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Local and Global Lipschitz Classes

Lokalne i globalne klasy Lipschitza

Локальные и глобальные классы Липшица

1. Introduction. A modulus of continuity is a concave positive increasing function $h : [0, \infty) \to R$. Let D be a nonempty domain in R^n. A function $u : D \to R$ belongs to the global Lipschitz class $Lip_h(D)$ if there exists a constant $M < \infty$ such that

$$(1.1) \qquad |u(x) - u(y)| \leq M\, h(x,y) \quad ,$$

whenever x and y belong to the domain D ; here and later on $h(x,y) := h(|x-y|)$. We say that the function u belongs to the corresponding local Lipschitz class $loc\ Lip_h(D)$ if there exists constants $b \in (0, 1)$ and $M = m_b$ such that (1.1) holds for each $x \in D$ and $y \in B_b(x) := B(x, b\,dist(x,\partial D))$. As a metter of fact, in [L, Theorem 2.17 and 2.19] it is shown that it is equivalent to require that the condition holds for $b = 1/2$. And more generally, if a modulus of continuity h is "smooth" enough, then $u \in loc\ Lip_h(D)$ if and only if (1.1) holds whenever x and y belong to an open ball contained in D ; see [L, Theorem 4.29] . It should be remarked that this definition differs from the standart definitions of local Hölder spaces. In fact, the class $\mathrm{loc}\,\mathrm{Lip}_h(D)$ is not a local space but in some sense semiglobal.

In this paper we study the domains where local and global Lipschitz classes are equal. These domains are called Lip_h -extension domains. More generally, let h and g be two moduli of continuity. We say that the domain D is a $\mathrm{Lip}_{h,g}$ -extension domain if there is a constant $E = E(D,h,g)$ such that every function u in the class $\mathrm{loc}\,\mathrm{Lip}_h(D)$ with a constant m, belongs also to the class $\mathrm{Lip}_g(D)$ with a constant $M = Em$. We write Lip_α for the Hölder spaces, where $h(t) = t^\alpha$, $0 < \alpha \leq 1$.

For quasiconformal mappings and solutions of elliptic partial differential equations it is easy to derive local Lipschitz results and hence in $\mathrm{Lip}_{h,g}$ -extension domains global Lipschitz bounds are obtained. See [GM1] and [GM2] for these applications. Some applications for Sobolev embedding theorems is found from [LL] and [Le]. Applications to analytic functions is presented in [GM1] and [L, Section 7].

2. $\mathrm{Lip}_{h,g}$ -extension domains. Let h and g be two metrics in the domain $D \subset R$. We say that g dominates h in D and write $h \prec g$ if there is a constant $A < \infty$ such that for each $x,y \in D$

$$h(x,y) \leq Ag(x,y) \quad .$$

The modulus of continuity h defines in D the metric

$$(2.1) \qquad h_D(x,y) := \inf_{\gamma(x,y)} \int_\gamma \frac{h(\mathrm{dist}(\gamma(s)\,,\,\partial D))}{\mathrm{dist}(\gamma(s)\,,\,\partial D)}\, ds \,,$$

where $\gamma(x,y)$ is a rectifiable curve joining x to y in D and $\operatorname{dist}(\gamma(s), \partial D)$ is the euclidean distance from the point $\gamma(s)$ (arc length representation for γ) to the boundary of the domain D.

For the proof of the following characterisation of $\mathrm{Lip}_{h,g}$ -extension domains see [L, Theorem 3.6] or [GM2, Theorem 2.2].

<u>Theorem 2.1. Let</u> h <u>and</u> g <u>be two moduli of continuity. A domain</u> D <u>is a</u> $\mathrm{Lip}_{h,g}$ <u>-extension domain if and only if</u> $h_D \prec g$.

Notice that if $h_D \prec g$ in D, then $h \prec g$ in D, see [L, Lemma 3.8].

<u>Corollary 2.2. A domain</u> D <u>is a</u> Lip_h <u>-extension domain if and only if</u> $h_D \prec h$.

The next inclusion theorem is proved in [L, Theorem 4.6].

<u>Theorem 2.3. Let</u> D <u>be a</u> Lip_h <u>-extension domain and</u> g <u>a modulus of continuity such that the function</u> h/g <u>is decreasing. Then</u> D <u>is also a</u> Lip_g <u>- extension domain.</u>

If $h(t) = t^{\alpha}$, $g(t) = t^{\beta}$ and $0 < \alpha < \beta \leq 1$ then every Lip_{α} -extension domain is also a Lip_{β} -extension domain, because the function $h(t)/g(t) = t^{\alpha-\beta}$ is decreasing. On the other hand, we can construct Lip_{β} -extension domains, which are not Lip_{α} -extension domains. For one construction see [L, Countexample 6.7].

Let us recall that a domain D in R^n is c -quasiconvex if every $x, y \in D$ can be joined by a rectifiable curve γ in D with

$$\ell(\gamma) \leq c|x-y| ,$$

where $\ell(\gamma)$ is the length of the curve γ.

Theorem 2.4. A Lip_h -extension domain is quasiconvex.

Proof. Let $g(t) = t$. Using the properties of the modulus of continuity, we see that the function $h(t)/g(t) = h(t)/t$ is decreasing. Thus if D is a Lip_h -extension domain, then D is also a Lip_g -extension domain, and by Corollary 2.2 there exists $c < \infty$ such that

$$\inf_{\gamma(x,y)} \ell(\gamma) = g_D(x,y) \geq cg(x,y) = c|x - y| .$$

So D is quasiconvex.

Notice that a $\mathrm{Lip}_{h,g}$ -extension domain need not to be quasiconvex, see [L, Theorem 5.1].

Olli Martio introduced uniform domains in [M]. A domain D in R^n is c-uniform if every $x, y \in D$ can be joined with a rectifiable curve γ in D such that

$$\ell(\gamma) \leq c|x - y| , \tag{2.2}$$

and

$$\operatorname{dist}(\gamma(t), \partial D) \geq \frac{1}{c} \min(t, \ell(\gamma)-t) . \tag{2.3}$$

A domain is called uniform, if it is c-uniform for some $c < \infty$. The snowflake or the Koch curve described in Mandelbrot [Ma, p.42] is an example of a uniform domain whose boundary is very irregular.

If the modulus of continuity h increases too fast near 0, then there is no $\mathrm{Lip}_{h,g}$ -extension domain; see [L, Lemma 4.11]. As a matter of fact, we can prove the following theorem for the existence of the $\mathrm{Lip}_{h,g}$ -extension domains.

Theorem 2.5. Let h and g be two moduli of continuity. Then the following conditions are equivalent:

(i) There are constants $K < \infty$ and $t_K > 0$ such that for every $0 < t \leq t_K$

$$\int_0^t \frac{h(s)}{s} ds \leq K g(t) .$$

(ii) All bounded uniform domains are $Lip_{h,g}$ -extension domains.

(iii) The unit ball in R^n is a $Lip_{h,g}$ -extension domain.

(iv) There exists at least one $Lip_{h,g}$ -extension domain.

Proof. Suppose first that (i) holds. From the properties of the modulus of continuity h it follows that $h'(t) \leq h(t)/t$ and so $h(t) \leq K g(t)$ whenever $0 < t \leq t_K$. Let D be a c -uniform domain with the diameter d_D . Choose $x , y \in D$ and γ as in the definition for c -uniform domain. Since $h(t)/t$ is decreasing we obtain from the definition of h_D

$$h_D(x,y) \leq \int_0^{\ell(\gamma)} \frac{h(dist(\gamma(s) , \partial D))}{dist(\gamma(s) , \partial D)} ds \leq$$

$$\leq 2 \int_0^{\frac{\ell(\gamma)}{2}} \frac{h(s/c)}{s/c} ds \leq$$

$$\leq 2c \int_0^{\frac{\ell(\gamma)}{2}} \frac{h(s)}{s} ds ,$$

because $c \geq 1$. If $\ell(\gamma)/2 \leq t_K$, then

$$h_D(x,y) \leq 2cKg(\ell(\gamma)/2) \leq 2cKg(c|x-y|) \leq 2c^2Kg(|x-y|) ,$$

and by Theorem 2.1 D is a $Lip_{h,g}$ -extension domain.

If $\ell(\gamma)/2 > t_K$, then

$$h(\ell(\gamma)) \leqslant h(c|x-y|) \leqslant ch(d_D) = ch(\frac{d_D}{t_K}\cdot t_K) \leqslant$$

$$\leqslant c \max(\frac{d_D}{t_K}, 1)\, h(t_K) \leqslant c\, AKg(t_K) \leqslant c\, AKg(\ell(\gamma)) ,$$

where $A = \max(d_D/t_K , 1)$, and so $h_D \prec g$ in D :

$$2c\int_0^{\ell(\gamma)/2} \frac{h(s)}{s}ds = 2c\int_0^{t_K} \frac{h(s)}{s}ds + 2c\int_{t_K}^{\ell(\gamma)/2} \frac{h(s)}{s}ds \leqslant$$

$$\leqslant 2cK\, g(t_K) + 2c\, h(\ell(\gamma))\int_{t_K}^{\ell(\gamma)/2} \frac{1}{s}\, ds \leqslant$$

$$\leqslant 2c\left(K\, g(\ell(\gamma)) + cAK\, g(\ell(\gamma))\ln\frac{\ell(\gamma)}{2t_K}\right) \leqslant$$

$$\leqslant 2cK\left(1 + cA\ln\frac{c|x-y|}{2t_K}\right) g(c|x-y|) \leqslant$$

$$\leqslant 2c^2K\left(1 + cA\ln\frac{cd_D}{2t_K}\right) g(|x-y|) .$$

Thus D is a $\mathrm{Lip}_{h,g}$ -extension domain.

Next, we show that (iv) implies (i). Let D be a $\mathrm{Lip}_{h,g}$ - extension domain. Take a point $y_o \in D$ and choose a point $x_o \in \partial D$ such that the line segment $J(x_o , y_o) \subset D \cup \{x_o\}$. Let G be the complement of x_o and $t_K := |x_o - y_o|$. Let $0 < t \leqslant t_K$ and $0 < \varepsilon < t$. Choose points $x,y \in J(x_o , y_o)$ such that $|x - x_o| = \varepsilon$ and $|y - x_o| = t$. By Theorem 2.1

there exists a constant K such that $h_D(x,y) \leqslant K\, g(x,y)$ and trivially $h_G(x,y) \leqslant h_D(x,y)$. So

$$\int_{\varepsilon}^{t} \frac{h(s)}{s}\, ds = h_G(x,y) \leqslant h_D(x,y) \leqslant K\, g(x,y) = K\, g(t-\varepsilon) \quad ,$$

and (i) holds by letting $\varepsilon \longrightarrow 0$.

This completes the proof, since the fact that a unit ball is uniform and (iv) follows trivially from (iii).

Modifying the previous proof we have the same theorem for unbounded domains:

Theorem 2.6. *Let h and g be two moduli of continuity. Then the following conditions are equivalent:*

(i) There is a constant $K < \infty$ such that for every $t > 0$

$$\int_{0}^{t} \frac{h(s)}{s}\, ds \leqslant K\, g(t) \quad .$$

(ii) All uniform domains are $Lip_{h,g}$ -extension domains.

(iii) The complement of a point in R^n is a $Lip_{h,g}$ -extension domain.

By Theorem 2.5 and Theorem 2.6 one could think that $Lip_{h,g}$ -extension domains are exactly uniform domains. But that is not the case even with Lip_h -extension domains. In [L, Lemma 4.28] and [GM2, Example 2.26(c)] there are examples of non-uniform Lip_h -extension domains.

For an other characterization for $Lip_{h,g}$ -extension domains based on the maximum derivative see [L, Theorem 7.3] and [GM1].

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STRESZCZENIE

Mówimy, że funkcja rzeczywista u określona w niepustym obszarze $D \subset \mathbb{R}^n$ należy do klasy globalnej Lipschitza $Lip_h(D)$, jeśli istnieje stała $M < \infty$, taka u (*): $|u(x) - u(y)| \leq M\, h(|x-y|)$ dla wszystkich $x, y \in D$, gdzie $h : (0; +\infty) \to R$ jest funkcją rosnącą i wklęsłą (moduł ciągłości). Funkcja u należy do klasy lokalnej Lipschitza loc $Lip_n(D)$, jeśli istnieją stałe $b \in (0; 1)$, $M = m_b$ takie, że (*) ma miejscedlakażdego $x \in D$ i każdego y w kuli $B(x, b \operatorname{dist}(x, \partial D))$. W pracy bada się obszary, dla których obie klasy pokrywają się.

РЕЗЮМЕ

Вещественная функция u определенная в непустой области $D \subset \mathbb{R}^n$ принадлежит к глобальному классу Липшица $Lip_h(D)$, если существует постоянная $M < \infty$, такая, что $|u(x) - u(y)| \leq M\, h(|x-y|)$ для всех $x, y \in D$ где $h : (0; +\infty) \to \mathbb{R}$ возрастающая и вогнутая (модуль непрерывности). Функция принадлежит к локальному классу Липшица loc $Lip_h(D)$, если существуют постоянные $b \in (0 : 1)$, $M = m_b$, такие, что (*) имеет место для произвольного $x \in D$ и всех y в шаре $B(x, b \operatorname{dist}(x, \partial D))$. В этой работе были исследованы области, для которых эти классы совпадают.