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Department of Mathematics
Dawson College, Montreal

R. FOURNIER

A Growth Theorem for a Class of Convex Functions

Twierdzenie o wzroście dla funkcji wypukłych

Теорема о возрастании для выпуклых функций

INTRODUCTION

Let A denote the class of analytic functions f in the unit disc $E = \{z \mid |z| < 1\}$ with $f(0) = f'(0)-1 = 0$. Also let S, S^*, and C designate the subsets of A containing respectively the univalent, starlike univalent, and convex univalent functions. We also define, for each $t > \frac{1}{2}$,

$$(S^*)_t = \{f \in S^* \mid |\frac{zf'(z)}{f(z)} - t| < t \ , \ z \in E\}$$

$$(C)_t = \{f \in C \mid |\frac{zf''(z)}{f'(z)} + 1 - t| < t \ , \ z \in E\} .$$

The classes $(S^*)_t$ and $(C)_t$ were studied by R. and V. Singh ([6]) and by Ruscheweyh and Singh ([5]).

In this paper we mainly deal with the following problem: "Let H be any of the subsets mentionned above and $f \in H$. Also let

$$u,v \in E \ , \ 0 < |v| < |u| < 1 \ \text{and} \ \arg(f(v)) = \arg(f(u)) . \qquad (1)$$

Wath is a good upper bound for the quotient $\frac{|f(u)|}{|f(v)|}$?". In the case where $H = S$, S^* or C the region

$$\{\frac{f(u)}{f(v)} \mid f \in H\}$$

is well known (see for example [2] and [4]) for each $u,v \in E$ and it follows easily that, under conditions (1),

$$\frac{|f(u)|}{|f(v)|} \le \frac{|u|/(1-|u|)^2}{|v|/(1-|v|)^2} \ \text{if} \ f \in S^*$$

and

$$\frac{|f(u)|}{|f(v)|} \le \frac{|u|/(1-|u|)}{|v|/(1-|v|)} \ \text{if} \ f \in C .$$

Hovever it seems very difficult to obtain the variability region (2) in the case where $H = (S^*)_t$ or $H = (C)_t$. Nevertheless we can prove

<u>THEOREM 1</u>: <u>Let</u> $t > \frac{1}{2}$, $w_t = \frac{1}{t} - 1$ <u>and</u> $f \in (C)_t$. <u>Then, under the conditions (1),</u>

$$\frac{|f(u)|}{|f(v)|} \le \frac{(1+w_t|u|)^{1/w_t} - 1}{(1+w_t|v|)^{1/w_t} - 1} .$$

<u>THEOREM 2</u>: <u>Let</u> $t > \frac{1}{2}$, $w_t = \frac{1}{t} - 1$ <u>and</u> $f \in (S^*)_t$. <u>Then, under the conditions (1),</u>

$$\frac{|f(u)|}{|f(v)|} \le \frac{|u|(1+w_t|u|)^{1/w_t - 1}}{|v|(1+w_t|v|)^{1/w_t - 1}} .$$

In our conclusion we indicate how Theorem 1 can be used to obtain some results on the growth of $\frac{zf'(z)}{f(z)}$ when $f \in (C)_t$.

REMARK ON THEOREMS 1 AND 2

Our proof of the Theorems depends on a "real variable" method known as the Theorem of Kuhn and Tucker (see [3], pages 232-234). We give here a brief account of this method adapted to our needs. Let $P(x,y)$, $Q(x,y)$, $R_1(x,y)$ and $R_2(x,y)$ be continuously differentiable real functions on some open set $O \subset \mathbb{R}^2$ and let (x^*,y^*) be a relative maximum point for the problem

"Maximise $P(x,y)$ subject to the constraints $Q(x,y) = 0$ and

$R(x,y) = (R_1(x,y),R_2(x,y)) \le 0$" . (3)

We say that the point (x^*,y^*) is a regular point of the constraints $Q(x,y) = 0$ and $R(x,y) \le 0$ if $R_1(x^*,y^*) \ne 0$ and if the vectors $(\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y})$ and $(\frac{\partial R_2}{\partial x}, \frac{\partial R_2}{\partial y})$ evaluated at (x^*,y^*) are linearly independant in $\mathbb{R}^2$. It is then possible to prove the following

THEOREM (Kuhn-Tucker conditions): Let P,Q,R_1,R_2 as above and (x^*,y^*) be a relative maximum point for the problem (3). Then there exist two real numbers λ and μ such that, at the point (x^*,y^*) ,

$$- (\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}) + \lambda(\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}) + \mu(\frac{\partial R_2}{\partial x}, \frac{\partial R_2}{\partial y}) = (0,0)$$

$$- \mu R_2(x^*,y^*) = 0 ,$$

if (x^*,y^*) is a regular point of the given constraints.

PROOF OF THEOREMS 1 AND 2

We first prove Theorem 1. We need the following lemma, essentially due to Ruscheweyh and Singh ([5]):

LEMMA 1: Let $t > \frac{1}{2}$, $w_t = \frac{1}{t} - 1$ and $f \in (C)_t$. Then

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) \geq \frac{(1+w_t|z|)^{1/w_t} - 1}{|z|(1+w_t|z|)^{1/w_t - 1}}, \quad z \in E$$

and the equality is possible only if $f(z)$ is a rotation of $f_t(z) = (1+w_t z)^{1/w_t} - 1$.

PROOF OF LEMMA 1

It was proved in ([5]) that $\frac{zf'(z)}{f(z)}$ is subordinate to $\frac{zf_t'(z)}{f_t(z)}$ if $f \in (C)_t$. It is also known that $\frac{f_t(z)}{zf_t'(z)} = 1 + (1-w_t)g_t(z)$ where

$g_t(z) = -1 + \frac{1-(1+w_t z)^{1-1/w_t}}{(1-w_t)z}$ is a convex univalent (non normalized) function.

Since $g_t(E)$ is convex and symmetrical with respect to the real axis we obtain

$$\min_{\substack{|z|=r<1 \\ f \in (C)_t}} \operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) = \min_{|z|=r} \left(\frac{f_t(z)}{zf_t'(z)}\right) \tag{4}$$

$$= 1 + (1-w_t)\min(g_t(r), g_t(-r))$$

and a simple calculation shows that

$$g_t(-r) > g_t(r) = -1 + \frac{1-(1+w_t r)^{1-1/w_t}}{(1-w_t)r}. \tag{5}$$

The combination of (4) and (5) completes the proof of Lemma 1.

In order to prove Theorem 1 we define, for each $\rho \in (0,1)$ and $\varphi \in [0,2\pi)$,

$$E_{\rho,\varphi} = \{re^{i\theta} \in E \mid 0 < r \le \rho \text{ and } \arg(f(re^{i\theta})) = \arg(f(\rho e^{i\varphi}))\}.$$

Since the function f is convex univalent, it follows that $E_{\rho,\varphi}$ is a Jordan arc intercepting, at a unique point, each circle with center at the origin and radius $\le \rho$. The statement of Theorem 1 is equivalent to

$$re^{i\theta} \in E_{\rho,\varphi} \Rightarrow ((1+w_t r)^{1/w_t} - 1)/|f(re^{i\theta})| \le ((1+w_t\rho)^{1/w_t} - 1)/|f(\rho e^{i\varphi})| \quad (6)$$

and in order to prove (6) it is clearly enough to show that if the maximum of the function

$$P(r,\theta) = \ln\left(\frac{(1+w_t r)^{1/w_t} - 1}{r}\right) - \operatorname{Re} \ln\left(\frac{f(re^{i\theta})}{re^{i\theta}}\right)$$

under the constraints

$$Q(r,\theta) = \operatorname{Im}\left(\ln\left(\frac{f(re^{i\theta})}{f(\rho e^{i\varphi})}\right)\right) = 0$$

and

$$R(r,\theta) = (-r, r-\rho) \le 0$$

is attained at (r^*,θ^*), then $r^* = \rho$ and $\theta^* = \varphi$.

We are going to show that this is indeed the case when $f(z)$ is not a rotation of $f_t(z)$. We remark first that $r^* \neq 0$; otherwise

$$re^{i\theta} \in E_{\rho,\varphi} \Rightarrow ((1+w_t r)^{1/w_t} - 1)/|f(re^{i\theta})| \leq 1$$

$$\Rightarrow (1+w_t r)^{1/w_t} - 1 \leq |f(re^{i\theta})| ,$$

which is possible only if f is a rotation of $f_t(z)$ (see [6]). Moreover the vectors $(\frac{\partial Q}{\partial r}, \frac{\partial Q}{\partial \theta})$ and $(\frac{\partial R_2}{\partial r}, \frac{\partial R_2}{\partial \theta})$ are linearly independant in $\mathbb{R}^2$ because $\frac{\partial R_2}{\partial \theta} = 0$, $\frac{\partial R_2}{\partial r} = 1$ and

$$\frac{\partial Q}{\partial \theta} = \mathrm{Re}\left(r^* e^{i\theta^*} \frac{f'(r^* e^{i\theta^*})}{f(r^* e^{i\theta^*})}\right) > 0 ,$$

since $f \in (C)_t \subset S^*$. It follows that the point (r^*, θ^*) is a regular point of the given constraints. In view of the Kuhn-Tucker conditions, there exist real numbers λ and μ such that, if

$$\xi = r^* e^{i\theta^*} \frac{f'(r^* e^{i\theta^*})}{f(r^* e^{i\theta^*})} ,$$

then

$$\frac{r^*(1+w_t r^*)^{1/w_t - 1}}{(1+w_t r^*)^{1/w_t} - 1} - \mathrm{Re}(\xi) + \lambda\, \mathrm{Im}(\xi) + \mu r^* = 0 , \tag{7}$$

$$\mathrm{Im}(\xi) + \lambda\, \mathrm{Re}(\xi) = 0 , \tag{8}$$

$$\mu(r^* - \rho) = 0. \tag{9}$$

If $\mu = 0$ we obtain from (7) and (8) that $\mathrm{Re}(\frac{1}{\xi}) = \dfrac{(1+w_t r^*)^{1/w_t} - 1}{r^*(1+w_t r^*)^{1/w_t - 1}}$. This is impossible in view of Lemma 1, because f is not a rotation of f_t. Therefore $\mu \neq 0$ and, by (9), $r^* = \rho$. Since $E_{\rho,\varphi}$ intersects the circle $|z| = \rho$ only at the point $\rho e^{i\varphi}$, it must also follow that $\theta^* = \varphi$. This completes the proof of Theorem 1 in the case where the function f is not a rotation of f_t, and the general result follows by continuity. The bound given for the quotient $\frac{|f(u)|}{|f(v)|}$ is sharp, as seen by choosing $f(z) = f_t(z)$ and $0 < v < u < 1$.

The proof of Theorem 2 will be omited; it follows essentialy the pattern given above except that Lemma 1 is replaced by an appropriate result on the growth of $\frac{zf'(z)}{f(z)}$ where $f \in (S^*)_t$.

CONCLUSION

We want to point out two possible applications of Theorem 1 to the classes $(C)_t$ and $(S^*)_t$. Note first that for $\rho \in (0,1)$ and $f \in (C)_t$,

$$\frac{1}{\rho} = \frac{f(z)}{\rho f(z)} = \frac{f(z)}{f(f^{-1}(\rho f(z)))}$$

and by Theorem 1,

$$\frac{1}{\rho} \le \frac{(1+w_t|z|)^{1/w_t} - 1}{(1+w_t|f^{-1}(\rho f(z))|)^{1/w_t} - 1}.$$

This last inequality is equivalent to

$$f \in (C)_t \Rightarrow |f^{-1}(\rho f(z))| \le \frac{(1+\rho[(1+w_t|z|)^{1/w_t} - 1])^{w_t} - 1}{w_t}, \quad z \in E. \tag{10}$$

The statement (10) is crucial in the proof (omitted here) of the sharp inequalities:

COROLLARY 1.1: Let $t \geq 1$, $w_t = \frac{1}{t} - 1$ and $f \in (C)_t$. Then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{1-(1+w_t|z|)^{1/w_t} + |z|(1+w_t|z|)^{1/w_t - 1}}{(1+w_t|z|)^{1/w_t} - 1}, \quad z \in E. \tag{11}$$

COROLLARY 1.2: Let $\frac{1}{2} < t \leq 1$, $w_t = \frac{1}{t} - 1$ and $f \in (C)_t$. Then

$$\left|(1+w_t)\frac{zf''(z)}{f(z)} - 1\right| \leq \frac{1-(1-|z|)(1+w_t|z|)^{1/w_t - 1}}{(1+w_t|z|)^{1/w_t} - 1}, \quad z \in E. \tag{12}$$

Remark finally that (11) implies that $(C)_t \subset (S^*)_1$ if and only if $\frac{1}{2} < t \leq \frac{1}{1+x}$ where x is the unique root in the interval $(-\frac{1}{2}, 0)$ of the equation $(1+2x)(1+x)^{1/x - 1} = 2$. Note also that (12) is a refinement of the well known inclusion $(C)_t \subset (S^*)_t$, in the case where $\frac{1}{2} < t \leq 1$. A special case of (11) and (12), when $t = 1$, was presented in ([1]).

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STRESZCZENIE

Otrzymano dla pewnych podklas S_t^* (C_t) klasy S^* (względnie klasy C) unormowanych funkcji gwiaździstych (wypukłych) oszacowanie stosunku $\frac{f(u)}{f(v)}$, gdzie $0 < |v| < |u| < 1$ oraz $\arg f(v) = \arg f(u)$.

РЕЗЮМЕ

Полученные оценки величины $\left|\frac{f(u)}{f(v)}\right|$, где $0 < |v| < |u| < 1$, $\arg f(v) = \arg f(u)$, для $f \in S_t^*$, C_t (S_t^*, C_t некоторые классы нормированных звездообразных или выпуклых функций).