ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XL, 8 SECTIO A 1986

Sektion Mathematik/Physik
d. Pädagogischen Hochschule „N. K. Krupskaja"

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Remarks on Extremal Problems in a Class of Quasiconformal Mappings in the Mean

Uwagi o problemach ekstremalnych dla odwzorowań średnio quasikonforemnych

Замечания об экстремальных проблемах для отображений квазиконформных в среднем

1. Introduction. Extremal problems for quasiconformal mappings in the mean are closely related to the extremal problems for quasiconformal mappings with a prescribed dilatation bound which is a bounded function of a complex variable, whereas now the dilatation is only bounded in the mean (6) . It was P.A. Biluta [1], [2] who first investigated such problems. He derived a necessary condition for extremal functions if they do exist. R. Kühnau [8] proved, that under some further conditions this necessary condition is also sufficient. This condition means, that the extremal function is connected with a quasilinear elliptic system of differential equations and inequalities which appears in gas dynamics, see [7], [8]. In his paper [9] R. Kühnau used this necessary and sufficient condition to construct analytically the extremal function and to determine the extreme value of the considered functional in a special case.

But it is also possible to go just the other way. Our main effort centers on sharp upper and lower estimates of the extreme value of the considered functional and on the geometrical characterization of those ranges of integration G' in (6) for which the extremal problem admits solution and construction of the extremal function and the extreme value of the functional is equal to the upper, or lower bound.

In general the proof for the existence of solution of these problems is rather complicated. The class of quasiconformal mappings satisfying (6) is usually not compact. The existence of extremal functions depends on the mean function in (6) and on the boundary of G'. To illustrate this we consider the case where G' is a square and Φ is a linear function. In order to make clear the main ideas we choose as an example a functional of Grötzsch - Teichmüller type whose associated quadratic differential is a complete square.

2. Notations and the Problem. Let $G \ni \infty$ be a n-tuply connected Jordan domain in the complex plane $\mathbb{C}$ with the boundary $\Gamma = \Gamma_1 + \ldots + \Gamma_n$, $G' := \mathbb{C} \setminus \bar{G}$, $I := I(G')$ is the area of G' and σ the two-dimensional Lebesgue measure. Further, let $p := p(z) \geq 1$ be a real valued bounded and measurable function which is defined on $\mathbb{C}$ and identical 1 in G.

We denote by g_θ the uniquely determined quasiconformal mapping of $\mathbb{C}$ onto $\mathbb{C}$ with hydrodynamical normalization $z + a_{1,\theta} z^{-1} + \ldots$ near $z = \infty$, where $W = U + iV := ie^{-i\theta} g_\theta$ satisfies

$$W_{\bar{z}} = - \frac{p-1}{p+1} \overline{W_z} \quad \sigma\text{- almost everywhere.} \tag{1}$$

Moreover, we denote by G_θ the univalent conformal mapping of

G onto a θ - parallel slit domain, $0 \leq \theta \leq \pi$, with hydrodynamical normalization $z + A_{1,\theta} z^{-1} + \ldots$ near $z = \infty$.

We put

(2) $$u_\theta + iv_\theta := ie^{-i\theta} \ (g_\theta(z) - z) \ , \quad z \in \mathbb{C} \ ,$$

(3) $$\hat{u}_\theta := \begin{cases} \mathrm{Re}\ (ie^{-i\theta}(G_\theta - z)) \ , & z \in G \\ R_\vartheta - \mathrm{Re}\ (ie^{-i\theta} z) \ , & z \text{ within } \Gamma_\vartheta \ , \ \vartheta = 1,2,\ldots,n \ , \end{cases}$$

where $R_\vartheta := \mathrm{Re}\ (ie^{-i\theta} G_\theta(\Gamma_\vartheta)) = \mathrm{const.}$, $\vartheta = 1,2,\ldots,n$,

(4) $$\hat{v}_\theta := \begin{cases} \mathrm{Im}\ (ie^{-i\theta}(G_{\theta+\pi/2} - z)) \ , & z \in G \\ I_\vartheta - \mathrm{Im}\ (ie^{-i\theta} z) \ , & z \text{ within } \Gamma_\vartheta \ , \ \vartheta = 1,2,\ldots,n \ , \end{cases}$$

where $I_\vartheta := \mathrm{Im}\ (ie^{-i\theta} G_{\theta+\pi/2}(\Gamma_\vartheta)) = \mathrm{const.}$, $\vartheta = 1,2,\ldots,n$

and

(5) $$\varphi + \psi := ie^{-i\theta} z \ .$$

Let $\mathcal{L}_2^1$ be the class of all real valued piecewise smooth functions $u=u(z)$ defined on $\mathbb{C}$ with finite Dirichlet integral and $\lim_{z\to\infty} u(z) = 0$. Denote by A_Φ the class of all univalent conformal mappings of G with quasiconformal continuation into G' and hydrodynamical normalization $z + a_1 z^{-1} + \ldots$ near $z = \infty$. The dilatation $p(z)$ satisfies

(6) $$\int_{G'} \Phi(p(z))\, d\sigma_z \leq C \ ,$$

where the constant $C > \Phi(1) \cdot I(G')$ and $\Phi : [1,\infty) \to R^1$ is a prescribed continuous, monotone and convex function for which Φ'' exists. Denote by Φ^{-1} the inverse function of Φ.

We study the following extremal problem:

(7) $\operatorname{Re}(e^{-2i\theta} a_1) \longrightarrow \sup =: s_\theta$,

where the supremum is taken over A_Φ . Because of the extremal property of g_θ , see [5,p.98], it is clear, that for maximization it is sufficient to consider only such $g \in A_\Phi$ for which $ie^{-i\theta}g$ satisfies the system (1).

3. A variational characterization for u_θ and v_θ .

To begin with let p and G be sufficiently smooth, so that Gaus's Theorem is applicable. Because of (1) u_θ satisfies the equation $\operatorname{div}(p \nabla(u_\theta + \varphi)) = 0$ in $\mathbb{C}$.

Therefore we have

(8) $$\int_{\mathbb{C}} (p \Delta u_\theta + \nabla p \nabla u_\theta + \nabla p \nabla \varphi) u \, d\sigma = 0 \quad , \quad u \in \mathcal{L}_2^1 .$$

By applying Gaus's Theorem (8) yields

(9) $$(u , u_\theta)_p - l(u) = 0 \quad , \quad u \in \mathcal{L}_2^1$$

with $(u , v)_p := \int_{\mathbb{C}} p \nabla u \nabla v \, d\sigma$, $l(u) := \int_{\mathbb{C}} \nabla p \nabla \varphi \, u \, d\sigma$, $u,v \in \mathcal{L}_2^1$.

Hence we obtain

(10) $$0 \leq \|u - u_\theta\|_p^2 = F(u) - F(u_\theta) \quad , \quad u \in \mathcal{L}_2^1 ,$$

where $F(u) := \|u\|^2 - 2l(u)$.

Now we compute $F(u_\theta)$. Because of (9) we have for $u = u_\theta$

(11) $$F(u_\theta) = -l(u_\theta) = - \int_{\mathbb{C}} \nabla p \nabla \varphi \cdot u_\theta \, d\sigma = - \int_{\mathbb{C}} \nabla p \nabla \varphi (U - \varphi) d\sigma =$$

$$= \int_{\mathbb{C}} \nabla(p-1) \nabla \varphi \cdot \varphi \, d\sigma - \int_{\mathbb{C}} \nabla p \nabla \varphi \, U \, d\sigma =$$

$$= - \int_G (p-1) d\sigma + \int_{|z|=R} (\varphi \frac{\partial U}{\partial n} - U \frac{\partial \varphi}{\partial n}) \, ds =$$

$$= - \int_{G'} (p-1)\, d\sigma - \operatorname{Im} \int_{|z|=R} (ie^{-i\theta} G_\theta)\, d(ie^{-i\theta} z) =$$

$$= - \int_{G'} (p-1)\, d\sigma + 2\pi \operatorname{Re} (e^{-2i\theta} a_{1,\theta}) ,$$

where $(|z| \leqslant R)$ contains Γ and n is the outward pointing unit normal vector on $|z| = R$.

Putting $u = \hat{u}_\theta$ we obtain

$$\|u\|^2 = \int_{\mathbb{C}} (p-1)|\nabla u|^2 d\sigma + \int_{\mathbb{C}} |\nabla u|^2 d\sigma = \int_{G'} (p-1) d\sigma + I(G') + \int_{G} |\nabla u|^2 d\sigma .$$

Further calculation shows

$$\int_G |\nabla u|^2 d\sigma = \int_\Gamma u \frac{\partial u}{\partial n}\, ds = \int_\Gamma \varphi \frac{\partial \varphi}{\partial n}\, ds - \int_\Gamma \Big(\varphi \frac{\partial \operatorname{Re}(ie^{-i\theta} G_\theta)}{\partial n} -$$

$$- \operatorname{Re}(ie^{-i\theta} G_\theta) \frac{\partial \varphi}{\partial n}\Big)\, ds = - I(G') + \operatorname{Im} \int_{|z|=R} (ie^{-i\theta} G_\theta)\, d(ie^{-i\theta} z) =$$

$$= - I(G') + 2\pi \operatorname{Re}(e^{-2i\theta} A_{1,\theta}) ,$$

hence $\|\hat{u}_\theta\|^2 = \int_{G'} (p-1)\, d\sigma + 2\pi \operatorname{Re}(e^{-2i\theta} A_{1,\theta})$

and $l(\hat{u}_\theta) = \int_{\mathbb{C}} \nabla p \nabla \varphi\, \hat{u}_\theta\, d\sigma = - \int_{\mathbb{C}} \nabla(p-1) \nabla \varphi \cdot \varphi\, d\sigma = \int_{G'} (p-1)\, d\sigma .$

Putting $\lambda := l(\hat{u}_\theta) / \|\hat{u}_\theta\|^2 =$

$$= \int_{G'} (p-1)\, d\sigma \Big/ \Big(\int_{G'} (p-1) d\sigma + 2\pi \operatorname{Re}(e^{-2i\theta} A_{1,\theta})\Big)$$

we have

$$F(\lambda \hat{u}_\theta) = - \lambda \int_{G'} (p-1)\, d\sigma . \tag{12}$$

Thus (10), (11) and (12) yield

$$0 \leqslant \|\lambda \hat{u}_\theta - u_\theta\|_p^2 = 2\pi \Big[\lambda \operatorname{Re}(e^{-2i\theta} A_{1,\theta}) - \operatorname{Re}(e^{-2i\theta} a_{1,\theta})\Big] . \tag{13}$$

Taking into account that v_θ satisfies the equation

$\operatorname{div}\left(\frac{1}{p}\nabla(v_\theta+\psi)\right)=0$, we obtain analogously

(14) $$0\leqslant\|-\hat{\lambda}\hat{v}_\theta-v_\theta\|^2_{1/p}=$$
$$=-2\pi\left[\hat{\lambda}\operatorname{Re}(e^{-2i(\theta+\pi/2)}A_{1,\theta+\pi/2})-\operatorname{Re}(e^{-2i\theta}a_{1,\theta})\right]$$

where

$$\hat{\lambda}:=\int_{G'}(1-\tfrac{1}{p})d\sigma\Big/\Big(2\pi\operatorname{Re}(e^{-2i(\theta+\pi/2)}A_{1,\theta+\pi/2})-\int_{G'}(1-\tfrac{1}{p})d\sigma\Big).$$

Also under more general assumptions on p and Γ we have the following variational characterization of u_θ and v_θ in

<u>Lemma 1. If p and G satisfy the assumptions stated in sect. 2, then we have for all θ, $0\leqslant\theta\leqslant\pi$, the inequalities</u>

(15) $$\hat{\lambda}\operatorname{Re}(e^{-2i(\theta+\pi/2)}A_{1,\theta+\pi/2})\leqslant\operatorname{Re}(e^{-2i\theta}a_{1,\theta})\leqslant$$
$$\leqslant\lambda\operatorname{Re}(e^{-2i\theta}A_{1,\theta}),$$

<u>where λ, $\hat{\lambda}$ are the same constants as in (12), (13) and (14). The equality on the right and left holds iff</u>

(16) $\lambda\hat{u}_\theta=u_\theta$ <u>on</u> $\mathbb{C}$

<u>and</u>

(17) $-\hat{\lambda}\hat{v}_\theta=v_\theta$ <u>on</u> $\mathbb{C}$ <u>respectively.</u>

<u>Proof.</u> Because of (13) and (14) the inequalities (15) are valid for sufficiently smooth p_n and G_n, and also for p and G. The latter case is obtained by applying well-known Theorems [11, I.3] about the convergence for conformal mappings of sequences of domains G_n and quasiconformal mappings $g_{\theta,n}$ for which $ie^{-i\theta}g_{\theta,n}$ satisfies (1) for $p:=p_n$, whereas

(18) $G_n\to G$ in the sense of kernel convergence and $p_n\to p$ σ- almost everywhere with $\operatorname{supp}(p_n^{-1})\subset G'_n$.

We now prove (16). Let e be an arbitrary compact set in $\mathfrak{G}$. There exists a natural number $n_0(e)$ such that $e \subset\subset G_n$ for all $n > n_0(e)$. Considering (13) we have the inequality

$$(19)\qquad \int_e |\nabla(\lambda_n \hat{u}_{\theta,n} - u_{\theta,n})|^2 d\sigma \leqslant$$

$$\leqslant 2\pi \left[\lambda_n \mathrm{Re}\,(e^{-2i\theta} A_{1,\theta}(G_n)) - \mathrm{Re}\,(e^{-2i\theta} a_{1,\theta}(G_n))\right]$$

for $n > n_0(e)$. Because of (18) and by applying the theorem on the convergence for sequences of quasiconformal mappings [12], [11,I.3] and the theorem on kernel convergence for conformal mappings we obtain from (19)

$$(20)\qquad \int_e |\nabla(\lambda \hat{u}_\theta - u_\theta)|^2 d\sigma \leqslant$$

$$\leqslant 2\pi \left[\lambda \mathrm{Re}(e^{-2i\theta} A_{1,\theta}(G)) - \mathrm{Re}(e^{-2i\theta} a_{1,\theta}(G))\right] ,$$

for all compact $e \subset \mathfrak{C}\setminus\Gamma$. Therefore, if the equality on the right-hand side in (15) holds, there must be necessarily $\lambda\hat{u}_\theta - u_\theta \equiv 0$ on every compact $e \subset \mathfrak{C}\setminus\Gamma$ bearing in mind the hydrodynamical normalization of g_θ and G_θ near $z=\infty$ and the continuity of G_θ in $\bar{G}$. If on the other hand (16) is valid we conclude that $g_\theta = (1-\lambda)z + \lambda G_\theta$ in G. Therefore we have $\mathrm{Re}(e^{-2i\theta} a_{1,\theta}) = \lambda \mathrm{Re}(e^{-2i\theta} A_{1,\theta})$.

Analogously one can prove the assertion in connection with (17).

4. Sharp estimates for the extreme value s_θ. In the following we use the inequality of Jensen [3,p.150] in the form

$$(21)\qquad \int_{G'} p\, d\sigma \leqslant I\Phi^{-1}\left(\frac{1}{I}\int_{G'} \Phi(p(z))\, d\sigma\right) .$$

Here the equality holds iff $p \equiv$ const. in G', or $\Phi'' > 0$.

Accordingly we have

(22) $$\int_{G'} p \, d\sigma \leq I\Phi^{-1}(C/I)$$

for all p satisfying (6) and by applying the inequality of Schwarz

$$I^2 = \Big(\int_{G'} p\cdot p^{-1} d\sigma\Big)^2 \leq \int_{G'} p\,d\sigma \cdot \int_{G'} p^{-1} d\sigma \leq \int_{G'} p^{-1} d\sigma \cdot I \cdot \Phi^{-1}(C/I) .$$

Thus we obtain

(23) $$\int_{G'} \Big(1-\frac{1}{p}\Big) d\sigma \leq I\cdot\Big(1 - \frac{1}{\Phi^{-1}(C/I)}\Big)$$

for all p satisfying (6) . Equality in (22) and (23) holds iff $p \equiv \Phi^{-1}(C/I) = \text{const.}$ in G' and $\Phi'' > 0$. From this in connection with Lemma 1 , (15), we have except for the assertion on the equality the following sharpend form of (31) in [8] .

Theorem 1. If G and Φ satisfy the assumptions under 2., then for all θ , $0 \leq \theta \leq \pi$,

(24) $$\hat{a}\hat{\lambda} := \frac{\hat{a}I(1 - 1/\Phi^{-1}(C/I))}{2\pi\hat{a} - I(1-1/\Phi^{-1}(C/I))} \leq s_\theta \leq \frac{a\, I(\Phi^{-1}(C/I)-1)}{2\pi a + I(\Phi^{-1}(C/I)-1)} =: a\lambda ,$$

where $a := \mathrm{Re}(e^{-2i\theta} A_{1,\theta})$, $\hat{a} := \mathrm{Re}(e^{-2i(\theta+\pi/2)} A_{1,\theta+\pi/2})$.
If in addition $\Phi'' > 0$ and the extremal problem is solvable, then the equalities on the right and left in (24) always hold simultaneously. This is the case iff

(25) $$\alpha G_\theta(z) + \beta G_{\theta+\pi/2} \equiv z \quad \text{in} \quad \overline{G} ,$$

where $\alpha = \lambda/(\lambda + \hat{\lambda})$, $\beta = \hat{\lambda}/(\lambda + \hat{\lambda})$. The extremal function g_θ with the representation

(26) $$g_\theta = \begin{cases} (1-\lambda)z + \lambda G_\theta(z) , & z \in G \\ \Big(1-\frac{\lambda-\hat{\lambda}}{2}\Big)z + \frac{\lambda+\hat{\lambda}}{2} e^{2i\theta}\bar{z} - I_{\partial}\hat{\lambda}e^{i\theta} - iR_{\partial}\lambda e^{i\theta} , & \end{cases}$$

z within Γ_ν , $\nu=1,2,\dots,n$,

where $R_\nu := \mathrm{Re}(ie^{-i\theta} G_\theta(\Gamma_\nu))$, $I_\nu := \mathrm{Im}(ie^{-i\theta} G_{\theta+\pi/2}(\Gamma_\nu))$, $\nu=1,2,\dots,n$, is uniquely determined.

Proof. If the equality on the right (left) in (24) holds, then by Lemma we have necessarily (16) ((17)) and because of (22) ((23)) $p \equiv \Phi^{-1}(C/I) = \text{const.}$ in G'. From (16) ((17)) in connection with (1) we conclude, that $g_\theta \in A_\Phi$ is an affine mapping within Γ_ν of the form

$$\frac{1}{2}(1-\lambda)(1+p)(z+qe^{2i\theta}\bar{z}) + C_\nu \quad ,$$

with $q := (p-1)/(p+1)$, C_ν constant , $\nu=1,2,\dots,n$. Therefore (25) follows from [6, Theorem 1, p.237]. Because of [4, Theorem 2 and 3] (25) is valid iff the equality on the right and left in (24) holds simultaneously. From (16) and (17) we obtain (26). Conversly, one can prove that $g_\theta \in A_\Phi$ is represented by (26) as in [6] by considering (25).

In the case Φ is linear, the class of domains G for which the equality in (24) holds is wider then in the strict convex case of Φ. A complete geometrical characterization of those domains is given in

Theorem 2. Let G be a domain bounded by analytic closed Jordan curves, $\Phi(p) \equiv p$.

I. The extremal problem (7) is solvable and the extreme value

$s_\theta = \dfrac{a\,(C-I)}{2\pi a + C - I} =: a\cdot\lambda$ (the upper bound in (24)), where $a := \mathrm{Re}(e^{-2i\theta} A_{1,\theta})$ iff the following three conditions are fulfilled :

(i) there is no tangent on Γ subtending the angle $\theta+\pi/2$ with the positive real axis except for those points on Γ

which correspond to the end points of boundary segments of the ϑ - parallel slit domain $G_\vartheta(G)$;

(ii) for every pair of points z', $z'' \in \Gamma_\vartheta$ satisfying $Re(e^{-i\theta}(z' - z'')) = 0$ and every $\vartheta = 1,2,\ldots,n$

$G_\theta(z') = G_\theta(z'')$;

(iii) in all exceptional points under (i) Γ has a non-vanishing curvature .

The extremal function is uniquely determined and has the representation

$$(27) \qquad g_\theta = \begin{cases} (1-\lambda)z + \lambda G_\vartheta(z) , & z \in G \\ (1-\lambda)z + \lambda G_\theta(z') , & z \in G' . \end{cases}$$

Here z' is one of the two points of intersection of the line through $z \in G'$ subtending the angle $\theta + \pi/2$ with the positive real axis and the closed curve Γ_ϑ containing z inside.

II. The extremal problem (7) is solvable and the extreme value

$s_\vartheta = \frac{\hat{a}\, I(O - I)}{2\pi \hat{a} O - I(O-I)} =: \hat{\lambda}\,\hat{a}$ (the lower bound in (24)) ,

where $\hat{a} := Re(e^{-2i(\theta + \pi/2)} A_{1,\theta+\pi/2})$, iff G fulfils (25). The uniquely determined extremal function is given by (26). The constants λ, $\hat{\lambda}$ are given under I. and II.

Remark 1. Domains G satisfying (i), (ii) and (iii) are for instance those with the property (25) or analytic bounded domains G fulfilling (i) and (iii), which are symmetric with respect to an arbitrary fixed line subtending the angle θ with the positive real axis and intersecting every closed curve Γ_ϑ , $\vartheta = 1,2,\ldots,n$.

Remark 2. Now by considering domains bounded by piecewise analytic closed Jordan curves with the property of symmetry as noted in Remark 1 , Theorem 2,I is also valid in the case of analytic corners (exterior angle γ, $\pi < \gamma < 2\pi$) corresponding to the end points of the straight lines of the θ - parallel slit domain $G_\theta(G)$.

Proof of Theorem 2. I. Let $g_\theta \in A_p$ be an extremal function for which a_θ is equal to the upper bound in (24). Because of $p \equiv 1$ in G and Lemma 1, (17), we have for g_θ the following representation

(28) $\quad g_\theta = (1-\lambda)z + \lambda G_\theta(z) \quad$ in G

and

$U := Re(ie^{-i\theta} g_\theta) = (1-\lambda) Re(ie^{-i\theta} z) + \lambda R_\vartheta$, R_ϑ = const. ,

for z within Γ_ϑ , $\vartheta = 1,2,\dots,n$. Because $U + iV := ie^{-i\theta} g_\theta$ satisfies the system (1) for which the corresponding dilatation p realizes the equality in (6) the level lines (U = const.) which are straight lines are necessarily orthogonal to (V = const) in G'. Therefore by considering (28) and the continuity of g_θ we conclude

$V := Im(ie^{-i\theta} g_\theta) = (1-\lambda)Im(ie^{-i\theta} z) + \lambda Im(ie^{-i\theta} G_\theta(z'))$ for $z \in G'$.

Here z' is one of the two points of intersection of the line through z G`subtending the angle $\theta + \pi/2$ with the positive real axis and the closed curve Γ_ϑ containing z inside. Particularly to every z within Γ_ϑ there may be at most two such points of intersection z' and z'' satisfying (ii) from Theorem 2. Those points $z \in \Gamma$ for which $z' = z''$ obviously correspond to the end points of the straight lines of the θ - parallel slit domain $G_\theta(G)$. This yields the representation (27). Evidently g_θ maps rectangles with sides parallel to the axis of the co--ordinate system after a rotation $\zeta := \xi + i\eta = e^{-i\theta} z$ into rectangles with sides parallel to the axes in the image plane. By considering infinitesimal rectangles we obtain for the ratio of the side-lengths

$$(29)\quad p(z) = 1 + \frac{\lambda}{1-\lambda}\frac{d}{d}\frac{Re(e^{-i\theta}G_\theta(z'))}{\xi} = 1 + \frac{\lambda}{1-\lambda}\frac{|G_\theta'(z')|}{\cos(\alpha-\theta)},$$

$$z \in G',$$

whereas the larger side is parallel to the real axis of the rotated co-ordinate system. We denote by α the angle between the tangent on Γ at z' and the positive real axis. Obviously $p(z)$ is the dilatation of the mapping g_θ in G'. From (29) we conclude necessarily (i), otherwise $p(z)$ would not be bounded. Taking into account that every exceptional point z_0 is a simple zero of $G_\theta'(z)$, we deduce from (29) and $p < \infty$

$$\lim_{z\to z_0} p(z') = \lim_{z\to z_0} 1 + \frac{|G_\theta'(z')|}{\cos(\alpha-\theta)} = 1 + \left|\frac{C_0}{k}\right| < \infty ,$$

where $C_0 = \lim_{z\to z_0}\left|\frac{G_\theta'(z)}{z}\right| \neq 0, \infty$ and k denotes the curvature of Γ at z_0. From this (iii) follows.

Conversly, if G fulfils the conditions (i), (ii), (iii) of Theorem 2, one proves as in [9] that for every λ, $0 < \lambda < 1$, g_θ given by (27) is a hydrodynamically normalized quasiconformal mapping of $\mathbb{C}$ onto $\mathbb{C}$. Using (29) and writting $\zeta = \xi + i\eta = e^{-i\theta} z$ we obtain after a short calculation

$$(30)\quad \int_{G'} p\,d\sigma = I + \frac{\lambda}{1-\lambda}\,Re\int_{\zeta(\Gamma)} e^{-i\theta}G_\theta\,d\eta =$$

$$= I + \frac{\lambda}{1-\lambda}\,Re\left(\frac{1}{2i}\int_{\zeta(\Gamma)} e^{-i\theta}\,G_\theta\,d(\zeta-\bar{\zeta})\right) =$$

$$= I + \frac{2\pi\lambda}{1-\lambda}\,Re(e^{-2i\theta}A_{1,\theta}) = I + \frac{2\pi a\lambda}{1-\lambda} ,$$

where $a := \operatorname{Re}(e^{-2i\theta} A_{1,\theta})$. If we choose λ so that the equality in (6) holds, then g_θ shows to be an admissible mapping for which the equality on the right-hand side in (24) holds.

II. The assertion under II. can be proved in the same manner as in Theorem 1.

Remark 3. R. Kühnau [6] proved that in the case $G = (|z|>1)$ and $\lim_{Q\to\infty} \frac{\Phi(Q)}{Q} = 0$ an extremal function can not exist because $s_\theta = \operatorname{Re}(e^{-2i\theta} A_{1,\theta})$. Because of (24) this situation is obviously not possible in the case of an arbitrary domain G and convex Φ.

In the following we illustrate the dependence of the solvability of the extremal problem (7) on the boundary of G in the case $\Phi(p) \equiv p$.

Theorem 3. Let G be the exterior of a square with the center at the origin and the sides of length ℓ parallel to the axes of the coordinate system, $\Phi(p) \equiv p$, $\theta = 0$.

Then extremal function for the problem (7) does not exist and we have

$$(31) \qquad s_0 := \sup \operatorname{Re} a_1 = \frac{a(C - \ell^2)}{2\pi a + C - \ell^2} ,$$

where the supremum is taken over A_p and $a := \operatorname{Re} A_{1,0} = = \frac{\ell^2 \Gamma^4(1/4)}{16\pi^3}$, $\Gamma(.)$: Gamma - function.

Proof. At first we prove (31). Because of (24) for $\Phi(p) \equiv p$ it is sufficient to construct a maximizing sequence $(g_{0,n})$, $g_{0,n} \in A_p$, where $(\operatorname{Re} a_1[g_{0,n}])$ converges to the expression on the right-hand side of (31). Let G_n be the exterior of the piecewise analytic closed Jordan curve given by the equation

$$\left|\frac{2y}{\ell}\right| + \left|\frac{2x}{\ell}\right|^n = 1 \quad , \quad n \geq 2 \ , \quad z = x+iy \in \mathbb{C} \ .$$

The two analytic arcs of $\Gamma_n = \partial G_n$ meet at $z_{1,2} = \pm 1/2$ under the same exterior angle $\gamma = 2(\pi - \arctan n)$. Obviously (G_n) converges to G in the sense of the kernel convergence. Hence $(a_n := \mathrm{Re}\, A_{1,0}(G_n))$ converges to $a := \mathrm{Re}\, A_{1,0}(G)$ and $(I(G_n'))$ to $I(G')$ for $n \to \infty$.

According to Remark 2 and Theorem 2, (27), the mapping $g_{0,n} \in A_p$:

$$g_{0,n} = \begin{cases} (1-\lambda_n)z + \lambda_n G_{0,n}(z) \ , & z \in G_n \\ (1-\lambda_n)z + \lambda_n G_{\theta,n}(z') \ , & z \in G_n' \ , \end{cases}$$

is admissible if λ_n is chosen so that

$$C = \ell^2 - I(G_n') + \int_{G_n'} p_n \, d\sigma = \ell^2 + \frac{\lambda_n}{1-\lambda_n} 2\pi a_n \quad ,$$

which is obtained by using (29) and (30).

Consequently we have

$$\lim_{n\to\infty} \mathrm{Re}\, a_1[g_{0,n}] = \lim_{n\to\infty} \lambda_n \cdot a_n = \frac{a(C - \ell^2)}{2\pi a + C - \ell^2} = s_0 \ .$$

Because of the symmetrical configuration of G evidently $a := A_{1,0} = d^2$. We denote by d the exterior conformal radius of G whose numerical value is $d = \ell \cdot \Gamma^2(1/4)/(4\pi^{3/2}) = \ell \cdot 0{,}59017\ldots$, see [10].

Suppose there exists an extremal function $g_0 \in A_p$. Then according to Theorem 2.I g_0 would have necessarily the representation (27). But from (29) one concludes that the dilatation $p(z)$ of g_0 would be unbounded if $z \in G'$ converges to $z_1 = 1/2$. Moreover, g_0 would be discontinuous along the vertical sides of the square G. Accordingly $g_0 \notin A_p$.

5. Geometrical bounds for the domains of the values a_1 and $w(z_1)$, $w \in A_\Phi$. Let be

$K_\Phi := \{a_1 : w \in A_\Phi\}$, Φ', $\Phi'' > 0$. Because of the fact, that

$$a := \mathrm{Re}(e^{-2i\theta} A_{1,\theta}) = r + \mathrm{Re}(e^{-2i\theta} m) \leq r + |m| ,$$

where $r := (A_{1,0} - A_{1,\pi/2})/2$, $m := (A_{1,0} + A_{1,\pi/2})/2$, and that the upper and lower bound of s_θ in (24) increases and decreases by increasing a and $\hat{a}$ respectively we obtain the following

Corollary 1. The boundary of K_Φ lies within the closed annulus with centre at the origin and the interior and exterior radii

$$R_i := \frac{(r+|m|)\cdot I\cdot(1 - 1/\Phi^{-1}(C/I))}{2\pi(r+|m|) - I(1 - 1/\Phi^{-1}(C/I))} \tag{32}$$

and

$$R_e := \frac{(r+|m|)\cdot I\cdot(\Phi^{-1}(C/I) - 1)}{2\pi(r+|m|) + I(\Phi^{-1}(C/I) - 1)} ,$$

whereas $R_i = R_e$ iff

$$G_0(z) + G_{\pi/2}(z) \equiv 2z \quad \text{for all} \quad z \in \overline{G} .$$

In this case $K_\Phi = (|z| \leq R_i = R_e)$ is a closed disc.

Remark 4. It is well-known that for instance the domain of values a_1 over the class of quasiconformal mappings with a prescribed dilatation bound which is a bounded function of a complex variable is always a closed disc.

In the case of the class K_Φ this is in general not true except for the special case of Corollary 1, for instance. The example in Theorem 3 shows that $K_{\Phi=p}$ is i.g. closed.

Remark 5. Because of $\mathrm{Re}(e^{-2i\theta} A_{1,\theta}) \leq r + |m| \leq R^2$, where R is the radius of the smallest circle K which contains Γ and by an argument of monotonicity one can replace $\mathrm{Re}(e^{-2i\theta} A_{1,\theta})$ or $(r + |m|)$ by R^2 in all estimates. After this replacement equality holds in every estimate iff G is the exterior of K.

Applying the square root transformation $\zeta = \sqrt{z - z_1}$, $z_1 \in \mathbb{C}$ fixed, in (15) we obtain by Remark 5

Corollary 2. Put $\Phi(p,z) := (p-1)/|z-z_1|$, $z_1 \in \mathbb{C}$ fixed. Then we have the inequality

$$|w(z_1) - z_1| \leq \frac{2D(z_1) \cdot C}{4\pi D(z_1) + C}, \quad w \in A_\Phi, \tag{33}$$

where $D(z_1) := \max_{z \in \Gamma} |z-z_1|$. In the case G is the exterior of a circle centered at z_1 the exact domain of values $w(z_1)$, $w \in A_\Phi$, is a closed disc given by (33). See also [4].

Remark 6. Analogously to Corollary 2 a reasoning as in [4] enables us to obtain sharp estimates for the functionals of Grunsky and Golusin type by using mean functions Φ adapted to the functional.

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STRESZCZENIE

Praca jest poświęcona istnieniu funkcji ekstremalnej realizującej kresy górny i dolny pewnego funkcjonału na klasie odwzorowań średnio quasikonforemnych, tzn. homeomorfizmów, których dylatacja ma ograniczoną średnią polową.

Szczegółowo rozpatrzono przypadek, kiedy dziedziną odwzorowania jest kwadrat, a różniczka kwadratowa związana z funkcjonałem jest zupełnym kwadratem.

РЕЗЮМЕ

Работа посвящена существованию экстремальной функции, которая дает точную верхнюю или нижнюю грань некоторого функционала в классе отображений квазиконформных в среднем, т.е. гомеоморфизмов, дилатация которых имеет ограниченное среднее по площади. Подробнее рассмотен случай отображений заданных на квадрате и функционала сопряженного с квадратическим дифференциалом, который является полным квадратом.