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On Some Extremal Problem in the Class S of Functions Holomorphic and Univalent in the Unit Disc

O pewnym problemie ekstremalnym w klasie S funkcji holomorficznych jednolistnych w kole jednostkowym

Об одной экстремальной проблеме для класса S однолистных в единичном круге функций

1. Introduction. The investigations taken up in the present paper aim at the obtaining of an estimate of the functional

$$H(f) = \left|a_2^{\,2}\,(a_3 - \alpha a_2^{\,2})\right| \quad , \quad \alpha \in R \quad , \tag{1}$$

considered in the well-known class S of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

holomorphic and univalent in the unit disk Δ.

As one knows, each of the factors $|a_2|$ and $|a_3 - \alpha a_2^{\,2}|$ occuring in (1) was an ealier the object of investigations in various classes of holomorphic functions. This rich literature devoted to the estimation of these functionals also contains results obtained in the class S. They are the classical theorems of Bieberbach, Fekete and Szegö, Bazilevich, Goluzin, Jenkis; to this series also belongs the estimate of the functional $|a_3 - \alpha a_2^2|$, $\alpha \in C$, obtained by Szwankowski [6], generalizing previous result.

The reasons for which one seeks an estimate of the functional $H(f)$ in the class S, as well as in other classes of functions, are analogous to the case of the functional $|a_3 - \alpha a_2^2|$ -namely, expressions of type (1) occur in relations between suitable coefficients of series (2) of functions of the same class (cf. [2] [4], [7]) or in relations between such coefficients of functions of different classes, suitably connected with one another (cf.[4]). Such a situation gives the possibility of using the estimate of functional (1) for estimations of other functionals depending on the coefficients of series (2).

Let us pay attention to one more aspect of the investigations of functional (1). As known, the factor $|a_2^2|$ is maximized by the Koebe functions, while the other factor $|a_3 - \alpha a_2^2|$, when $\alpha \in (0;1)$, attains its maximum for functions which are not Koebe ones. So, the question arises whether there exist $\alpha \in$ $\in (0;1)$ with which the extremal functions for (1) are the Koebe functions.

2. Discussion of the form of the equation for extremal functions.

Let us consider the functional

$$H^*(f) = \operatorname{re}\left[a_2^2(a_3 - \alpha a_2^2)\right] \tag{3}$$

defined in the class S, where $\alpha \in R$. The family S is compact, whereas functional (3) is continuous, thus, for each $\alpha \in R$, there exists a function $f_\alpha \in S$ for which $H^*(f_\alpha) = \max_{f \in S} H^*(f)$. In the sequel, the functions $f = f_\alpha$ will be called extremal.

Note that from the well-known estimates of the functionals $H(f) = |a_2|$ and $H(f) = |a_3 - \alpha a_2^2|$ in the class S it follows that, for $\alpha \in R \setminus (0;1)$, the extremal functions for functional (3) are the Koebe functions. Hence it is sufficient that our investigations be carried out for $\alpha \in (0;1)$.

Let us next observe that none of the functions of the class S, whose coefficient a_2 equals zero, is extremal; therefore we shall further assume that $a_2 \neq 0$. At the same time, this assumption guarantees that, for the extremal functions f, we have $\operatorname{grad} H^*(f) \neq 0$ (cf. [3]).

Consequently, the functional under considerations satisfies the assumptions of the Schaeffer-Spencer theorem [5], hence each extremal function fulfils the following equation:

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{lf(z) + k}{(f(z))^2} = \frac{\bar{k}z^4 + \bar{l}z^3 + 2B_0 z^2 + lz + k}{z^2}, \quad z \in \Delta \tag{4}$$

where

$$B_0 = a_2^2(a_3 - \alpha a_2^2), \tag{5}$$

$$l = a_2\left[a_3 + (1 - 2\alpha)a_2^2\right], \tag{6}$$

$$k = \frac{1}{2} a_2^2. \tag{7}$$

Besides, it is known [5] that $B_0 > 0$, and that the right-hand side of (4) is nonnegative on the circle $|z| = 1$ and possesses at least one double root z_0 such that $|z_0| = 1$.

It is evident that (4) is a differential-functional equation. The determination of the upper bound of functional (3) for an arbitrarily fixed $\alpha \in (0;1)$ is therefore reduced to that of finding suitable functions which satisfy this equation. It is worth recalling that the fulfilment of equation (4) by a function is only a necessary condition for this function to be extremal for the functional being examined.

For $z \in \Delta$, $z \neq 0$, let us put

(8) $$N(z) = (\bar{k}z^4 + \bar{l}z^3 + 2B_0 z^2 + lz + k)/z^2 ,$$

(9) $$M(w) = (lw + k)/w^2 , \quad w = f(z) .$$

Since $N(z)$ possesses at least one double root and is nonnegative on the circle $|z| = 1$, therefore function (8) is factorized in the following way:

(10) $$N(z) = \bar{k}(z-e^{i\psi})^2(z^2-tze^{-i\varphi}+e^{-2i\varphi})/z^2$$

where $\varphi, \psi \in (-\pi;\pi\rangle$, $t \geq 2$.

Further, note that if the function $f(z)$ is extremal with respect to the functional considered, then also the functions $-f(-z)$ and $\overline{f(\bar{z})}$ are extremal. Hence it appears that, in our further considerations, it is enough to assume that $\psi \in \langle 0;\pi/2\rangle$ (cf. [6]).

The discussion about the shape of equation (4) according to the type of the factorization of function (10) and the cases

$l \neq 0$ or $l = 0$ in (9) leads to only four possible forms of this equation, namely:

$$\text{(a)} \qquad \left[\frac{zf'(z)}{f(z)}\right]^2 \frac{lf(z)+k}{(f(z))^2} = \bar{k}\,\frac{(z-z_0)^2(z-z_1)(z-z_2)}{z^2}\,, \quad l\neq 0\,,$$

or

$$\text{(b)} \qquad \left[\frac{zf'(z)}{f(z)}\right]^2 \frac{lf(z)+k}{(f(z))^2} = \bar{k}\,\frac{(z-z_0)^2(z-z_3)^2}{z^2}\,, \quad l\neq 0\,,$$

or

$$\text{(c)} \qquad \left[\frac{zf'(z)}{f(z)}\right]^2 \frac{lf(z)+k}{(f(z))^2} = \bar{k}\,\frac{(z-z_0)^4}{z^2}\,, \quad l\neq 0\,,$$

or

$$\text{(d)} \qquad \left[\frac{zf'(z)}{f(z)}\right]^2 \frac{k}{(f(z))^2} = \bar{k}\,\frac{(z-z_0)^2(z-z_3)^2}{z^2}\,, \quad l=0\,,$$

where $z_0 = e^{i\psi}$, $z_1 = \rho e^{i\varphi}$, $z_2 = 1/\bar{z}_1$, $\rho \in (0;1)$, $\psi \in \langle 0;\pi/2\rangle$, $\varphi \in (-\pi;\pi\rangle$, $|z_3| = |z_0| = 1$, $z_3 \neq z_0$.

Sections 3,4,5,6 of the paper will be devoted to the investigation of solutions of equations (a), (b), (c), (d), respectively. The main result will be inserted in section 7.

3. Equation of form (a). Let us first consider the case when equation (4) is of form (a). After a transformation we have

$$\text{(11)} \qquad \left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1 - f(z)/w}{(f(z))^2} = \frac{(1-\bar{z}_0 z)^2(1-z/z_1)(1-z/z_2)}{z^2}$$

where

(12) $$\tilde{w} = -k/l \ .$$

Denote

(13) $$k/\bar{k} = e^{-2i\gamma} \ , \qquad \gamma \in R \ .$$

Comparing (4) and (11) and taking account of (13), we obtain

(14) $$z_1 = \rho e^{-i\gamma} \bar{z}_0 \ , \quad z_2 = \frac{1}{\rho} e^{-i\gamma} \bar{z}_0 \ , \quad \rho \in (0;1) \ ,$$

(15) $$2z_0 + z_1 + z_2 = -\bar{l}/\bar{k} \ ,$$

(16) $$z_0^2 + 2z_0(z_1+z_2) + z_1 z_2 = 2B_0/\bar{k} \ .$$

Integrating (11) (cf. [6]), and next, expanding the function $f(z)$ inaa series in a neighbourhood of $z = 0$ and comparing the coefficients at equal powers of z, in view of (6), (7), (12) and (15), we have

(17) $$4(a_3 - \alpha a_2^2) = e^{2i\gamma} z_0^2 + 2(\rho + \frac{1}{\rho})e^{i\gamma} + \bar{z}_0^2$$

and

(18) $$\log \frac{2 + (\rho + \frac{1}{\rho})e^{i\gamma} z_0^2}{(\frac{1}{\rho} - \rho)e^{i\gamma} z_0^2} + \frac{\rho + \frac{1}{\rho} + 2e^{i\gamma} z_0^2}{2 + (\rho + \frac{1}{\rho})e^{i\gamma} z_0^2} \log\frac{1-\rho}{1+\rho} =$$

$$= \frac{2(a_2 + 2\bar{z}_0)z_0}{2 + (\rho + \frac{1}{\rho})e^{i\gamma} z_0^2} \ .$$

Relation (17) can also be obtained directly from (16) in view of (5), (7), (13) and (14). From (15) and (17) we can determine a_2 depending on α, ρ, γ and z_0. Taking this dependence into account in (18), we find that, for α real, equation (16) is true if and only if

(19) $$e^{i\gamma} z_0^2 = \pm 1 .$$

From (19) and (18) we have

(20) $$a_2 = -2\bar{z}_0 ,$$

whence, in consequence,

(21) $$a_3 = 3\bar{z}_0^2 .$$

From (17), in view of (19), (20) and (21), we next get

$$2(3 - 4\alpha) = 1 + \rho + \frac{1}{\rho} \quad \text{or} \quad 2(3 - 4\alpha) = 1 - (\rho + \frac{1}{\rho}) .$$

Since $\rho \in (0;1)$, we get the following conditions:

$$\alpha < 3/8 \quad \text{or} \quad \alpha > 7/8 .$$

Summing up, we have obtained

(22) $$a_2^2(a_3-a_2^2) = \begin{cases} 4(3-4\alpha) & \text{for } 0 < \alpha < 3/8 , \\ 4(4\alpha-3) & \text{for } 7/8 < \alpha < 1 . \end{cases}$$

We have thus proved

Lemma 1. If, for $\alpha \in (0;3/8) \cup (7/8;1)$, the extremal function $f(z)$ satisfies equation (a), then it is of the form

$$f(z) = z/(1+\bar{z}_0 z)^2 \quad , \quad |z_0| = 1 \quad ,$$

and the maximum of functional (3) is expressed by formula (22). What is more, for $\alpha \in \langle 3/8;7/8 \rangle$, the extremal function does not satisfy equation (a).

4. Equation of form (b). Let us next consider the case when equation (4) is of form (b). After some transformations we get

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1-\frac{1}{\tilde{w}}f(z)}{(f(z))^2} = \frac{(1-\bar{z}_0 z)^2(1-\bar{z}_3 z)^2}{z^2} \quad , \tag{23}$$

$$|z_0| = |z_3| = 1 \; , \quad z_0 \neq z_3 \quad .$$

Comparing (4) and (23), we obtain, among other things, the relation

$$\bar{l}^2/\bar{k} = l^2/k \quad . \tag{24}$$

From (24), taking account of formulae (6) and (7), we have

$$\mathrm{im}\left[a_3 + (1-2\alpha)a_2^2\right] = 0 \tag{25}$$

or

$$\mathrm{re}\left[a_3 + (1-2\alpha)a_2^2\right] = 0 \quad . \tag{26}$$

Let us also notice that the condition $B_0 > 0$ implies

$$\text{(27)} \qquad \text{im}\left[a_2^2(a_3 - \alpha a_2^2)\right] = 0 \quad .$$

Making use of (27), one can prove that relation (26) is not possible.

From (25) and (27) it follows that, for the extremal function satisfying equation (b), only two conditions

$$\text{(28)} \qquad \text{im}\ a_2^2 = 0$$

or

$$\text{(29)} \qquad \text{re}\ (a_3 - a_2^2) = 0$$

are possible.

Denote by ψ_1, $\psi_1 \in (0;\pi/2)$, the only solution of the equation

$$8\cos^2\psi(1 - \log\cos\psi) - 2\cos^2\psi - 1 = 0 \quad .$$

After integrating equation (23) (cf. [6]) and making use of the fact that there exists an $x \in R$ for which $f(e^{ix}) = \tilde{w}$ we obtain, respectively, in cases (28) and (29):

$$\text{(30)} \qquad a_2^2(a_3-\alpha a_2^2) = 2\cos^2\psi(1+2\cos^2\psi)(1-\log\cos\psi)^2$$

where $\psi = \psi(\alpha)$ is the inverse function of

$$\text{(31)} \qquad \alpha = 1 + \frac{1+2\cos^2\psi - 8\cos^2\psi(1-\log\cos\psi)}{8\cos^2\psi(1-\log\cos\psi)^2} \quad , \qquad \psi \in (0;\psi_1),$$

and

$$(32)\qquad a_2^{\,2}(a_3 - \alpha a_2^{\,2}) = \frac{1}{1-\alpha}\left(\frac{1}{2} + e^{\frac{1-2\alpha}{1-\alpha}}\right)^2$$

where α satisfies the inequality

$$(33)\qquad e^{\frac{1-2\alpha}{1-\alpha}} < \frac{1-\alpha}{2\alpha} .$$

Besides, we find that function (31) is increasing, whereas the set of its values is the interval $(3/8;1)$.

We have thus proved

Lemma 2. If, for $\alpha \in (3/8;1)$, the extremal function satisfies equation (b) and condition (28), then the maximum of functional (3) is expressed by the formula (30); in case (28), for $\alpha \in (0;3/8)$, there is no extremal function satisfying equation (b). Whereas if, for a given α, (33) holds and the extremal function satisfies equation (b) and condition (29), then the maximum of functional (3) is expressed by formula (32); in case (29), for α not satisfying (33), there is no extremal function being a solution of equation (b).

5. Equation of form (c). Equation (c) is represented, after some transformations, in the following equivalent form:

$$(34)\qquad \left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1 - \frac{1}{\tilde{w}} f(z)}{(f(z))^2} = \frac{\bar{k}}{k}\,\frac{(z - z_0)^4}{z^2} .$$

Integrating (34), we shall get

$$(35)\qquad \frac{\sqrt{1-4\bar{z}_0 w}}{w} + 2\bar{z}_0 \log \frac{-4\bar{z}_0 w}{z(1 + \sqrt{1-4\bar{z}_0 w})^2} = \frac{1}{z} - \bar{z}_0^{\,2} z + C$$

where C is a constant.

From the comparison of (34) and (4) and from (35) we have

(36) $$(1-\alpha)z_0^2a_2^2 + 2z_0a_2 + \frac{3}{2} = 0 \ .$$

Next, using in (35) the fact that there exists an $x \in R$ such that $f(e^{ix}) = w$, we shall obtain

(37) $$a_2 = -2\bar{z}_0 \ .$$

From (36) and (37) it follows that $\alpha = 3/8$. Consequently, we have proved

Lemma 3. If, for $\alpha = 3/8$, the extremal function f satisfies the equation of form (c), then

(38) $$H^*(f) = 6 \ .$$

For $\alpha \neq 3/8$, the extremal function does not satisfy equation (c).

6. Equation of form (d). Let us finally consider the equation of form (d). After transforming it we shall get

(39) $$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1}{(f(z))^2} = \bar{z}_0^{\,4} \frac{(z^2 - z_0^2)^2}{z^2} \ .$$

After integrating (39) we have

(40) $$\frac{1}{f(z)} = \frac{1}{z} + \bar{z}_0^{\,2}z + C$$

where C is a constant. From the condition $l = 0$ and from (40) it follows that

(41) $$a_3 = (2\alpha - 1)a_2^2$$

and

(42) $$|a_2|^2 = \frac{1}{2(1-\alpha)} .$$

From (41) and (42) we next have

(43) $$H^*(f) = 1/\,4(1-\alpha) .$$

Since $|a_2| \leqslant 2$ in the class S, therefore (42) implies the inequality $\alpha \leqslant 7/8$. So, we have proved

Lemma 4. If, for $\alpha \in (0;7/8\rangle$, the extremal function satisfies the equation of form (d), then the maximum of functional (3) is expressed by formula (43). For $\alpha \in (7/8;1)$, the extremal function does not satisfy equation (d).

7. The main theorem. Basing ourselves on the previous considerations, we shall prove

Theorem. For any function $f \in S$,

(44) $$|a_2^2(a_3 - \alpha a_2^2)| \leqslant \begin{cases} 4(3-4\alpha) & \text{for } \alpha \leqslant 3/8 \\ 2\cos^2\psi(1+2\cos^2\psi)(1-\log\cos\psi)^2 & \text{for } 3/8 \leqslant \alpha \leqslant \alpha_0 , \end{cases}$$

(45)

$$|a_2^2(a_3 - \alpha a_2^2)| \leqslant \begin{cases} \\ 1/4(1-\alpha) & \text{for } \alpha_0 \leqslant \alpha \leqslant 7/8 \quad (46) \\ 4(4\alpha - 3) & \text{for } \alpha \geqslant 7/8 \quad (47) \end{cases}$$

where $\psi = \psi(\alpha)$ is the inverse function of the function $\alpha = \alpha(\psi)$ of the form

$$\alpha = 1 + \frac{1+2\cos^2\psi - 8\cos^2\psi(1-\log\cos\psi)}{8\cos^2\psi(1-\log\cos\psi)^2}, \qquad \psi \in \langle 0; \psi_0 \rangle ,$$

ψ_0 being the smallest positive root of the equation

$$(16\cos^4\psi + 8\cos^2\psi)(1 - \log\cos\psi) - (1 + 2\cos^2\psi)^2 - 1 = 0 \ ;$$

moreover, $\alpha_0 = \alpha(\psi_0)$. Estimate (44)-(47) is sharp.

Proof. Note first that, together with the function $f(z)$, also the function $e^{-i\theta} f(e^{i\theta} z)$, $\theta \in R$, belongs to the class S . In consequence, the maximum of the functional $|a_2^2(a_3 - \alpha a_2^2)|$ is identical in this class with that of functional (3). We shall therefore confine ourselves to the latter. Besides, as we have already observed, it is enough to carry out the proof for $\alpha \in (0;1)$.

In all the cases considered below we make use of lemmas 1-4.

Let $0 < \alpha < 3/8$. Then the maximum of functional (3) is expressed by (22) or (43) or (32). However, case (32) is impossible since, for $\alpha \in (0;3/8)$, we have $|a_3 - \alpha a_2^2| > 2e^{-2\alpha/(1-\alpha)} + 1$ (cf. [1]). Next, note that, for $\alpha \in (0;3/8)$, the inequality

$$1/4(1-\alpha) < 4(3-4\alpha)$$

is true. Hence, in the interval $(0;3/8)$, estimate (44) holds true. In view of the continuity of the functional H with respect to the variable α , estimate (44) holds true and is identical with (38) also for $\alpha = 3/8$.

Let $3/8 < \alpha \leqslant 7/8$. Then the maximum of the functional being examined is expressed by (30) or (32) or (43) . Using again the continuity of the functional as well as the inequalities

$$1/4(1-\alpha) \leqslant \frac{1}{1-\alpha}\left(\frac{1}{2} + e^{\frac{1-2\alpha}{1-\alpha}}\right)^2$$

and

$$2\cos^2\psi(1 + 2\cos^2\psi)(1 - \log\cos\psi)^2 <$$
$$< \frac{1}{1-\alpha}\left(\frac{1}{2} + e^{\frac{1-2\alpha}{1-\alpha}}\right)^2$$

for $\alpha \in (3/8;7/8)$ and satisfying inequality (33), we obtain that, for those values of α , formula (32) cannot be valid. If we next compare (43) and (30), then we get estimates (45) and (46).

For $7/8 < \alpha < 1$, the maximum of the functional under consideration is expressed by (22) or (30) or (32). From the inequality

$$2\cos^2\psi(1 + 2\cos^2\psi)(1 - \log\cos\psi)^2 < 4(4\alpha - 3) <$$
$$< \frac{1}{1-\alpha}\left(\frac{1}{2} + e^{\frac{1-2\alpha}{1-\alpha}}\right)^2$$

and from the continuity of the functional with respect to α we obtain estimate (47), which completes the proof.

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Summary. In the paper the maximum of the functional $|a_2^2(a_3 - \alpha a_2^2)|$, $\alpha \in R$, is determined in the well-known class S of holomorphic and univalent functions.

STRESZCZENIE

W pracy tej wyznaczono maksimum funkcjonału $|a_2^2(a_3-\alpha a_2^2)|$, $\alpha \in \mathbb{R}$, w dobrze znanej klasie S funkcji holomorficznych jednolistnych.

РЕЗЮМЕ

В этой работе определен максимум функционала $|a_2^2(a_3-\alpha a_2^2)|$, $\alpha \in \mathbb{R}$ в хорошо знакомом классе S голоморфных однолистных функций.