ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XL, 17 SECTIO A 1986

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Interval Averages

Średnie po przedziałach

Среднее по интервалах

Abstract. It is known that for every H^1 function f in $|z|<1$ and every z_0, $|z_0|<1$, there are $e^{i\vartheta}$, $\vartheta\in R$, and ε, $0<\varepsilon\leqslant\pi$, such that $f(z_0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta}e^{it})\,dt$. We show that if f is a conformal mapping from $|z|<1$ onto a Jordan domain with analytic boundary, then $\varepsilon \geqslant c_f(1-|z_0|)^{1/2}$, where $c_f>0$ is a constant independent from z_0, $|z_0|<1$. The exponent $1/2$ is the best possible in this case.

Introduction. Suppose that f is a function of class H^1 in the open unit disk D. In [2], [3] it has been proved that every value $f(z_0)$, $z_0\in D$, is of the form

$$f(z_0) = f_I \equiv \frac{1}{|I|}\int_I f(e^{i\vartheta})\,d\vartheta, \text{ for some interval } I\subset T=\partial D$$

with length $|I|$, $0<|I|\leqslant 2\pi$. The above property has been used in order to evaluate the B.M.O. norm

$$_2\|\varphi\| = \sup_{I \subset T \text{ interval}} \left[\frac{1}{|I|}\int_I |(e^{i\vartheta}) - \varphi_I|^2 \, d\vartheta\right]^{1/2}$$

for all inner functions φ ; more precisely $_2\|\varphi\| = 1$, for every non-constant inner function.

A first extension of the above property of H^1 functions consists in replacing the Lebesgue measure $d\vartheta$ by any finite strictly positive continuous measure μ on $T = \partial D$. Then the same result holds, provided that f is in the disk algebra and $f(z_0) \notin f(T)$. An example given in [4] shows that the condition $f(z_0) \notin f(T)$ is not superflous. An open question, as for as I know, is to characterize the measures for which this condition is not needed.

In the same paper [4] the following has been proved:

Suppose $\varphi : T \longrightarrow C - \{w\}$, $w \in C$, is a continuous function. Then (i) and (ii) are equivalent:

(i) The winding number of φ with respect to w is non zero.

(ii) For every finite strictly positive continuous measure μ on T , there is an interval $I \subset T$ with lenght $|I|$, $0 < |I| \leq 2\pi$, such that

$$f(z) = \frac{1}{\mu(I)} \int_I \varphi(e^{i\vartheta}) \, d(e^{i\vartheta}) \quad .$$

The proof (i) $\Longrightarrow$ (ii) is purely topological. For the converse a rather delicate construction is needed.

The above equivalence allows us to determine the range of the BMO norms $_2\|\varphi \circ u\|$, when u varies in the set of all topological homeomorphisms of T onto T and φ is any given continuous unimodular function $\varphi : T \longrightarrow T$.

Some of the above results have been extended in the case of functions of several complex variables [5]. For instance we have the following:

Suppose $F : B_n \longrightarrow C^n$ is of class H^1 in the open unit ball B_n of C^n. Let $z_0 \in B_n$ be such that the set $\{ z \in B_n : F(z) = F(z_0) \}$ contains at least one isolated point. Then $F(z_0)$ is of the form $F(z_0) = \frac{1}{\lambda(S_{(j,\varepsilon)})} \int_{S_{(j,\varepsilon)}} F(\xi) d\lambda(\xi)$

where λ is the Lebesgue measure on ∂B_n, $j \in \partial B_n$, $0 < \varepsilon \leq 2$ and $S_{(j,\varepsilon)} = \{ \xi \in \partial B_n : \|\xi - j\| \leq \varepsilon \}$, with the Euclidean norm.

The condition that the set $\{ z \in B_n : F(z) = F(z_0) \}$ contains at least one isolated point does not appear in the case $n=1$; in this case this condition is automatically fulfilled or the function in question is constant.

The proofs of the above results do not give any essential quantitative information. A natural question, as S. Pichorides and others suggested, is to compare ε with the distance of z_0 from the boundary. In the present article we prove the following quantitative result.

Theorem. Let $f : D \longrightarrow C$ be a conformal mapping from the open unit disk D onto a Jordan domain with analytic boundary. Then there is a constant $C_f > 0$, such that the following holds:

If $z_0 \in D$, $e^{i\vartheta} \in T = \partial D$ and ε, $0 < \varepsilon \leq \pi$, are related by $f(z_0) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta} e^{it}) dt$, then $\varepsilon \geq C_f (1 - |z_0|)^{1/2}$.

The exponent $1/2$ in the above case is the best possible, as one can easilly check by the trivial example $f(z) \equiv z$.

A more technical argument gives $\varepsilon \geqslant C_f(1-|z_0|)^{5/2}$ in the more general case of univalent H^1 functions. We do not include the proof of this fact, because the sharp result $\varepsilon \geqslant L(1-|z_0|)$, with $L > 0$ an absolute constant, has recently been obtained ([6]).

Proof of the theorem. First we observe that every conformal mapping from D onto a Jordan domain with analytic boundary can be extended by reflexion to a univalent map in a larger disk $|z| < r$, $r > 1$ ([1]). Then by compactness

$|f'(z)| \geqslant C_1 = C_1(f) > 0$, for all z , $|z| \leqslant 1$ and

$$\left|\frac{\partial}{\partial t} f(e^{i\vartheta} e^{it})\right| \leqslant C_2(f) < +\infty ,$$

$$\left|\frac{\partial^2}{\partial t^2} f(e^{i\vartheta} e^{it})\right| \leqslant C_3(f) < +\infty \quad \text{and}$$

$$\left|\frac{\partial^3}{\partial t^3} f(e^{i\vartheta} e^{it})\right| \leqslant C_4(f) < +\infty \quad \text{for all } e^{i\vartheta}, e^{it} \in T .$$

Applying the $1/4$ -Koebe Theorem ([7]) to the function $g_{z_0}(j) = f(z_0 + (1-|z_0|)j)$, $|j| < 1$, we find

$$\operatorname{dist}(f(z_0), f(T)) \geqslant \frac{1}{4}\left|f'(z_0)\right|(1-|z_0|) \geqslant \frac{C_1(f)}{4}(1-|z_0|) .$$

On the other hand, since

$$f(z_0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta} e^{it})\, dt , \text{ we have}$$

$$\operatorname{dist}(f(z_0), f(T)) \leqslant \left|f(z_0) - f(e^{i\vartheta})\right| =$$

$$= \left| \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta} e^{it})dt - f(e^{i\vartheta}) \right| =$$

$$= \left| \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \left[f(e^{i\vartheta} e^{it}) - f(e^{i\vartheta}) \right] dt \right| .$$

Thus, we find

$$\text{(1)} \quad \frac{C_1(f)}{4} (1 - |z_0|) \leq \left| \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \left[f(e^{i\vartheta} e^{it}) - f(e^{i\vartheta}) \right] dt \right| .$$

We use now the following finite Taylor development:

$$f(e^{i\vartheta} e^{it}) - f(e^{i\vartheta}) = \left. \frac{\partial f(e^{i\vartheta} e^{it})}{\partial t} \right|_{t=0} \cdot t + \left. \frac{\partial^2 f(e^{i\vartheta} e^{it})}{\partial t^2} \right|_{t=0} \cdot \frac{t^2}{2} +$$

$$+ \frac{t^3}{6} R(\vartheta, t) .$$

The Lagrange formula yields that

$$\left| R(\vartheta, t) \right| \leq 2 \sup_{-\pi \leq j \leq \pi} \left| \left. \frac{\partial^3 f(e^{i\vartheta} e^{it})}{\partial t^3} \right|_{t=j} \right| \leq 2C_4(f) .$$

Therefore we find

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \left[f(e^{i\vartheta} e^{it}) - f(e^{i\vartheta}) \right] dt = \left. \frac{\partial f(e^{i\vartheta} e^{it})}{\partial t} \right|_{t=0} \cdot \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} t dt +$$

$$+ \frac{1}{2} \left. \frac{\partial^2 f(e^{i\vartheta} e^{it})}{\partial t^2} \right|_{t=0} \cdot \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} t^2 dt + \frac{1}{6} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} t^3 R(\vartheta, t) dt .$$

Since $\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} t dt = 0$, $\left| \frac{\partial^2 f}{\partial t^2} \right| \leq C_2(f)$ and $\left| R(\vartheta, t) \right| \leq 2C_4(f)$, we find

$$\left|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left[f(e^{i\vartheta}e^{it})-f(e^{i\vartheta})\right]dt\right| \leq \frac{C_2(f)}{2}\,\frac{\varepsilon^2}{3}+\frac{C_4(f)}{6}\varepsilon^3 \leq$$

$$\leq C_5(f)\cdot\varepsilon^2 , \quad \text{where} \quad C_5(f) = \max\left\{\frac{C_2(f)}{6}, \frac{C_4(f)}{3}\pi\right\} < +\infty .$$

Combining this with (1) we find $\frac{C_1(f)}{4}(1-|z_0|) \leq C_5(f)\varepsilon^2$,

which gives $\varepsilon \geq C_f(1-|z_0|)^{1/2}$, with $C_f = \sqrt{\frac{C_1(f)}{4C_5(f)}} > 0$.

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STRESZCZENIE

Jak wiadomo, dla każdej funkcji klasy H^1 w kole $|z|<1$ i dla każdego z_0, $|z_0|<1$, istnieją liczby $e^{i\vartheta}$, $\vartheta \in \mathbb{R}$, oraz ε, $0<\varepsilon\leqslant\pi$, takie, że $f(z_0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta}e^{it})\,dt$. Dowodzi się, że jeśli f jest odwzorowaniem konforemnym koła jednostkowego na obszar Jordana o brzegu analitycznym, to $\varepsilon \geqslant c_f\,(1-|z_0|)^{1/2}$, gdzie $c_f>0$ jest stałą niezależną od z_0, $|z_0|<1$. Wykładnik 1/2 jest możliwie najlepszy.

РЕЗЮМЕ

Как известно, для любой функции класса H^1 в единичном круге и для любой точки z_0, $|z_0|<1$, существуют числа $e^{i\vartheta}$, $\vartheta\in\mathbb{R}$, и ε, $0<\varepsilon\leq\pi$, такие что $f(z_0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta}e^{it})\,dt$. Доказывается, что для конформного отображения f единичного круга на Жорданову область с аналитической границей $\varepsilon \geq c_f(1-|z_0|)^{\frac{1}{2}}$ где $c_f>0$ постоянная независима от z_0, $|z_0|<1$. Экспонент $\frac{1}{2}$ самый лучший.