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Some Applications of the Hadamard Convolution in the Theory of Functions

Pewne zastosowania splotu Hadamarda w teorii funkcji analitycznych

Некоторые применения свертки Адамара в теории аналитических функций

1. Let $U_r = \{z \in C : |z| < r\}$, $U = U_1$ and let $H = H(U)$ be the family of all holomorphic functions in the unit disk U. By N we denote its subfamily of the functions f normalized by condition $f(0) = 0$, $f'(0) = 1$ and by B the family of the functions $w \in H$ such that $w(0) = 0$, $|w(z)| < 1$ for $z \in U$.

We say that f is subordinate to F in U and write $f \prec F$, if there exists a function $w \in B$ such that $f(z) = F(w(z))$.

Let $f, g \in H$ be of the form

$$f(z) = a_0 + a_1 z + \dots , \quad g(z) = b_0 + b_1 z + \dots .$$

The convolution or Hadamard product of the functions f and g is defined as follows

$$(f * g)(z) := a_0 b_0 + a_1 b_1 z + \dots .$$

Let, as usual, S^c denote the class of normalized convex univalent functions in the unit disk U.

Polya and Schoenberg [2] conjectured that:

Theorem 1. If F, $G \in S^c$, then $F * G \in S^c$.

Wilf [13] conjectured that a more general theorem is satisfied:

Theorem 2. If F, $G \in S^c$ and $f \prec F$, then $f * G \prec F * G$.

Wilf [13] proved only that Theorem 2 implies Theorem 1. These two theorems were proved by Ruscheweyh and Sheil-Small [5].

Q.I. Rahman and J. Stankiewicz studying the problems on subordination and convolution, conjectured that the following theorem holds:

Theorem 3. If F, $G \in S^c$ and $f \prec F$, $g \prec G$ then $f * g \prec F * G$.

In the special case $g \equiv G$ this theorem coincides with Theorem 2.

This problems were published in [9,10] and proved by Ruscheweyh and J. Stankiewicz [6]. In Theorem 3 we can dropp the normalization and obtain the following

Theorem 4. Let F, $G \in H$ be any convex univalent functions in U. If $f \prec F$ and $g \prec G$ then $f * g \prec F * G$.

2. St. Ruscheweyh [4] began investigations of the neighbourhoods of univalent function in connection with convolution of functions. He used some new definitions of the known classes

of functions. These definitions depend on the concept of convolution. These investigations were continued by Q.I. Rahman and J. Stankiewicz [3] and others [7,11,12].

Now we give some examples of such classes for which we can find the equivalent definitions depending on convolution.

Let us put

$$S = \left\{ f \in N : z_1, z_2 \in U , \quad z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2) \right\} ,$$

$$S^* = \left\{ f \in N : \operatorname{Re}(zf'(z)/f(z)) > 0 \quad \text{for} \quad z \in U \right\} ,$$

$$S^*_\alpha = \left\{ f \in N : \operatorname{Re}(zf'(z)/f(z)) > \alpha \quad \text{for} \quad z \in U \right\} , \quad \alpha \in \langle 0, 1) ,$$

$$S^*(G) = \left\{ f \in N : zf'(z)/f(z) \prec G(z) \right\} , \quad G \in H , \quad G(0)=1 , \quad 0 \notin G(U)$$

and

$$S' = \left\{ h(z) = \frac{z}{(1 - xz)(1 - yz)} : |x| = |y| = 1 \right\} ,$$

$$S^{*\prime}_\alpha = \left\{ h(z) = \frac{z}{(1 - z)^2} + (it - \alpha) \frac{z}{1 - z} : t \in R \right\} , \quad S^{*\prime} = S^{*\prime}_0 ,$$

$$S^{*\prime}(G) = \left\{ \frac{z}{(1 - z)^2} - G(x) \frac{z}{1 - z} : |x| = 1 \right\} .$$

Theorem 5. Let Q be one of the classes S, S^*, S^*_α, $S^*(G)$ and Q' the corresponding class with the prime. Then for every function $f \in N$ the following conditions are equivalent

(i) $f \in Q$;

(ii) for each $h \in Q'$ and for each $z \in U$, $\frac{1}{z}(f * h)(z) \neq 0$.

The second condition gives the new definition of the corresponding class Q which is expressed by the convolution.

The proofs of Theorem 5 for different classes we can find in [3, 4, 11, 12].

3. For $f(z) = z + a_2z^2 + \ldots \in N$ and $\delta > 0$ we put

$$\mathcal{N}_\delta(f) := \left\{ g(z) = z + b_2z^2 + \ldots \in N : \sum_{k=2}^{\infty} k|a_k - b_k| < \delta \right\} .$$

Such a set $\mathcal{N}_\delta(f)$ is called a δ-neighbourhood of a given function f (c.f. [4], [12]).

For a given $f \in N$ and arbitrary ε, n (ε-complex, n-natural) we define

$$f_{n,\varepsilon}(z) := \begin{cases} (f(z) + \varepsilon z)/(1 + \varepsilon) & \text{for } n = 1 , \\ f(z) + \varepsilon z^n & \text{for } n \geq 2 . \end{cases}$$

Using Theorem 5 we can obtain some interesting results about the neighbourhoods $\mathcal{N}_\delta(f)$. We can determine the numbers $\gamma = \gamma(n, \delta, Q)$ such that the following theorem holds:

Theorem 6. Let $Q \subset N$ be a given class of functions. Let $\delta > 0$, and n-natural number be fixed. If for every ε, $|\varepsilon| < \delta$ the functions $f_{n,\varepsilon}(z)$ belong to Q, then $\mathcal{N}_\gamma(f) \subset Q$, where $\gamma = \gamma(n, \delta, Q)$ not depended on f.

In particular we have

$$\gamma(n, \delta, S^*_\alpha) = (1 - \alpha)\delta ,$$

$$\gamma(n, \delta, \check{S}_\beta) := \begin{cases} \delta\cos\beta & \text{for } n = 1 \\ \delta\cos^2\beta & \text{for } n = 2, 3, \ldots \end{cases}$$

where

$$\check{S}_\beta = \left\{ f \in N : \operatorname{Re}\,(e^{i\beta} z f'(z)/f(z)) > 0 \quad \text{for} \quad z \in U \right\}$$

is the class of β-spirallike functions.

Some other results of this kind are given in [3,4,11,12]. The results on the neighbourhoods of functions are closely related to results of paper [8] where an influence of some changes of the coefficients on the properties of holomorphic functions are investigated.

4. Theorem 2 has an application in a problem of influence of some integral operators on subordination, when the majorized function is convex. In this direction the interesting result is given in [3]:

Theorem 7. Let $\alpha_4, \alpha_5, \dots$ be a sequence of complex numbers such that

$$\sum_{m=4}^{\infty} \lambda_m |\alpha_m| \leqslant \frac{3}{8} + |\alpha| - |\alpha - \tfrac{1}{8}| \,,$$

where

$$\alpha := \sum_{m=4}^{\infty} \alpha_m \,, \quad \lambda_m := \sum_{n=1}^{\infty} n^{2-m} \,, \quad m = 4,5,\dots \,.$$

If $F \in S^c$ and $f \prec F$ then

$$(B + \sum_{m=4}^{\infty} \alpha_m B^m) f \prec (B + \sum_{m=4}^{\infty} \alpha_m B^m) F$$

where

$$Bg(z) := \int_0^z g(t)/t \, dt \,, \quad B^m g := B(B^{m-1} g) \,.$$

Theorems 3 and 4 have also some application in the theory of complex functions. In [6] some simple applications of Theorem 4 are given:

Theorem 8. Let $f(z) = a_0 + a_1 z + \ldots \in H$ and let $F(z) = A_0 + A_1 z + \ldots \in H$ map univalently the unit disk U onto a convex domain. If $f \prec F$ then we have

$$\underbrace{f * f * \ldots * f}_{n} \prec \underbrace{F * F * \ldots * F}_{n}, \quad n = 2,3,\ldots$$

$$\sum_{k=0}^{\infty} |a_k|^2 z^k \prec \sum_{k=0}^{\infty} |A_k|^2 z^k ,$$

$$\sum_{k=0}^{\infty} a_k^m \bar{a}_k^n z^k \prec \sum_{k=0}^{\infty} A_k^m \bar{A}_k^n z^k , \quad m,n = 0,1,2,\ldots$$

and

$$\sum_{k=0}^{n} |a_k|^{2m} \leqslant \sum_{k=0}^{n} |A_k|^{2m} , \quad m,n = 0,1,2,\ldots .$$

5. Let Q, V be two fixed classes of holomorphic functions in U. If necessary we suppose that they are compact or convex. Denote by $Q * V$ the following class of functions

$$Q * V := \left\{ f = q * v : q \in Q , \ v \in V \right\} .$$

Thus by Theorem 1 we have

Remark 1. $S^c * S^c = S^c$.

For given class Q let sQ denote the class of all functions which are subordinate to any function of the class Q, that is

$$sQ := \{ h \in H : \text{there exists } f \in Q ,\ h \prec f \} .$$

Thus by Theorem 3 we have

Remark 2. $sS^c * sS^c = sS^c$.

It is natural to ask: Are there other classes with this property ? Is it easy to determine a class $Q * V$? What can we say about the extrem points of a class $Q * V$ when the extrem points of Q and V are given ? A partial answer gives the following theorem.

Theorem 9. Let EQ denote the set of the extrem points of Q . If Q , V are compact and convex then

$$E(Q * V) \subset EQ * EV .$$

Proof. By the definition of extrem points we have: $v \notin EV \Rightarrow$ there exist $\lambda \in (0,1)$, and $v_1 \in V$, $v_2 \in V$, $v_1 \neq v_2$ and $v = \lambda v_1 + (1-\lambda)v_2$. Thus if $f \in Q * V$ and $f \notin EQ * EV$ then there are functions $q \in Q$, $v \in V$ such that $v \notin EV$ (or $q \notin EQ$) and $f = q * v$. Therefore

$$f = q*(\lambda v_1 + (1-\lambda)v_2) = \lambda q * v_1 + (1-\lambda) q * v_2 = \lambda f_1 + (1-\lambda) f_2$$

where

$f_1 = q * v_1 \in Q * V$, $f_2 = q * v_2 \in Q * V$ and $f_1 \neq f_2$.
This implies that $f \notin E(Q * V)$.

Let P_α , $\alpha \leqslant 1$, denote the class of functions

$$P_\alpha = \left\{ p \in H : p(0) = 1 , \ \operatorname{Re} p(z) \geqslant \alpha \ \text{ for } \ z \in U \right\} .$$

<u>Theorem 9'. For</u> $\alpha = 1/2$ <u>we have</u>

$$P_{1/2} * P_{1/2} = P_{1/2}$$

$$E(P_{1/2} * P_{1/2}) = EP_{1/2} * EP_{1/2} = EP_{1/2} .$$

<u>Proof.</u> It is easy to observe that

$$P_{1/2} = \left\{ p \in H : p(z) \prec 1/(1 - z) \right\}$$

and that for every $h \in H$ we have

$$1/(1 - z) * h(z) = h(z) .$$

If $p , q \in P_{1/2}$ then by Theorem 4 we have

$$p * q \prec \frac{1}{1 - z} * \frac{1}{1 - z} = \frac{1}{1 - z}$$

and therefore $p * q \in P_{1/2}$. This gives $P_{1/2} * P_{1/2} \subset P_{1/2}$.
Since $1/(1 - z) \in P_{1/2}$ we have $P_{1/2} \subset P_{1/2} * P_{1/2}$.
It is known that $EP_{1/2} = \left\{ 1/(1 - xz) : |x| = 1 \right\}$. Thus

$$EP_{1/2} * EP_{1/2} = \left\{ \frac{1}{1 - xz} * \frac{1}{1 - yz} : |x| = |y| = 1 \right\} =$$

$$= \left\{ \frac{1}{1 - xyz} : |x| = |y| = 1 \right\} = \left\{ \frac{1}{1 - xz} : |x| = 1 \right\} = EP_{1/2} .$$

The last theorem is a special case of the following general theorem.

Theorem 10. Let $\alpha, \beta \in (-\infty, 1\rangle$ and $\gamma = 1-2(1-\alpha)(1-\beta)$

Then

$$P_\alpha * P_\beta = P_\gamma ,$$

$$E(P_\alpha * P_\beta) = EP_\alpha * EP_\beta = EP_\gamma .$$

Proof. We observe first that

$$P_\alpha = \left\{ p \in H : p(0) = 1 ,\ p(z) \prec 2\alpha - 1 + 2(1-\alpha)/(1-z) \right\}$$

and that functions

$$p_\alpha(z) := 2\alpha - 1 + 2(1-\alpha)/(1-z) = \alpha + (1-\alpha)\frac{1+z}{1-z}$$

are convex univalent in the unit disk U .

Thus for $p \in P_\alpha$ and $q \in P_\beta$ we have

$$p * q \prec p_\alpha * p_\beta = p_\gamma$$

and therefore $P_\alpha * P_\beta \subset P_\gamma$.

Now let $h \in P_\gamma$. Thus $h \prec p_\gamma$. This implies that

$$p(z) = \frac{1-\alpha}{1-\gamma}(h - 2\gamma + 1) + 2\alpha - 1 \prec \frac{1-\alpha}{1-\gamma}(p_\gamma - 2\gamma + 1) + 2\alpha - 1 =$$

$$= p_\alpha(z)$$

belongs to the class P_α . Putting $q(z) = p_\beta(z) \in P_\beta$ we have

$$p(z) * q(z) = h(z) .$$

This means that $P_\gamma \subset P_\alpha * P_\beta$ and the first part of theorem is proved.

To prove the second part it is enough to observe that

$$EP_\alpha = \left\{ p_\alpha(xz) : |x| = 1 \right\}$$

and

$$p_\alpha(xz) * p_\beta(yz) = p_\gamma(xyz) ,$$

where $\gamma = 1 - 2(1 - \alpha)(1 - \beta)$ is determined in Theorem 10.

Remark 3. For $\alpha = \beta = 0$ the class $P_0 \equiv P$ is the class of Caratheodory functions with positive real part and we have

$$P * P = P_{-1} .$$

If $p \in P$, $q \in P$ then $\operatorname{Re}(p*q)(z) > -1$ for $z \in U$.

If $p \in P$, $q \in P$ then $\operatorname{Re}(p*q)(z) > 0$ for $z \in U_{1/3}$.

Two first results follows immediately from Theorem 10. The last we obtain by the fact that

$$(p*q)(z) \quad p_{-1}(z) = -3 + 4/(1 - z) = (1 + 3z)/(1 - z)$$

and $p_{-1}(U_{1/3})$ lies entirely in the right half-plane.

6. The classes P_α may be related with some subclasses of N. Some of them are the subclasses of the class S of univalent functions:

$$R_\alpha = \left\{ f \in N : f'(z) \in P_\alpha \right\} = \left\{ f \in N : \operatorname{Re} f'(z) > \alpha \quad \text{for} \quad z \in U \right\}$$

$$K_\alpha = \left\{ f \in N : \frac{f(z)}{z} \in P_\alpha \right\} = \left\{ f \in N : \operatorname{Re} \frac{f(z)}{z} > \alpha \quad \text{for} \quad z \in U \right\}$$

where $\alpha < 1$.

For $\alpha \in \langle 0, 1 \rangle$ we have $R_\alpha \subset S$.

Theorem 11. Let $\alpha, \beta \in (-\infty, 1)$ and $\gamma = 1-2(1-\alpha)(1-\beta)$. Then

(1) $\quad K_\alpha * K_\beta = K_\gamma$

(2) $E(K_\alpha * K_\beta) = EK_\alpha * EK_\beta = EK_\gamma$,

and

(3) $K_\alpha * R_\beta = R_\gamma$,

(4) $E(K_\alpha * R_\beta) = EK_\alpha * ER_\beta = ER_\gamma$.

Proof. To prove this theorem we observe first that

(5) $\frac{1}{z}(f*g)(z) = \frac{f(z)}{z} * \frac{g(z)}{z}$; $z(f*g) = zf*zg$.

(6) $(f*g)'(z) = \frac{f(z)}{z} * g'(z) = f'(z) * \frac{g(z)}{z}$.

Now suppose that $h \in K_\alpha * K_\beta$, that is $h = f*g$ where $f \in K_\alpha$, $g \in K_\beta$. Thus by the definition of K_α $f(z)/z \in P_\alpha$, $g(z)/z \in P_\beta$. Using (5) we have by Theorem 10

$$\frac{h(z)}{z} = \frac{f(z)}{z} * \frac{g(z)}{z} \in P_\gamma$$

which implies that $h \in K_\gamma$.

Conversly, let $h \in K_\gamma$. Then $\frac{h(z)}{z} \in P_\gamma$ and by Theorem 10, there exists two functions p , q , $p \in P_\alpha$, $q \in P_\beta$ such that $p*q = \frac{h(z)}{z}$. If we put $f = zp$, $g = zq$ then $f \in K_\alpha$, $q \in K_\beta$ and $(f*g) = (zp)*(zq) = z(p*q) = z\,\frac{h(z)}{z} = h(z)$. It means that $h \in K_\alpha * K_\beta$ and the equality (1) is proved.

Using the result of paper [1] we see that

$$EK_\alpha = \left\{ zp_\alpha(xz) : |x| = 1 \right\}$$

and therefore

$$EK_\alpha * EK_\beta = \left\{ (zp_\alpha(xz)) * (zp_\beta(yz)) : |x| = |y| = 1 \right\} =$$

$$= \left\{ z(p_\alpha(xz) * p_\beta(yz)) : |x| = |y| = 1 \right\} = \left\{ zp_\gamma(xyz) : |x| = \right.$$

$$\left. = |y| = 1 \right\} = \left\{ zp_\gamma(xz) : |x| = 1 \right\} = EK_\gamma(z) ,$$

which gives (2).

Using (6) and Theorem 10 we can in an analogous way prove (3) and (4).

In particular case $\alpha = \beta = \frac{1}{2}$ we have:

Remark 4. $K_{1/2} * K_{1/2} = K_{1/2}$.

Since the class $K_{1/2}$ is the closed convex hull of the class S^c (see [1]), then we have

$$\mathrm{cl\ co}\, S^c * \mathrm{cl\ co}\, S^c = \mathrm{cl\ co}\, S^c$$

where cl co Q denotes the closed convex hull of Q .

7. Let Q , V be fixed subsets of H . (If necessary we suppose that Q , V are compact or convex). Let $\circledast$ denote any rule acting in H ($f \in H$, $g \in H \Rightarrow f \circledast g \in H$). We define the set $Q \circledast V$ as follows

$$Q \circledast V := \{h = f \circledast g : f \in Q, \ g \in V\} .$$

As some special cases we may take

a) $f \circledast g$ is the convolution $f * g$;

b) $f \circledast g$ is the sum of functions $(f + g)(z) = f(z) + g(z)$;

c) $f \circledast g$ is the product of functions $(f \cdot g)(z) = f(z)\, g(z)$.

It will be interesting to find a such additional condition that

$$E(Q \circledast V) = EQ \circledast EV$$

or

$$E(Q \circledast V) \subset EQ \circledast EV$$

where $\circledast$ is any rule of the kind a), b), c) .

For the known classes Q , V to determine the classes $Q \circledast V$ in a different way as by definition.

For the case b) we have

Remark 5. Let Q , V be given compact and convex subsets of H and let

$$Q + V := \left\{ f(z) = q(z) + v(z) : q \in Q , \quad v \in V \right\} .$$

Then $Q + V$ is compact and convex and

$$E(Q + V) = EQ + EV \quad .$$

Let $f \notin EQ + EV$ but $f \in Q + V$. Then there exist some functions $q \in Q$, $v \in V$, $q \notin EQ$ (or $v \notin EV$) such that $f = q + v$. Since $q = \lambda q_1 + (1 - \lambda)q_2$, where $\lambda \in (0,1)$ and $q_1 \neq q_2$, then $f = \lambda(q_1 + v) + (1 - \lambda)(q_2 + v) = \lambda f_1 + (1 - \lambda)f_2$ where f_1 , $f_2 \in Q + V$, $f_1 \neq f_2$, $\lambda \in (0,1)$. This implies that $f \notin E(Q + V)$ and therefore we have

$$E(Q + V) \subset EQ + EV \quad .$$

Conversly $f \in Q + V$ and $f \notin E(Q + V)$. Thus we have $f = \lambda f_1 + (1 - \lambda)f_2$ where $\lambda \in (0,1)$, $f_1 \neq f_2$ and f_1 , $f_2 \in$ $\in Q + V$. Thus $f_1 = q_1 + v_1$, $f_2 = q_2 + v_2$, where $q_k \in Q$, $v_k \in V$, $k = 1,2$ and therefore

$$f = (\lambda q_1 + (1 - \lambda)q_2) + (\lambda v_1 + (1 - \lambda)v_2) = q + v \quad .$$

If $q = \lambda q_1 + (1 - \lambda)q_2 \in EQ$ and $v = \lambda v_1 + (1 - \lambda)v_2 \in EV$ then $q_1 = q_2$ and $v_1 = v_2$ and therefore $f_1 = q_1 + v_1 =$ $= q_2 + v_2 = f_2$. This contradicts that $f_1 \neq f_2$. Thus q

and v can not be together the extremal points. It means that $f \notin EQ + EV$ which gives $EQ + EV \subset E(Q + V)$.

This proves Remark 5.

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STRESZCZENIE

W pracy tej przedstawiono kilka zastosowań splotu i podporządkowania w geometrycznej teorii funkcji analitycznych. Obok przeglądu najważniejszych wyników w tym kierunku podano kilka nowych wyników związanych ze splotami pewnych specjalnych funkcji analitycznych i ich punktami ekstremalnymi.

РЕЗЮМЕ

В данной работе представлены некоторые применения свертки и подчинения в геометрической теории аналитических функций. После обзора самых важных результатов в этом направлении представленых несколько новых результатов связанных со свертками специальных аналитических функций и их экстремальными точками.