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**Close-to-convex Functions, Univalence Criteria
and Quasiconformal Extension**

Funkcje prawie wypukłe, kryteria jednołistości i przedłużenie
kwazikonformne

Почти выпуклые функции и признаки однолиственности
и квазиконформное продолжение

Let S denote the family of univalent functions f of the unit disk D , normalized by

$$(1) \quad f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

Let K denote the subset of convex functions, i.e. functions that have a convex range. A function f , normalized by (1), is called close-to-convex of order β , $\beta \geq 0$, if there is a convex function φ such that $|\arg(e^{i\alpha} f'(z)/\varphi'(z))| \leq \beta \frac{\pi}{2}$ for some $\alpha \in \mathbb{R}$. Let $G(\beta)$ denote the family of close-to-convex functions of order β . For $\beta \leq 1$ it turns out that a function is close-to-convex of order β , if and only if it maps D univalently onto a domain whose complement is the union of rays which are pairwise disjoint up to their tips, such that every ray is the bisector of a sector of angle

$(1-\beta)\pi$ which wholly lies in E (see [9], p.176). Obviously $C(0)$ equals K .

Kenari showed that if a locally univalent function f satisfies a condition of the form

$$(2) \quad (1-|z|^2)^2 \left| \left(\frac{f''}{f'} \right)'(z) - \left(\frac{f''}{f'} \right)^2(z) \right| \leq \sigma$$

with $\sigma = 2$, then f is univalent [7]. Later he showed that each convex function satisfies the univalence criterion [8]. The next Theorem gives a generalized version of this result for close-to-convex functions.

Theorem 1. Let $\beta \geq 0$ and $f \in C(\beta)$. Then f satisfies a Kenari type condition (2) with

$$(3) \quad \sigma = \begin{cases} 2 + 4\beta & \text{if } \beta \leq 1 \\ 2\beta^2 + 4\beta & \text{if } \beta \geq 1 \end{cases},$$

and this result is sharp.

We indicate the idea of the proof, which is in [4]. The first step is to show the result at the origin, i.e.

$$(4) \quad |a_3 - a_2^2| \leq \sigma/6,$$

where σ is defined by (3). Therefore one uses the representation

$$f'(z) = e^{-i\alpha} \varphi'(z)(\cos\alpha \cdot p(z) + i\sin\alpha)^\beta$$

with some $q(z) = z + q_2 z^2 + q_3 z^3 + \dots \in \mathcal{K}$,
 $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ with positive real part and
 $\cos \alpha > 0$. With aid of the estimates

$$|q_3 - q_2^2| \leq \frac{1}{3} (1 - |q_2|^2)$$

[11] and

$$|p_2 - \frac{1}{2} p_1^2| \leq 2 - \frac{1}{2} |p_1|^2$$

(see e.g. [10], p.166, formula (10)) one gets using the triangle inequality

$$(5) \quad 3 |a_3 - a_2^2| \leq 1 - |q_2|^2 + \beta \cos \alpha \left(2 - \frac{|p_1|^2}{2} (1 - \sqrt{1 - \cos^2 \alpha (1 - \frac{\beta^2}{4})}) \right) + \beta \cos \alpha |p_1| |q_2| .$$

Now it is well known that $|p_1| \in [0, 2]$, $|q_2| \in [0, 1]$ and $\cos \alpha \in [0, 1]$. A careful analysis shows that the right hand side of (5) is maximized at $(|p_1|, |q_2|, \cos \alpha) = (0, 0, 1)$ if $\beta \in [0, 1]$ and at $(|p_1|, |q_2|, \cos \alpha) = (2, 1, 1)$ if $\beta \geq 1$ which gives (4).

Now the linear-invariance of $G(\beta)$ is used (see e.g. [3]). Composing with an automorphism of D the information at the origin is transported to an arbitrary point $z \in D$, which finishes the proof. The sharpness is easily verified.

Becker showed that if a locally univalent function f

satisfies a condition of the form

$$(6) \quad (1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right| \leq \lambda$$

with $\lambda = 1$, then f is univalent [1]. The Becker univalence criterion is often used to construct exotic univalent functions with "bad behaviour". In most cases these examples are gap series. We show conversely that m -fold symmetric close-to-convex functions with large m fulfill the Becker univalence criterion. A function f is called m -fold symmetric if it has the special form

$$f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \dots$$

Theorem 2.

(a) Let $f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \dots \in K$. Then f satisfies a Becker type condition (6) with $\lambda = 4/m$. In particular, if $m \geq 4$, then f fulfills the Becker univalence criterion.

(b) Let $\beta \in (0, 1)$ and let f be an m -fold symmetric close-to-convex function of order β . Then f satisfies a Becker type condition (6) with $\lambda = 4/m + 2\beta$. In particular, if $\beta < 1/2$ and $m \geq 4/(1-2\beta)$, then f fulfills the Becker univalence criterion. Moreover, the results are sharp.

The proof of Theorem 2 is in [5]. The idea for (a) is to use a subordination result for convex functions of the given form,

namely $f' \prec (1-z)^{-2/m}$. The statement $f \prec g$ means that $f = g \circ \omega$, where ω satisfies the hypotheses of Schwarz's Lemma. It implies that

$$(7) \quad \sup_{z \in D} (1 - |z|^2) |f'(z)| \leq \sup_{z \in D} (1 - |z|^2) |g'(z)|$$

(see e.g. [10], p.35, formula (4)), which is used to deduce (a). This result and a further application of (7) leads to (b). The sharpness is again easily verified.

From Theorem 2 one could deduce a condition which guarantees that an m -fold symmetric close-to-convex function of order β has a quasiconformal extension. A more precise result is

Theorem 3. Let $\beta \in [0, 1)$ and f be an m -fold symmetric close-to-convex function of order β .

- (a) If $m > 2/(1-\beta)$, then f has a rectifiable boundary;
 (b) if $m > 4/(1-\beta)$, then f has a quasiconformal extension.

This result is in [6]. For to prove (a) one uses an estimate on the integral means of the derivative of m -fold symmetric close-to-convex functions of order β ([6], Theorem 5), and for (b) a general condition, which implies the existence of a quasiconformal extension for Bazilevič functions [2], gives the result.

REFERENCES

- [1] Becker, J., Löwner'sche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen, *J. Reine Angew. Math.* 255(1972), 23-43.
- [2] Gall, U., Über das Randverhalten von Bazilevič-Funktionen, Dissertation an der Technischen Universität Berlin, 1986.
- [3] Koepf, W., Close-to-convex functions and linear-invariant families, *Ann. Acad. Sci. Fenn. Ser. A I Math.* 8(1983), 349-355.
- [4] Koepf, W., On the Fekete-Szegő problem and successive coefficients of close-to-convex functions II, 1985, to appear.
- [5] Koepf, W., Some remarks on the Becker univalence criterion, 1986, to appear.
- [6] Koepf, W., Extremal problems for close-to-convex functions, 1986, to appear.
- [7] Nehari, Z., The Schwarzian derivative and schlicht functions, *Bull. Amer. Math. Soc.* 55(1949), 545-551.
- [8] Nehari, Z., A property of convex conformal maps, *J. Anal. Math.* 30(1976), 390-393.
- [9] Pommerenke, Ch., On close-to-convex analytic functions, *Trans. Amer. Math. Soc.* 114(1965), 176-186.
- [10] Pommerenke, Ch., *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
- [11] Trimble, S.Y., A coefficient inequality for convex univalent functions, *Proc. Amer. Math. Soc.* 48(1975), 266-267.

STRESZCZENIE

Niech $C(\beta)$ będzie rodziną funkcji prawie wypukłych rzędu $\beta \in (0; 1)$. Znalaziono dokładną wartość normy szwarcjana $\|S_f\| = \sigma$ dla $f \in C(\beta)$ (tw. 1). Wykazano również, że m -symetryczna funkcja f klasy $C(\beta)$ spełnia warunek jednolistości Beckera, jeśli $\beta < 1/2$ oraz $m > 4/(1-2\beta)$. (tw. 2). Ponadto, jeśli $m > 4/(1-\beta)$, to f ma przedłużenie quasikonforemne na całą płaszczyznę (tw. 3).

РЕЗЮМЕ

Пусть $C(\beta)$ семейство почти выпуклых функций порядка $\beta \in (0; 1)$. Найдена точная оценка нормы шварцiana $\|S_f\| = \sigma$ для $f \in C(\beta)$ (теор. 1). Доказано тоже, что m -симметрическая функция $f \in C(\beta)$ исполняет признак Бекера для $\beta < \frac{1}{2}$, $m > 4/(1-2\beta)$ (теор. 2). Кроме того, если $m > 4/(1-\beta)$, тогда f квазиконформно продолжима на целую плоскость (теор. 3).

