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Jack's Lemma for Holomorphic Mappings in C^n

Lemat Jack'a dla odwzorowań holomorficznych w C^n

Лемма Джека для отображений голоморфных в C^n

Introduction. We let C^n denote the space of n-complex variables $z = (z_1, \ldots, z_n)$, with the Euclidean inner product

$$(1) \qquad \langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w}_j$$

and the norm $\|z\| = \sqrt{\langle z, z \rangle}$. The ball $\{z \in C^n : \|z\| < r\}$ will be denoted by B_r^n, for short we write B^n for B_1^n.

In the recent paper [2] S.S. Miller and P.T. Mocanu give the following generalization of the one-dimensional Jack's lemma [1].

Theorem A. Let $g : B^1 \to C$ be a holomorphic function, with $g(0) = 0$ and $g(\zeta) \not\equiv 0$. If

$$|g(\mathring{\zeta})| = \max_{|\zeta| \leq |\mathring{\zeta}|} |g(\zeta)| \quad , \quad \mathring{\zeta} \in B^1 \, ,$$

then there exists a real number $m_0 \geq 1$ such that

$$(2) \qquad \frac{g'(\mathring{\zeta})}{g(\mathring{\zeta})} = m_0$$

(3) $$\operatorname{Re}\frac{\mathring{\zeta}g''(\mathring{\zeta})}{g'(\mathring{\zeta})}+1\geqslant m_0 .$$

Let us observe that the relations (2), (3) can be presented in the following equivalent form

(4) $$\langle g'(\mathring{\zeta})\mathring{\zeta} , g(\mathring{\zeta})\rangle = m_0|g(\mathring{\zeta})|^2$$

(5) $$\operatorname{Re}\langle g''(\mathring{\zeta})\mathring{\zeta}^2 , g(\mathring{\zeta})\rangle \geqslant m_0(m_0-1)|g(\mathring{\zeta})|^2 ,$$

where $\langle\cdot,\cdot\rangle$ is the one-dimensional Euclidean inner product (1).

In the present paper we extend this result, with relations (4) and (5) to the case of holomorphic mappings in C^n .

Now let $H(B^n)$ denote the set of functions that are holomorphic in B^n with values in C^n .

The main result.

Proposition 1. Let $f\in H(B^n)$ with $f(0)=0$. If

(6) $$\|f(\mathring{z})\| = \max_{\|z\|\leqslant\|\mathring{z}\|}\|f(z)\| , \quad \mathring{z}\in B^n ,$$

then there exist real numbers m_0 , s_0 , $s_0\geqslant m_0\geqslant 1$, such that

(7) $$\langle Df(\mathring{z})(\mathring{z}) , f(\mathring{z})\rangle = m_0\|f(\mathring{z})\|^2$$

(8) $$\operatorname{Re}\langle D^2f(\mathring{z})(\mathring{z},\mathring{z}) , f(\mathring{z})\rangle \geqslant m_0(m_0-1)\|f(\mathring{z})\|^2$$

and

(9) $$\|Df(\mathring{z})(\mathring{z})\| = s_0\|f(\mathring{z})\| ,$$

where $Df(\overset{\bullet}{z})$ is the first and $D^2f(\overset{\bullet}{z})$ the second Frechet-derivative of f at $\overset{\bullet}{z}$. Moreover, for $n>1$, $m_0 = s_0$ if and only if

$$Df(\overset{\bullet}{z})(\overset{\bullet}{z}) = m_0 f(\overset{\bullet}{z}) \quad .$$

Proof. Since for $f(z) \equiv 0$ (7) and (8) hold, we may assume $f(z) \not\equiv 0$. Consider the function

$$g(\zeta) = \left\langle f\left(\zeta \frac{\overset{\bullet}{z}}{\|\overset{\bullet}{z}\|}\right), f(\overset{\bullet}{z}) \right\rangle \quad .$$

Then g is a complex valued holomorphic function in B^1, $g(0) = 0$, $g(\zeta) \not\equiv 0$. It is easy to check that from the assumptions and from Schwarz's inequality it follows that

$$\max_{|\zeta| \leq |\overset{\bullet}{\zeta}|} |g(\zeta)| = |g(\overset{\bullet}{\zeta})| \quad , \quad \overset{\bullet}{\zeta} = \|\overset{\bullet}{z}\| \quad .$$

All hypotheses of Theorem A are fulfilled, so there exists a real number $m_0 \geqslant 1$ such that (4) and (5) hold. As $g(\overset{\bullet}{\zeta}) = \|f(\overset{\bullet}{z})\|^2$, $g'(\overset{\bullet}{\zeta})(\overset{\bullet}{\zeta}) = \langle Df(\overset{\bullet}{z})(\overset{\bullet}{z}), f(\overset{\bullet}{z})\rangle$ and $g''(\overset{\bullet}{\zeta})(\overset{\bullet}{\zeta})^2 = \langle D^2f(\overset{\bullet}{z})(\overset{\bullet}{z},\overset{\bullet}{z}), f(\overset{\bullet}{z})\rangle$, thus (4) and (5) imply (7) and (8). The equality (9) and the last part of Proposition 1 follow immediately from (7) and from the Schwarz inequality.

This completes the proof of Proposition 1.

Remark. As for Euclidean norm of functions $f \in H(B^n)$ the maximum principle is true, so (6) is equivalent to the following equality

$$\|f(\overset{\bullet}{z})\| = \max_{\|z\| = \|\overset{\bullet}{z}\|} \|f(z)\| \quad .$$

Definition 1. Let M be a positive real number and G a domain C^{3n}, for which $(0,0,0) \in G$ and $\bigcup_{m \geqslant 1} G_{M,m} \subset G$, where

$$G_{M,m} = \left\{(u,v,w) \in C^{3n} : \|u\| = M , \langle v,u\rangle = mM^2 , \operatorname{Re}\langle w,u\rangle \geqslant m(m-1)M^2\right\}.$$

By $X(M,G)$ let us denote the set of all continuous functions h, $h : G \to C^n$ such that $\|h(0,0,0)\| < M$ and for any $(u,v,w) \in \bigcup_{m\geqslant 1} G_{M,m}$

(10) $$\|h(u,v,w)\| \geqslant M .$$

In the following theorem we give the first application of Proposition 1.

Theorem 1. Let $f \in H(B^n)$, $f(0) = 0$ and M be a positive real number. If there exists a function $h \in X(M,G)$ such that for each $z \in B^n$

$$(f(z) , Df(z)(z) , D^2f(z)(z,z)) \in G$$

and

(11) $$\|h(f(z) , Df(z)(z) , D^2f(z)(z,z))\| < M ,$$

then

$$\|f(z)\| < M .$$

Proof. Suppose that there exists a point $z^* \in B^n$ for which the inequality $\|f(z^*)\| \geqslant M$ holds, then according to the condition $f(0) = 0$ and in view of the continuity of the norm $\|f(z)\|$ we can find a ball $B^n_{r_0}$, $0 < r_0 < 1$, such that $z^* \notin B^n_{r_0}$ and $\max_{\|z\|=r_0} \|f(z)\| = M$. Let $\mathring{z}$ be a point for which $\|\mathring{z}\| = r_0$ and $\|f(\mathring{z})\| = \max_{\|z\| \leqslant \|\mathring{z}\|} \|f(z)\| = M$. As the assumptions of Proposition 1 are satisfied, then there exists a real number $m_0 \geq 1$ such that (7) and (8) hold. Let $\mathring{u} = f(\mathring{z})$, $\mathring{v} = Df(\mathring{z})(\mathring{z})$ and $\mathring{w} = D^2f(\mathring{z})(\mathring{z},\mathring{z})$. Then $(\mathring{u},\mathring{v},\mathring{w}) \in \bigcup_{m \geq 1} G_{M,m}$ and according to (10) we obtain the inequality $\|h(f(\mathring{z}), Df(\mathring{z})(\mathring{z}), D^2f(\mathring{z})(\mathring{z},\mathring{z}))\| \geq M$ which is contradictory to (11). Therefore our supposition was false, which means that $\|f(z)\| < M$ for $z \in B^n$.

<u>Corrolary 1. Let</u> $f(0) = 0$ <u>and let</u> $M > 0$. <u>If for</u> $z \in B^n$

<u>(i)</u> $$\|f(z) + Df(z)(z)\| < 2M$$

<u>or</u>

<u>(ii)</u> $$\|f(z) + D^2f(z)(z,z)\| < M$$

<u>or</u>

<u>(iii)</u> $$\|f(z)\| \exp(\|Df(z)(z)\| - M) < M ,$$

<u>then</u>

$$\|f(z)\| < M , \quad z \in B^n .$$

<u>Proof.</u> First we prove that if $M > 0$ and $G = C^{3n}$, then each of the following functions

$$h_1(u,v,w) = \frac{1}{2}(u + v)$$

$$h_2(u,v,w) = u + w$$

$$h_3(u,v,w) = u \exp(\|v\| - M)$$

is in $X(M,G)$. Indeed, if $(u,v,w) \in G_{M,m}$, then $\|v\| \geq mM$ and $\|w\| \geq m(m - 1)M$. Hence we have, in view of the definition of the norm in C^n, the inequalities $\|h_i(u,v,w)\| \geq M$ for $(u,v,w) \in G_{M,m}$ and $i=1,2,3$. Since h_i are continuous in G and $h_i(0,0,0) = 0$ we conclude that $h_i \in X(M,G)$ for $i=1,2,3$.

Now, it is sufficient to observe that applying inequalities (i), (ii), (iii) to the functions h_1, h_2, h_3 respectively, we have (11) in all cases. From the above and from Theorem 1 we obtain inequality $\|f(z)\| < M$.

It is interesting that Theorem 1 can be used to show that certain second order differential equations in C^n have bounded solutions.

Corollary 2. Let $F \in H(B^n)$ with $F(0) = 0$, $\|F(z)\| < M$ for $z \in B^n$, and let $h \in X(M,G)$ be holomorphic. If the differential equation

$$h(f(z), Df(z)(z), D^2f(z)(z,z)) = F(z), \quad f(0) = 0$$

has a solution $f \in H(B^n)$, then

$$\|f(z)\| < M, \quad z \in B^n.$$

The proof of the Corrolary 2 follows immediately from Theorem 1.

Now we shall give the definition of the subordination.

Let f, $F \in H(B^n)$. If there exists a Schwarz function ω, ($\omega \in H(B^n)$, $\|\omega(z)\| \leqslant \|z\|$ for $z \in B^n$), such that $f(z) = F(\omega(z))$ for $z \in B^n$, then we say that the function f is subordinated to the function F in the ball B^n and we write $f \prec F$.

For investigating the subordination, in case $n=1$, S.S. Miller and P.T. Mocanu ([3]) applied Jack's lemma and some differential inequalities for the first time. Now we shall show that this method can be adopted also to the case $n>1$.

Lemma 1. Let q be a biholomorphic map on $\overline{B^n}$ and $p \in H(B^n)$ with $p(0) = q(0)$. If p is not subordinated to q, then there exist points $\mathring{z} \in B^n$, $\mathring{\zeta} \in \partial B^n$ and a real number $s_0 \geq 1$ such that

(12) $$p(\mathring{z}) = q(\mathring{\zeta})$$

(13) $$p(B^n_{r_0}) \subset q(B^n) \quad , \quad r_0 = \|\mathring{z}\|$$

and

(14) $$s_0 \|(Dq(\mathring{\zeta}))^{-1}\|^{-1} \leq \|Dp(\mathring{z})(\mathring{z})\| \leq s_0 \|Dq(\mathring{\zeta})\| \quad .$$

Proof. The relations (12) and (13) are an immediate conclusion from the properties of the subordination. Now we prove inequality (14).

The function

(15) $$f(z) = q^{-1}(p(z))$$

is holomorphic in $\overline{B^n_{r_0}}$ and $f(0) = 0$, $\|f(\mathring{z})\| = 1$, $\|f(z)\| < 1$ for $z \in B^n_{r_0}$. Thus $f(z)$ satisfies the assumptions of Proposition 1 and consequently for the function f equality (9) holds. Moreover it is very simple to show that

(16) $$Df(\mathring{z})(\mathring{z}) = (Dq(\mathring{\zeta}))^{-1}(Dp(\mathring{z})(\mathring{z}))$$

and in respect to (9) we have

(17) $$s_0 \|(Dq(\mathring{\zeta}))^{-1}\|^{-1} \leq \|Dp(\mathring{z})(\mathring{z})\| \quad .$$

As the equality (16) is equivalent to the following equality

$$Dq(\mathring{\zeta})(Df(\mathring{z})(\mathring{z})) = Dp(\mathring{z})(\mathring{z})$$

we obtain, according to (9),

(18) $$\|Dp(\mathring{z})(\mathring{z})\| \leqslant s_0 \|Dq(\mathring{\zeta})\| .$$

The inequality (14) follows immediately from (17) and (18).

Definition 2. Let Q be a domain in C^n, q a biholomorphic mapping on B^n and E a domain in C^{2n} for which $(q(0), 0) \in E$ and

$$\bigcup_{\substack{s \geqslant 1 \\ \|\zeta\|=1}} E_{s,q(\zeta)} \subset E$$

where

$$E_{s,q(\zeta)} = \left\{(u,v) : u = q(\zeta) ,\ s\|(Dq(\zeta))^{-1}\|^{-1} \leqslant \|v\| \leqslant s\|Dq(\zeta)\|\right\} .$$

By $Y(E,q)$ let us denote the set of all continuous functions h, $h : E \to C^n$ such that $h(q(0), 0) \in Q$ and for any $(u,v) \in \bigcup_{\substack{s \geqslant 1 \\ \|\zeta\|=1}} E_{s,q(\zeta)}$

(19) $$h(u,v) \notin Q .$$

Theorem 2. Let $p \in H(B^n)$, $p(0) = a$. If there exists a function $h \in Y(E,q)$ with $q(0) = a$ such that for $z \in B^n$

(20) $$(p(z), Dp(z)(z)) \in E$$

and

(21) $$h(p(z), Dp(z)(z)) \in Q ,$$

then $p \prec q$ in B^n.

Proof. If the subordination $p \prec q$ does not hold then, in view of Lemma 1, there exist points $\mathring{z} \in B^n$, $\mathring{\zeta} \in \partial B^n$ and a real number $s_0 \geq 1$ such that (12), (13) and (14) hold. Let $\mathring{u} = p(\mathring{z})$, $\mathring{v} = Dp(\mathring{z})(\mathring{z})$. Then $(\mathring{u}, \mathring{v}) \in \bigcup_{\substack{s \geq 1 \\ \|\zeta\|=1}} E_{s,q(\zeta)}$ and, according to (19) $h(p(\mathring{z}), Dp(\mathring{z})(\mathring{z})) \notin Q$. But this contradicts (21) so we must have $p \prec q$. This completes the proof of this theorem.

Corollary 3. Let $p \in H(B^n)$, $p(0) = a$, q be a biholomorphic mapping on B^n and $q_\varrho(z) = q(\varrho z)$, for $\varrho \in (0,1)$, $z \in B^n$. If there exists a function $h \in \bigcap_{0<\varrho<1} Y(E, q_\varrho)$ such that (20) and (21) hold, then $p \prec q$ in B^n.

Proof. Let $p_\varrho(z) = p(\varrho z)$, then from the Theorem 2 it follows that $p_\varrho \prec q_\varrho$ for any $\varrho \in (0,1)$ and consequently $p \prec q$.

REFERENCES

[1] Jack, I.S., Functions starlike and convex of order α, J. London Math. Soc. 3(1971), 469-474.

[2] Miller, S.S., Mocanu, P.T., Second order differential inequalities in the complex plane, J.Math. Anal. Appl. 65(1978), 289-305.

[3] Miller, S.S., Mocanu, P.T., Differential subordinations and univalent functions, Michigan Math. J. 28(1981) No 2.

STRESZCZENIE

Podano n-wymiarową wersję lematu Jack'a i kilka jej zastosowań do badania podporządkowania i ograniczoności odwzorowań holomorficznych kuli jednostkowej w C^n.

РЕЗЮМЕ

Представлена n-мерная версия леммы Джека и несколько ее применений в исследовании ограниченности и подчинения для голоморфных отображений единичного шара в C^n.