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On the Radius of Spiral-convexity of a Subclass of Spiral-like Functions

O promieniu wypukłości spiralnej w klasie funkcji spiralnych

Радиус спиральной выпуклости в классе спиралеобразных функций

A function $f(z)$ analytic in $D = \{z : |z| < 1\}$ is said to be spiral like if $f(0) = 0$, $f'(0) = 1$ and

$$\operatorname{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\} > 0$$

for some fixed γ, $-\pi/2 < \gamma < \pi/2$. Let S^{γ} denote the class of such functions. It was shown by Spaček [1] that spiral-like functions are univalent.

A function $f(z)$ analytic in D belongs to the class $B(\alpha + i\beta)$, $\alpha > 0$, β real, if $f(0) = 0$, $\frac{f(z)\,f'(z)}{z} \neq 0$ and $\operatorname{Re} J[\alpha, \beta, f(z)] > 0$ in D, where

$$J[\alpha, \beta, f(z)] \equiv 1 + \frac{zf''(z)}{f'(z)} + (\alpha - 1 + i\beta)\frac{zf'(z)}{f(z)} .$$

The class $B(1+i\beta)$ has been considered by H. Yoshikawa [2] and the functions in this class were called spiral-convex. It is known [3] that if $f(z) \in B(\alpha+i\beta)$ then $f(z)$ is γ-spiral-like, where γ satisfies

$$\alpha + i\beta = (\alpha^2+\beta^2)^{1/2} e^{i\gamma} \quad , \quad -\frac{\pi}{2} < \gamma < \frac{\pi}{2} .$$

The radius of spiral-convexity for the class S^γ is defined as follows

$$R_{\alpha,\beta} \equiv R\left[\alpha,\beta, S^\gamma\right] = \sup\Big\{R : \operatorname{Re} J\left[\alpha,\beta, f(z)\right] > 0 ,$$
$$|z| < R , \ f(z) \in S^\gamma\Big\} .$$

This radius was determined by the author [4]. $R_{1,\beta}$ was determined by using different methods [5], [6].

Let A_n denote the class of normalized functions $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$, regular in D.

We denote by S_n^γ for $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$, n-natural number, the following class of functions

$$S_n^\gamma = \left\{ f(z) : f(z) \in A_n , \ \operatorname{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > 0 , \ z \in D \right\} .$$

The purpose of this note is to find the radius of spiral-convexity for the class S_n^γ. Denoting this radius by $R_{\alpha,\beta,n}$ we have

$$R_{\alpha,\beta,n} \equiv R\left[\alpha,\beta,S_n^{\gamma}\right] =$$

$$= \sup\left\{R : \operatorname{Re} J\left[\alpha,\beta,f(z)\right] > 0 \ , \ |z| < R \ , \ f(z) \in S_n^{\gamma}\right\}.$$

For the determination of this radius the method applied in [4] will be used.

Theorem. The radius of spiral-convexity of the class S_n^{γ} is

$$R_{\alpha,\beta,n} = \frac{\sqrt[n]{\sqrt{(\alpha+n)^2+\beta^2} - \sqrt{(\alpha+n)^2-\alpha^2}}}{\sqrt[2n]{\alpha^2+\beta^2}}$$

The result is sharp.

Proof. Let $f(z) \in S_n^{\gamma}$. Then there exists a function $p(z) \in P_n$ (the class of functions $p(z) = 1 + p_n z^n + \dots$ with positive real part) such that

$$e^{i\gamma}\frac{zf'(z)}{f(z)} = p(z)\cos\gamma + i\sin\gamma \quad .$$

Then

$$J\left[\alpha,\beta,f(z)\right] = 1 + \frac{zf''(z)}{f'(z)} + (\alpha - 1 + i\beta)\frac{zf'(z)}{f(z)} =$$

$$= \alpha p(z) + \frac{zp'(z)\cos\gamma}{p(z)\cos\gamma + i\sin\gamma} + i\beta \ ,$$

$$\operatorname{Re} J\left[\alpha,\beta,f(z)\right] = \operatorname{Re}\left\{\alpha p(z) + \frac{zp'(z)\cos\gamma}{p(z)\cos\gamma + i\sin\gamma}\right\} .$$

It is known [7] that if $p(z) \in P_n$ then on $|z| = r < 1$ and $n = 1,2,3,\dots$:

(1) $$\left|zp'(z)\right| \leqslant \frac{2n\, r^n}{1-r^{2n}} \operatorname{Re} p(z) .$$

Since $p(z) \prec \frac{1+z}{1-z}$ in D, the disc $|z| \leqslant r$ is transformed by functions $p(z)$ to the disc

$$\left|p(z) - a_n\right| \leqslant d_n , \quad a_n = \frac{1+r^{2n}}{1-r^{2n}} , \quad d_n = \frac{2r^n}{1-r^{2n}} ,$$

consequently

(2) $$\left|p(z)\cos\gamma - \frac{1+r^{2n}}{1-r^{2n}}\cos\gamma\right| \leqslant \frac{2r^n \cos\gamma}{1-r^{2n}}$$

since $\cos\gamma > 0$.

$$\left|p(z)\cos\gamma - \frac{1+r^{2n}}{1-r^{2n}}\cos\gamma\right| = \left|p(z)\cos\gamma + i\sin\gamma - i\sin\gamma - \frac{1+r^{2n}}{1-r^{2n}}\cos\gamma\right| \geqslant \left|-i\sin\gamma - \frac{1+r^{2n}}{1-r^{2n}}\cos\gamma\right| - \left|p(z)\cos\gamma + i\sin\gamma\right| .$$

(3) $$\left|p(z)\cos\gamma + i\sin\gamma\right| \geqslant \left|-i\sin\gamma - \frac{1+r^{2n}}{1-r^{2n}}\cos\gamma\right| - \left|p(z)\cos\gamma - \frac{1+r^{2n}}{1-r^{2n}}\cos\gamma\right| \geqslant$$

$$\geqslant \frac{\sqrt{(1-r^{2n})^2\sin^2\gamma + (1+r^{2n})^2\cos^2\gamma} - 2r^n\cos\gamma}{1 - r^{2n}} .$$

In view of (1), (2), (3) we get

$$\operatorname{Re} J\left[\alpha, \beta, f(z)\right] = \operatorname{Re}\left\{\alpha p(z) + \frac{zp'(z)\cos\gamma}{p(z)\cos\gamma + i\sin\gamma}\right\} \geqslant$$

$$\geqslant \alpha \operatorname{Re} p(z) - \frac{|zp'(z)|\cos\gamma}{|p(z)\cos\gamma + i\sin\gamma|} \geqslant$$

$$\geqslant \operatorname{Re} p(z)\left\{\alpha - \frac{2nr^n\cos\gamma}{\sqrt{(1-r^{2n})^2\sin^2\gamma + (1+r^{2n})^2\cos^2\gamma} - 2r^n\cos\gamma}\right\} =$$

$$= \operatorname{Re} p(z)\left\{\alpha - \frac{2nr^n\cos\gamma}{\sqrt{1 + 2r^{2n}\cos 2\gamma + r^{4n}} - 2r^n\cos\gamma}\right\}.$$

Now the radius of spiral-convexity $R_{\alpha,\beta,n}$ is the smallest positive root in $(0, 1]$ of the equation

$$\alpha - \frac{2nr^n\cos\gamma}{\sqrt{1 + 2r^{2n}\cos 2\gamma + r^{4n}} - 2r^n\cos\gamma} = 0 .$$

Since $\alpha + i\beta = (\alpha^2+\beta^2)^{1/2}e^{i\gamma}$ we have

$$(\alpha^2+\beta^2)r^{4n} + 2(\alpha^2-\beta^2-2(\alpha+n)^2)r^{2n} + \alpha^2 + \beta^2 = 0 .$$

The smallest positive root of the last equation is

$$R_{\alpha,\beta,n} = \frac{\sqrt[n]{\sqrt{(\alpha+n)^2+\beta^2} - \sqrt{(\alpha+n)^2-\alpha^2}}}{\sqrt[2n]{\alpha^2+\beta^2}} ,$$

and left-hand side of the equation is positive for $r < R_{\alpha,\beta,n}$.

The sharpness of the result follows by substituting the function

$$f_0(z) = \frac{z}{(1-z^n)^{\frac{1+e^{-2i\gamma}}{n}}} .$$

For this function

$$\mathrm{Re}\left\{1 + \frac{zf_0''(z)}{f_0'(z)} + (\alpha-1+i\beta)\frac{zf_0'(z)}{f_0(z)}\right\} = 0 \quad \text{for} \quad z = R_{\alpha,\beta,n} .$$

Hence $f(z)$ cannot be spiral-convex in any circle of radius greater than $R_{\alpha,\beta,n}$.

Putting in the above theorem $n = 1$ we get the result obtained by author [4] .

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STRESZCZENIE

Yoshikawa wyróżnił klasę odwzorowań konforemnych, spiralno-wypukłych. W pracy wyznaczono promień wypukłości spiralnej w klasie funkcji spiralnych, których współczynniki taylorowskie $a_2, \dots a_n$ znikają.

РЕЗЮМЕ

Йошикава ввел класс спирально-выпуклых функций. В этой работе определен радиус спиральной выпуклости в классе спиралеобразных функций, которых коэффициент $a_2, \dots a_n$ равны нулю.