = \{z \in \mathbb{C} : |z| < r\}$, $D_1 = D$, $C_r = \partial D_r$. Oznaczamy przez S klasę unormowanych funkcji jednolistnych w D; ponadto $L(r,f)$ oznacza miarę łukową zbioru $C_r \cap (D \setminus f(D))$, zaś $A(r,f)$ miarą płaską zbioru $D_r \setminus f(D)$. W pracy wyznaczono kresy górne $L(r,f)$, $A(r,f)$ dla funkcji f wypukłych.

РЕЗЮМЕ

Пусть $D_r = \{z \in C : |z| < r\}$, $D_1 = D$, $C_r = \partial D_r$. Через S обозначим класс нормированных функций однолистных в D ; пусть $L(r,f)$ – -мера системы дуг $C_r \cap (D \setminus f(D))$, $A(r,f)$ - плоская мера множества $D_r \setminus f(D)$. В данной работе определены точные верхние грани $L(r, f)$, $A(r,f)$ для функции f выпуклых.