ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XL, 14 SECTIO A 1986

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On Some Mappings Obtained by a Holomorphic Continuation from R^{2n} into C^{2n}

O pewnych odwzorowaniach powstałych przez przedłużenie holomorficzne z R^{2n} w C^{2n}

О некоторых отображениях полученных через голоморфное продолжение из R^{2n} в C^{2n}

1. Suppose that domain $D \subset R^2$ is simply-connected. We shall identify points $(x_1, x_2) \in D$ with points $z = x_1 + ix_2$ of the complex plane. Then the mapping

(1) $$F(x_1, x_2) = u_1(x_1, x_2) + iu_2(x_1, x_2) ,$$

where u_1, u_2 are harmonic in D can be written in the form

(2) $$F(z) = f(z) + \overline{g(z)} ,$$

where f, g are holomorphic in D.

Let $\tilde{D} = \{(z_1, z_2) \in C^2 : z_1 + iz_2 \in D , \ \bar{z}_1 + i\bar{z}_2 \in D\}$. We shall identify $\tilde{D}$ with the set

$$\tilde{D}' = \{(x_1, y_1, x_2, y_2) \in R^4 : (x_1-y_2, x_2+y_1) \in D , \ (x_1+y_2, x_2-y_1) \in D\}$$

and D with the set

$$D' = \{(x_1, y_1, x_2, y_2) \in R^4 : (x_1, x_2) \in D , \ y_1 = y_2 = 0\} .$$

The functions

$$\tilde{u}_1(z_1,z_2) = \frac{1}{2}\left[f(z_1+iz_2)+\overline{f(\overline{z}_1+i\overline{z}_2)}+g(z_1+iz_2)+\overline{g(\overline{z}_1+i\overline{z}_2)}\right] ,$$

(3)

$$\tilde{u}_2(z_1,z_2) = \frac{1}{2i}\left[f(z_1+iz_2)-\overline{f(\overline{z}_1+i\overline{z}_2)}-g(z_1+iz_2)+\overline{g(\overline{z}_1+i\overline{z}_2)}\right] ,$$

are holomorphic continuations of u_1, u_2 from D into $\tilde{D}$ i.e. $\tilde{u}_k \mid D = u_k$, $k=1,2$.

Denote

(4) $$\tilde{F} = (\mathrm{Re}\,\tilde{u}_1 , \mathrm{Im}\,\tilde{u}_1 , \mathrm{Re}\,\tilde{u}_2 , \mathrm{Im}\,\tilde{u}_2) .$$

I want to find the conditions under which the mapping (4) will be quasiregular or quasiconformal in $\tilde{D}$.

There are different equivalent definitions of quasiregular mapping but in this case the following one is most convenient.

The mapping $F = (F_1 ,\dots, F_p) : M \to N$ (M,N domains in R^p) is called quasiregular if the following conditions are satisfied

a) $F \in W^1_{p,loc}(M,R^p)$ i.e. for every compact subset $M_1 \subset M$ functions F_j , $j=1,\dots,p$ and their first distributional derivatives belong to the class $L_1(M_1 , R)$,

b) $\left\| p^{-1/2} J_F^{-1/p} |DF| \right\|_{\infty ,M} = K_F < \infty$,

where $|DF|$ denotes the norm of the matrix DF , J_F is a jacobian of F .

K_F is called the maximal dilatation of F .

The mapping F is quasiconformal if it is quasiregular and homeomorphic.

2. For the mapping $\tilde{F}$ defined above by the formulae (2)-(4) using the Cauchy-Riemann equations we get after some calculations that

$$J_{\tilde{F}}(x_1,y_1,x_2,y_2) = \left|\overline{f'(\zeta)}f(\eta) - \overline{g'(\zeta)}g'(\eta)\right|^2 ,$$

$$|D\tilde{F}(x_1,y_1,x_2,y_2)|^2 = 2(|f'(\zeta)|^2+|f'(\eta)|^2+|g'(\zeta)|^2+|g'(\eta)|^2) ,$$

$$\zeta = x_1-y_2+i(x_2+y_1) , \quad \eta = x_1+y_2+i(x_2-y_1) .$$

Thus we arrive at

Theorem 1. *The mapping $\tilde{F} : \tilde{D}' \longrightarrow R^4$ defined by the formulae (2)-(4) is quasiregular if and only if*

$$K_{\tilde{F}} = \sup_{(\zeta,\eta)\in D\times D} \left[\frac{1}{2}\,\frac{|f'(\zeta)|^2+|f'(\eta)|^2+|g'(\zeta)|^2+|g'(\eta)|^2}{|f'(\zeta)f'(\eta)-g'(\zeta)g'(\eta)|}\right]^{\frac{1}{2}} < \infty .$$

If F is 1:1 in D then $\tilde{F}$ is not 1:1 in $\tilde{D}'$ generally. For example let

$$F(z) = z\sqrt{z}+z+\overline{z\sqrt{z}-z} ,$$

where $D = C \setminus (-\infty,0\rangle$ and $\sqrt{1} = 1$. Then

$$\tilde{u}_1(z_1,z_2) = (z_1+iz_2)\sqrt{z_1+iz_2} + (z_1-iz_2)\sqrt{z_1-iz_2} ,$$

$$\tilde{u}_2(z_1,z_2) = 2z_2$$

and $\tilde{F}(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0) = \tilde{F}(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, 0) = (-2, 0,0,0)$

Jarnicki proved [1] that if f or g is constant and F defined by (1) is univalent in D then $\tilde{F}$ is 1:1 in $\tilde{D}'$. Thus for instance in the case g = const we get the following

Corollary. *If F is holomorphic and univalent in D then $\tilde{F}$ is quasiconformal in $\tilde{D}'$ if and only if*

$$\sup_{(\zeta,\eta)\in D\times D}\left|\frac{f'(\zeta)}{f'(\eta)}\right| = K$$ *is finite. Then* $K_{\tilde{F}} = \left[\frac{1}{2}(K+(1/K))\right]^{\frac{1}{2}}$

(cf. [2]) .

3. Analogous results can be obtained for higher dimensions. As the formulae are much more complicated I shall restrict myself to the following generalization of Theorem 1.

Theorem 2. Let $D = D_1 \times \ldots \times D_n$, where D_k, $k=1,\ldots,n$, is a simply-connected domain on the plane of variables (x_{2k-1}, x_{2k}), points (x_{2k-1}, x_{2k}) will be identified with points $x_{2k-1} + ix_{2k}$ of the complex plane. If $F = (F_1, \ldots, F_n)$: $: D \longrightarrow C^n$,

$$F_k(x_1, \ldots, x_{2n}) = f_k(x_{2k-1} + ix_{2k}) + \overline{g_k(x_{2k-1} + ix_{2k})} ,$$

f_k, g_k being holomorphic in D_k then the mapping

$$\tilde{F} = (\mathrm{Re}\,\tilde{u}_1, \mathrm{Im}\,\tilde{u}_1, \ldots, \mathrm{Re}\,\tilde{u}_{2n}, \mathrm{Im}\,\tilde{u}_{2n})$$

where $\tilde{u}_j$, $j=1,\ldots,2n$, are defined as previously, is quasiregular in the domain

$$\tilde{D}' = \{(x_1,y_1,\ldots,x_{2n},y_{2n}) : (x_1-y_2, x_2+y_1, \ldots, x_{2n-1}-y_{2n}, x_{2n}+y_{2n-1}) \in D,\ (x_1+y_2, x_2-y_1, \ldots, x_{2n-1}+y_{2n}, x_{2n}-y_{2n-1}) \in D\}$$

if and only if

$$K_{\tilde{F}} = \sup_{\substack{(\zeta_k,\eta_k)\in D_k \times D_k \\ k=1,\ldots,n}} \left[\frac{1}{2n} \frac{\sum_{k=1}^{n} |f_k'(\zeta_k)|^2 + |f_k'(\eta_k)|^2 + |g_k'(\zeta_k)|^2 + |g_k'(\eta_k)|^2}{\left[\prod_{k=1}^{n} \left|\overline{f_k'(\zeta_k) f_k'(\eta_k)} - \overline{g_k'(\zeta_k) g_k'(\eta_k)}\right|\right]^{1/n}} \right]^{\frac{1}{2}} < \infty .$$

The proof is analogous to that in the case $n=1$.

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STRESZCZENIE

W pracy badana jest quasiregularność i quasikonforemność odwzorowań otrzymanych przez przedłużenie holomorficzne odwzorowań harmonicznych z obszaru $D \subset \mathbb{R}^2$ do obszaru $\tilde{D} \subset \mathbb{C}^2$ oraz przedłużenie pewnych odwzorowań pluriharmonicznych z obszaru $D \subset \mathbb{R}^{2n}$ do obszaru $\tilde{D} \subset \mathbb{C}^{2n}$. Podany jest przykład różnowartościowego odwzorowania harmonicznego, które po przedłużeniu nie jest iniekcją.

РЕЗЮМЕ

В данной работе исследованы квазирегулярность и квазиконформность отображений полученных через голоморфное продолжение гармонических отображений из области $D \subset \mathbb{R}^2$ в область $\tilde{D} \subset \mathbb{C}^2$ и некоторых плюригармонических отображений из области $D \subset \mathbb{R}^{2n}$ в область $\tilde{D} \subset \mathbb{C}^{2n}$. Представлен пример гармонического отображения, которое 1 : 1, а продолженное отображение неинъективно.