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Two Applications of Mappings onto the Complement of Spirals

Dwa przykłady zastosowania odwzorowań na obszary będące dopełnieniem łuku spirali

Два применения отображения на дополнение дуги спирали

1. Introduction. This paper reports on two recent results in which conformal mappings onto the complement of spirals play an important role. In each case suitable constructions of mappings are carried out. For the first result, α-spirallike mappings are used to obtain a function related to the solution of a boundary peaking and interpolation problem for univalent functions. For the second result, counter-examples are found to a conjecture about the representation of univalent functions as Koebe transforms with respect to complex measures on the unit circle.

2. Peaking and interpolation of univalent functions.

Let $\Delta=\{z\in\mathbb{C}: |z|<1\}$ and suppose that $z_k=e^{i\alpha_k}$, $w_k=e^{i\beta_k}$ $(k=1,2,\ldots,n)$ where $\alpha_1<\alpha_2<\ldots<\alpha_n<\alpha_1+2\pi$ and $\beta_1<\beta_2<\ldots<\beta_n<\beta_1+2\pi$.

THEOREM 1. *There exists a function* f *that is analytic and univalent in the union of* Δ *and a neighborhood of* $\{z_1,z_2,\ldots,z_n\}$, *continuous on* $\bar{\Delta}$ *and satisfies* $f(z_k)=w_k$ for $k=1,2,\ldots,n$. *Also,* $|f(z)|=1$ *for* $|z|=1$ *and* z *sufficiently near any* z_k.

This theorem is proved in [5]. It is directly related to considerations in [2] where the following result is obtained.

THEOREM 2. *There exists a function* f *that is analytic and univalent in* $\bar{\Delta}$ *and satisfies* $|f(z)|<1$ *for* $|z|\le 1$ *and* $z\neq z_k$ *and* $f(z_k)=w_k$ $(k=1,2,\ldots,n)$.

The proof of Theorem 2 given in [2] is quite long. One consequence of Theorem 1 is a simple and more constructive argument for this result. This approach also relies on an elementary argument in [2, p.561] involving the construction of a map which is a finite composition of functions each of which is a power function, an exponential or a Möbius transformation. It is worth noting that [2] also contains a proof that there is a polynomial f satisfying Theorem 2.

The main step in the proof of Theorem 1 involves a construction about α-spriallike mappings. The definition and basic properties of such functions are given in [3, p.52].

An inductive procedure is set up through the following result. Let m be an integer, $m\geq 2$, $\zeta_k=e^{i\gamma_k}$ $(k=1,2,\ldots,m)$ where $\gamma_1<\gamma_2<\ldots<\gamma_m<\gamma_1+2\pi$ and $\zeta'=e^{i\gamma'}$ where $\gamma_{m-1}<\gamma'<\gamma_1+2\pi$. A function g exists which is analytic and univalent in the union of Δ and a neighborhood of $\{\zeta_1,\zeta_2,\ldots,\zeta_{m-1},\zeta'\}$ such that $g(\zeta_k)=\zeta_k$ $(k=1,2,\ldots,m-1)$ and $g(\zeta')=\zeta_m$.

The way in which g is obtained is as follows. Choose ζ'' on $\partial\Delta$ on the counter clockwise arc from ζ_{m-1} to ζ_1 such that ζ_m is between ζ' and ζ''. Let h be an α-spirallike function mapping Δ onto a domain which is complementary to m-1 spirals (joined only at ∞) such that the singularities of h on $\partial\Delta$ are at $\zeta_1,\zeta_2,\ldots,\zeta_{m-1}$. It is possible to write down such a function h and to choose α such that $h'(\zeta'')=0$. This ensures that one of the spirals has its endpoint at $h(\zeta'')$. Since $h(\Delta)$ is α-spirallike, the function $h_t(z)= h^{-1}[\exp(-e^{i\alpha}t)\cdot h(z)]$ is analytic in Δ and fixes $\zeta_1,\zeta_2,\ldots,\zeta_{m-1}$ for each $t>0$. Also $h_t(z)\to 0$ as $t\to\infty$ for $|z|\leq 1$ and $z\neq\zeta_1,\zeta_2,\ldots,\zeta_{m-1}$, and hence for a suitable t we also have $h_t(\zeta')=\zeta_m$. This h_t serves for g. The functions h_t map Δ onto Δ less m-1 slits one of which has ζ'' as an endpoint. As t increases the slit at ζ'' has the effect of moving ζ' along $\partial\Delta$ until it reaches ζ_m.

3. Koebe transforms of measures. Let U denote the set of functions that are analytic and univalent in Δ, and let S denote the subset of U such that $f(0)=0$ and $f'(0)=1$. Let Λ denote the set of (finite) complex valued Borel measures on $\partial\Delta$. Also, let F denote the set of functions f for which

$$f(z) = \int_{|x|=1} \frac{1}{(1-xz)^2}\, d\mu(x) \tag{1}$$

for $|z|<1$ and for some $\mu \in \Lambda$.

The second application of mappings involving spirals concerns the question: is $U \subset F$? Such a question was probably first asked by D.R. Wilken around 1970 and has interested a number of mathematicians more recently. If $f(0)=0$ then (1) is equivalent to

$$f(z) = \int_{|x|=1} \frac{z}{(1-xz)^2}\, d\mu(x) \tag{2}$$

for another $\mu \in \Lambda$. In particular, the question above can be stated: does each $f \in S$ have the representation (2)?

Many functions in U do have the representation (1). In particular, this includes the spirallike mappings and the close-to-convex mappings. It also includes functions in U for which $M(r)=\max_{|z|=r}|f(z)|$ is restricted by the condition $(1-r)M(r)$ is Lebesgue integrable on $(0,1)$. Proofs of these results are in [4]. An independent proof about close-to-convex mappings is given in [1].

The general conjecture $U \subset F$ is false.

Counter-examples are presented in [4] and they now will be described. Let $F(z)=\exp(g(z))$ where g is a conformal mapping of Δ onto the region defined by $w=u+iv$ where $\psi(u)<v<\psi(u)+2\pi$. The function ψ is required to have the properties: ψ' is bounded, ψ'^2 is integrable and $\frac{\psi(u_2)-\psi(u_1)}{u_2-u_1}\to 0$ as u_1 and u_2 simultaneously tend to $+\infty$. Specifically, ψ can be chosen such that $\psi(u)=u^b$ for $u\geq 1$ and where $0<b<1/2$.

The function F maps Δ onto the complement of a spiral which turns fairly slowly toward ∞. Normalizing F such that $z=1$ corresponds to ∞ then the Ahlfors distortion theorem and related results of Warschawski provide sufficiently precise information about $|F(r)|$ and $\arg F(r)$ for $0<r<1$. In particular, $\lim_{r\to 1}(1-r)^2M(r)>0$. Also, for a sequence $\{r_n\}$ with $r_n\to 1$ there is a sequence $\{b_n\}$ of positive real numbers such that if $\alpha_k=\arg F(r_k)$ then the sequence of continuous linear functionals defined by $L_n(f)=\sum_{k=1}^{n} b_k e^{-i\alpha_k} f(r_k)$ satisfies $L_n(F)\to+\infty$ as $n\to\infty$. Moreover, $\sum_{k=1}^{\infty} b_k e^{-i\alpha_k}\frac{1}{(1-xr_k)^2}$ is uniformly convergent for $|x|=1$. These two facts imply that F does not have the representation (1) for some $\mu\in\Lambda$.

References

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STRESZCZENIE

Podano dwa zastosowania odwzorowania konforemnego koła na dopełnienie łuku spirali.

Pierwsze z nich dotyczy problemu interpolacji dla funkcji jednolistnych, drugie jest związane z kontrprzykładem na hipotezę o możliwości przedstawienia funkcji jednolistnych w postaci transformatu funkcji Koebego za pomocą miary zespolonej rozłożonej na okręgu jednostkowym.

РЕЗЮМЕ

Представлено два примеры конформного отображения круга на дополнение дуги спирали. Первое применение относится к интерполационной проблеме для однолистных функций; второе дает контрпример на возможность представления однолистной функции в виде трансформаты по мере функции Кёбе.