ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLIV, 9 SECTIO A 1990

Department of Mathematics
University of Helsinki

O. TAMMI

The Maximum of $|a_3| + \lambda|a_2|$ for Bounded Univalent Functions

Maksimum wyrażenia $|a_3| + \lambda|a_2|$ dla funkcji jednolistnych ograniczonych

Abstract. In the class $S(b)$ of bounded univalent functions the coefficient body (a_2, a_3) is thoroughly analyzed. This allows estimating $|a_3|$ in terms of $|a_2|$. Hence, instead of the classical linear combination $|a_3 + \lambda a_2|$ one is able to maximize also $|a_3| + \lambda|a_2|$. This slight modification appears to give rise to involved estimations which for certain b-intervals remain necessarily computer based. Moreover, strong tangential effects exist, yielding some endpoints needed with unsatisfactory accuracy.

1. Introduction. Consider the class

$$S(b) = \{f \mid f(z) = b(z + a_2 z^2 + \ldots), |f(z)| < 1, \ 0 < b < 1\}$$

of bounded normalized univalent functions analytic in the unit disc $U: |z| < 1$. The leading coefficient b, constant in $S(b)$, characterizes the class. The limit process $b \to 0$ allows a uniform approximation of

$$S = \{F \mid F(z) = z + a_2 z^2 + \ldots\},$$

the class of not necessarily bounded normalized univalent functions. Thus, in this sense,

$$S = S(0).$$

In $S(b)$ the coefficient problems are essentially harder than those in S. This is mainly due to the fact that in $S(b)$ extremal functions usually vary with the index n depending, of course, on the problem and the value of b in question. Already the first indexes may offer quite involved estimations, as can be seen in what follows.

There exist certain traditional functionals which have been used in testing the knowledge available. The "founding" one is $|a_3 + \lambda a_2^2|$ which is maximized in $S(b)$ for real λ, e.g. in [3] and for complex λ in [2]. More recently use is made of the functional $|a_3 + \lambda a_2|$ which in $S(b)$ was studied in [5]. The aim of the present paper is to discuss such a test case which needs the complete characterization of the first non-trivial coefficient body (a_2, a_3) in $S(b)$. In [6] first tries in this direction were made when estimating $|a_3|$ in terms of $|a_2|$. It appears that the functional $|a_3| + \lambda|a_2|$ serves us well. The maximizing of it requires, indeed, all the facts available for (a_2, a_3) and is just on the limit of solvability. Altogether, the distance between the present and previously mentioned functionals seems to be large enough to be publicized.

If the maximum of $|a_3|$ in $|a_2|$ is available the same holds also for the maximum of $|a_3| + \lambda|a_2|$. This is finally to be maximized in the variable left i.e. in $x = |a_2| \in [0, 2(1-b)]$.

Let us start by mentioning those basic facts of the coefficient body (a_2, a_3) which yield the estimation of $|a_3|$ to be needed. It appears that this estimation is straightforward save in the interval $0.5 < b < e^{-1/2}$. There the most complicated part of the coefficient body, with nonsymmetric boundary functions, is involved and necessitates computer based comparisons.

2. The boundary of the coefficient body (a_2, a_3). In [4], [5] and [6] the coefficient body (a_2, a_3) was normalized by rotation

$$\tau^{-1} f(\tau z), \quad \tau = e^{iv},$$

so that $a_2 = |a_2| \geq 0$. Thus, it is located in the upper half of the space (X, Y, Z) with $X = \operatorname{Re} a_3$, $Y = \operatorname{Im} a_3$, $Z = a_2$. The plane $Z = a_2 = constant$ yields the intersection $N(a_2)$, the boundary of which can be presented by aid of three types of arcs to be called I, II and III. Let us consider these arcs more closely.

In I the boundary function f is of the type 2:2. This means that $f(U)$ is a slit domain where the slit system has 2 starting points and 2 endpoints. The corresponding notation will be applied for other extremal functions and domains, too.

We may parametrize the points of I by using the rotation angle v. According to [5], p. 11, we can summarize the result as follows.

Summary 1. *The boundary points* I $\subset \partial N(a_2)$ *are connected with functions* 2:2 *with two unequal diametral radial slits.* I *is a circular arc:*

$$\left|a_3 - \left(1 + \frac{1}{2\ln b}\right)|a_2|^2\right| = R = 1 - b^2 + \left(1 + \frac{1}{2\ln b}\right)|a_2|^2 \quad (\geq 0).$$

The points of this can be located by using v as follows:

$$\begin{cases} \dfrac{\pi}{2} \leq v \leq v_0 = \begin{cases} \pi, & |a_2| \leq 2b|\ln b|; \\ \arccos \dfrac{2b\ln b}{|a_2|}, & |a_2| \geq 2b|\ln b|, \end{cases} \\ \operatorname{Re} a_3 = \left(1 + \dfrac{1}{2\ln b}\right)|a_2|^2 + R\cos 2v, \\ \operatorname{Im} a_3 = -R\sin 2v. \end{cases}$$

Thus, I *is a whole circle for $|a_2| \leq 2b|\ln b|$ and a part of a circle for $|a_2| > 2b|\ln b|$.*

The gap left in $I \subset \partial N(a_2)$ is filled by a more complicated arc II ([5], p. 19):

Summary 2. *The boundary points on the arc* II $\subset \partial N(a_2)$ *belong to the functions* 1:2 *with a forked slit. The points of the upper half of* II, *parametrized in* v, *are determined through the formulae*

$$\begin{cases} \overline{\operatorname{arc}}\cos\dfrac{2b\ln b}{|a_2|} = v_0 \le v \le \pi, \\ \sigma\ln\sigma - \sigma + b - \dfrac{|a_2|\cos v}{2} = 0; \quad \sigma = \sigma(a_2) \in [b,1], \\ \operatorname{Re} a_3 = |a_2|^2 + 2\sigma|a_2|\cos v + (1 - b^2 + 2(\sigma - b)^2)\cos 2v, \\ \operatorname{Im} a_3 = -2\sigma|a_2|\sin v - (1 - b^2 + 2(\sigma - b)^2)\sin 2v; \\ E(\sigma) = \sqrt{1-\sigma^2} - \sigma\,\overline{\operatorname{arc}}\cos\sigma - \frac{1}{2}|a_2|\sin v \ge 0. \end{cases}$$

The existence condition $E(\sigma) \ge 0$ ([5], p. 19) *yields the interval* $2b|\ln b| < |a_2| \le |\tilde{a}_2|$ *for which the whole* II $\subset \partial N(a_2)$. *The double root of* $E(b) = 0$ *determines* $|\tilde{a}_2|$:

$$\begin{cases} E(\sigma) = \sqrt{1-\sigma^2} - \sigma\,\overline{\operatorname{arc}}\cos\sigma - (\sigma\ln\sigma - \sigma + b)\dfrac{\ln\sigma}{\overline{\operatorname{arc}}\cos\sigma} = 0, \\ |\tilde{a}_2| = -\dfrac{2(\sigma\ln\sigma - \sigma + b)}{\overline{\operatorname{arc}}\cos\sigma}\sqrt{\ln^2\sigma + (\overline{\operatorname{arc}}\cos\sigma)^2}. \end{cases}$$

For the remaining interval $|\tilde{a}_2| < |a_2| < 2(1-b)$ *there is a gap in* II *(in the upper and lower parts which are symmetric with respect to the* X *axis) which is filled by the arc* III $\subset \partial N(a_2)$).

The final arc III is governed by the results of [5], p. 45. The limiting values v_{01}, v_{02} of the gap in II are obtained from the above existence condition.

Summary 3. *The boundary points on the arc* III $\subset \partial N(a_2)$ *are connected with the functions* 1:1 *with one curved slit. Again, take the upper half of* III, *parametrized in* v.

$$(*)\qquad \begin{cases} v \in\,]v_{01}, v_{02}[; \quad v_{01} \text{ and } v_{02} \text{ satisfy:} \\ E(b) = \sqrt{1-\sigma^2} - \sigma\overline{\operatorname{arc}}\cos\sigma - \dfrac{1}{2}|a_2|\sqrt{1-\cos^2 v} = 0, \\ \cos v = \dfrac{2}{|a_2|}(\sigma\ln\sigma - \sigma + b). \end{cases}$$

The points of III *are expressed in two variables* α, ω, *located in a triangle* T ([5], p. 46). *With the normalization to be stated below for* U *and* V *the triangle* $T \subset$ *the* *first quadrant of the $\alpha\omega$-plane. The connection between α and ω for a given $|a_2|$ can be deduced from $|a_2|^2 = \tilde{U}^2 + \tilde{V}^2$, where*

$$\begin{cases} \tilde{U} = |a_2|\cos v = C_1 \ln \dfrac{\cos\alpha}{\cos\omega} + C_2(\cot\alpha - \cot\omega + \alpha - \omega) \le 0, \\ \tilde{V} = |a_2|\sin v = C_2 \ln \dfrac{\sin\alpha}{\sin\omega} + C_1(\tan\alpha - \tan\omega - \alpha + \omega) \ge 0; \end{cases}$$

$$\begin{cases} C_1 = 2\dfrac{\sin\alpha - b\sin\omega}{\sin(\alpha-\omega)}\cos\alpha\cos\omega, \\ C_2 = 2\dfrac{\cos\alpha - b\cos\omega}{\sin(\alpha-\omega)}\sin\alpha\sin\omega. \end{cases}$$

For the points of III *holds finally:*

$$\begin{cases} \operatorname{Re} a_3 = |a_2|^2 + |a_2|(C_1\cos v + C_2\sin v) + \cos 2v[1 - b^2 + C_1C_2(\tan\alpha - \tan\omega) \\ \qquad - \dfrac{C_2^2}{2}(\sin^{-2}\alpha - \sin^{-2}\omega)], \\ \operatorname{Im} a_3 = |a_2|(C_2\cos v - C_1\sin v) - \sin 2v[1 - b^2 + C_1C_2(\tan\alpha - \tan\omega) \\ \qquad - \dfrac{C_2^2}{2}(\sin^{-2}\alpha - \sin^{-2}\omega)]. \end{cases}$$

With respect to the endpoints of the arc III there are two alternatives (cf. Figure 3) 1° v_{01} belongs to the intersection II $\cap$ III or 2° v_{01} belongs to I $\cap$ III. In the case 1° v_{01} is obtained from (*) and also from the C_1, C_2-formulae for $\omega = 0$. In the case 2° we are on the boundary arc of T where $\cos\alpha = b\cos\omega$ and hence (cf. [5] pp. 33–35):

$$\delta = a_3 - a_2^2 = C_1\tau^{-1}a_2 + \tau^{-2}(1-b^2),\ C_2 = 0,\ C_1 = 2\sigma;$$

$$\sigma = \cos\alpha = b\cos\omega.$$

For the points of I Summary 1 yields

$$\delta = \delta^o + Re^{-i2v},\ \delta^o = \frac{|a_2|}{2\ln b},\ R = 1 - b^2 + \frac{|a_2|^2}{2\ln b}.$$

Hence for τ at the intersection:

$$\sigma = 2\sigma\tau^{-1}|a_2| + \tau^{-2}(1-b^2) = \delta^o + \left(1 - b^2 + \frac{|a_2|^2}{2\ln b}\right)\tau^{-2}$$

$\Rightarrow$

$$2\cos\alpha = 2\sigma = \frac{|a_2|}{2\ln b}(\tau + \tau^{-1});$$

$$\cos v = \frac{2\ln b}{|a_2|}\cos\alpha.$$

Thus we have in the above cases:

Completion of Summary 3. *In the case* II $\cap$ *III* v_{01} *is determined directly from* (*). *In the case* I $\cap$ III *for the endpoint* (α,ω) *and for the corresponding* $v_{01} = v$ *holds*

$$\begin{cases} |a_2|^2 = \bar{Y}^2 + \bar{V}^2, \\ \bar{U} = 2\sigma \ln b, \\ \bar{V} = 2\sigma(\tan\alpha - \tan\omega - \alpha + \omega), \\ \sigma = \cos\alpha = b\cos\omega, \\ \cos v = \dfrac{2b\ln b}{|a_2|}\cos\alpha. \end{cases}$$

3. The sharp estimates of $|a_3|$ in $|a_2|$. According to the analysis in [6], the following sharp upper bounds, connected with I, are valid.

1) $e^{-1/2} < b < 1$

(1) $$|a_3| \le 1 - b^2 - |a_2|^2, \quad 0 \le |a_2| \le 2(1-b).$$

Equality holds for the whole $|a_2|$-interval, at the left diametral point of $N(a_2)$ and the equality function is of the type 2:2 with two unequal radial slits along the same diameter.

2) $0 \le b < e^{-1/2}$

(2) $$|a_3| \le 1 - b^2 + \left(1 + \frac{1}{\ln b}\right)|a_2|^2, \quad 0 \le |a_2| \le 2b|\ln b|.$$

On the above sharpness interval equality is reached for 2:2-mappings with two symmetric curved slits at the right diametral point of $N(a_2)$. The inequality (2) remains to be true, but unsharp, up to the point $2(1-b)$.

The cases 1) and 2) are united to yield (2) at

3) $b = e^{-1/2}$

In this case there is a one parametric family of extremal functions which belong to the points of I. The family starts from unsymmetric radial slit case 2:2 mentioned above and evolves through unsymmetric curved 2:2–cases up to the final one which is either symmetric curved 2:2–case, symmetric or unsymmetric limiting 1:2–case and finally curved 1:1–case with one slit shrinked to a point. All these extremal domain types are schematically presented in Figure 1.

If $0 \le b < e^{-1/2}$ and $|a_2| > 2b|\ln b|$ the arcs II and III yield the following result

$$0 \le b \le 0.5 < e^{-1/2}, \quad |a_2| \ge 2b|\ln b:$$

(3) $$\begin{cases} |a_3| \le |a_2|^2 - 2|a_2|\sigma + 1 - b^2 + 2(\sigma - b)^2, \\ \sigma\ln\sigma - \sigma + b + \frac{1}{2}|a_2| = 0; \quad \sigma = \sigma(|a_2|) \in [b, 1]. \end{cases}$$

The estimation is sharp for $2b|\ln b| \le |a_2| \le 2(1-b)$ and the equality holds at the right diametral point of $N(a_2)$ i.e. for the symmetric 1:2-mapping.

Finally, the interval left, $0.5 < b < e^{-1/2}$, with $|a_2| > 2b|\ln b|$ requires analyzing thoroughly the points of III, by using the formulae in Summary 3. The results are best expressed in connection of the final combination $|a_3| + \lambda|a_2|$ we now turn to maximize.

4. Maximizing $|a_3| + \lambda|a_2|$ by symmetric extremal functions. The maximum of $|a_3|$ for a fixed $|a_2|$ implies similarly maximum for the functional $|a_3| + \lambda|a_2|$, $\lambda \in \mathbf{R}$. This is finally to be maximized in the variable left: $|a_2| = x \in [0, 2(1-b)]$. Let us consider this problem on the intervals found in Section 3.

a) $e^{-1/2} \le b < 1$; F_1

According to (1)

$$(4)\qquad |a_3| + \lambda|a_2| \le 1 - b^2 + \lambda x - x^2 = 1 - b^2 + \frac{\lambda^2}{4} - \left(x - \frac{\lambda}{2}\right)^2 = F_1(x)$$

$$\le 1 - b^2 + \frac{\lambda^2}{4}.$$

Equality in the last estimation is reached for $x = \frac{\lambda}{2} \in [0, 2(1-b)]$ provided

$$0 \le \lambda \le 4(1-b).$$

Both estimations are sharp simultaneously, because $x = |a_2| = \frac{\lambda}{2} \in [0, 2(1-b)]$. If $e^{-1/2} < b < 1$ the extremal function is uniquely of radial 2:2-type with usually unequal slits. At $b = e^{-1/2}$ there hold the following one-parametric extremal families:

$0 \le \lambda \le 2e^{-1/2}$: The extremal function starts from radial 2:2-function proceeding to symmetric curved-slit 2:2-function which, at $\lambda = 2e^{-1/2}$ is the limiting symmetric 1:2-case.

$2e^{-1/2} < \lambda < 4(1 - e^{-1/2})$: Again, the extremal type starts from radial 2:2-function and ends up to unsymmetric limiting 1:2-case which finally is of limiting 2:2-type with one slit shrinked to a point (cf. schematic presentation in Figure 1).

The upper bound $F_1(x) = 1 - b^2 + \lambda x - x^2$ in (4) is maximized at $x = 2(1-b)$ if $\lambda > 4(1-b) \Leftrightarrow \frac{\lambda}{2} > 2(1-b)$. Similarly, if $\lambda < 0$ F_1 is maximized at $x = 0$. The former extremal case is the radial slit-mapping and the latter one the symmetric radial slit-mapping 2:2. In Figure 1 the extremal domains for $e^{-1/2} \le b < 1$, $\lambda \in \mathbf{R}$ are schematically drawn.

b) $0 \le b < e^{-1/2}$; F_2

Next apply the estimation (2) which is valid for the whole $x = |a_2|$-interval $[0, 2(1-b)]$ but sharp only for the interval $[0, 2b|\ln b|]$:

$$(5)\qquad |a_3| + \lambda|a_2| \le 1 - b^2 + \lambda x + \left(1 + \frac{1}{\ln b}\right)x^2 = F_2(x)$$

$$= 1 - b^2 - \frac{\lambda^2}{4\left(1 + \frac{1}{\ln b}\right)} + \left(1 + \frac{1}{\ln b}\right)\left(x + \frac{\lambda}{2\left(1 + \frac{1}{\ln b}\right)}\right)^2.$$

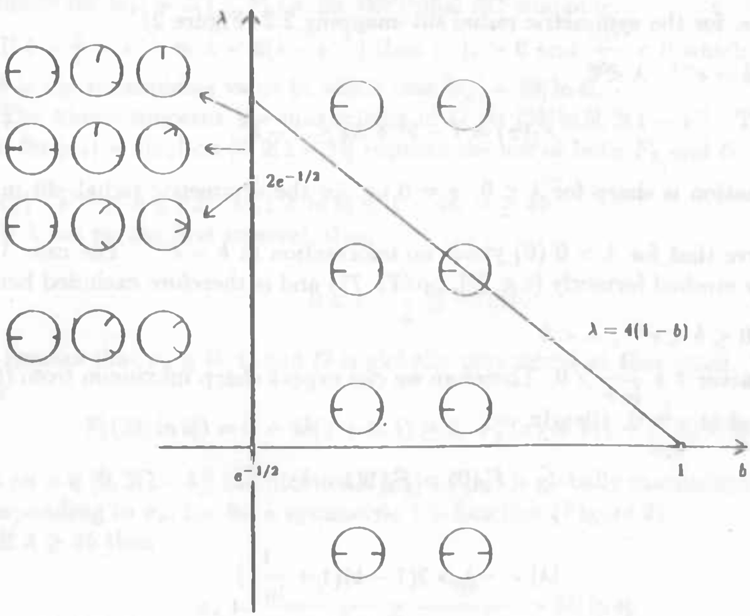

Fig. 1.

Depending on the values of λ we obtain three different cases.

b$_1$) $e^{-1} < b < e^{-1/2}$, $0 < \lambda \le 4b(1 + \ln b)$; $\lambda < 0$

In this interval $(1 + \dfrac{1}{\ln b}) < 0$. Therefore

$$F_2(x) \le 1 - b^2 - \frac{\lambda^2}{4(1 + \frac{1}{\ln b})}$$

with the equality at $x = -\dfrac{\lambda}{2(1 + 1/\ln b)}$. Taking the sharpness interval into consideration we see, that F_2 is sharply estimated on the whole interval $x \in [0, 2(1-b)]$ provided that

$$x_o = -\frac{\lambda}{2(1 + \frac{1}{\ln b})} \le -2b \ln b$$

$\Rightarrow$

(6) $$0 < \lambda \le 4b(1 + \ln b).$$

The extremal mapping is of curved symmetric 2:2–type reducing to a limiting 1:2–type at $\lambda = 4b(1 + \ln b)$ (Figure 2).

If $\lambda < 0$ the number $x_o = -\dfrac{\lambda}{2(1+1/\ln b)} < 0$ which implies that F_2 is maximized at $x = 0$ i.e. for the symmetric radial slit-mapping 2:2 (Figure 2).

b$_2$) $b = e^{-1}$, $\lambda < 0$

$$F_2(x) = 1 - b^2 + \lambda x \leq 1 - b^2.$$

This estimation is sharp for $\lambda < 0$, $x = 0$ i.e. for the symmetric radial-slit mapping 2:2.

Observe that for $\lambda > 0$ (6) yields no information at $b = e^{-1}$. The case $\lambda = 0$ is thoroughly studied formerly (e.g. [5], pp. 71–77) and is therefore excluded here.

b$_3$) $0 \leq b < e^{-1}$, $\lambda < 0$

The factor $1 + \dfrac{1}{\ln b} > 0$. Therefore we can expect sharp maximum from (5) only if $\lambda < 0$ and at $x = 0$. Clearly

$$F_2(0) > F_2(2(1-b))$$

for

$$|\lambda| = -\lambda > 2(1-b)(1 + \frac{1}{\ln b}).$$

Thus, the symmetric 2:2 radial-slit mapping yields the maximum if $|\lambda|$ is big enough. The exact result requires, however, also the information implied by (3).

c) $0 \leq b \leq 0.5 < e^{-1/2}$; F_2, G

From (3) we obtain

$$(7)\quad \begin{cases} |a_3| + \lambda|a_2| \leq 1 - b^2 + 4(\sigma - b - \sigma\ln\sigma)^2 + 2(\lambda - 2\sigma)(\sigma - b - \sigma\ln\sigma) \\ \qquad\qquad + 2(\sigma - b)^2 = G, \\ |a_2| = 2(\sigma - b - \sigma\ln\sigma) \in [2b|\ln b|, 2(1-b)];\ \sigma = \sigma(|a_2|) \in [b, 1]. \end{cases}$$

The connection between $|a_2|$ and σ is one-to-one. Therefore G depends on $|a_2|$ and can be interpreted as a function of σ too, for which

$$\frac{dG}{d\sigma} = 8\ln\sigma(\sigma\ln\sigma + b - \frac{\lambda}{4})_o.$$

If $b - \frac{\lambda}{4} \in [0, e^{-1}]$, then $(\ )_o = 0$ has a root, to be called σ_o, which lies in the interval $[e^{-1}, 1]$. From the sign of $\dfrac{dG}{d\sigma}$ we see that σ_o is at least a locally maximizing point of G for the λ-values for which

$$0 < b - \frac{\lambda}{4} < e^{-1}$$

$\Longleftrightarrow$

$$4(b - e^{-1}) < \lambda < 4b.$$

If $b - \frac{\lambda}{4} \le 0 \Leftrightarrow \lambda \ge 4b$, then $(\ )_o < 0$ and $\frac{dG}{d\sigma} > 0$ and hence $\sigma = 1$ yields the maximum for $|a_2| = 2(1-b)$ i.e. for the radial slit–mapping.

If $b - \frac{\lambda}{4} > e^{-1} \Leftrightarrow \lambda < 4(b - e^{-1})$ then $(\ )_o > 0$ and $\frac{dG}{d\sigma} < 0$ which implies that $\sigma = b$ is the maximizing value in which case $|a_2| = 2b|\ln b|$.

The above concerns the maximizing of G on $[2b|\ln b|, 2(1-b)]$. The complete result for $|a_3| + \lambda|a_2|$ on $[0, 2(1-b)]$ requires the use of both F_2 and G.

c$_1$) $e^{-1} \le b \le 0.5$, $4b(1+\ln b) \le \lambda < 4b$; $\lambda \ge 4b$

If λ lies on the first interval, then

$$0 < b - \frac{\lambda}{4} \le -b \ln b.$$

This implies that $\sigma_o \in [b, 1]$ and G is globally maximized at that point. For F_2 there holds

$$F_2(2b|\ln b|) = \lambda - 4b(1+\ln b) \ge 0, \ F_2''(x) \equiv 2(1 + \frac{1}{\ln b}) < 0.$$

Thus on $x \in [0, 2(1-b)]$ the functional $|a_3| + \lambda|a_2|$ is globally maximized at the point corresponding to σ_o, i.e. for a symmetric 1:2–function (Figure 2).

If $\lambda \ge 4b$ then

$$x_o = \frac{\lambda}{2|1 + \frac{1}{\ln b}|} \ge \frac{2b}{|1 + \frac{1}{\ln b}|} > 2b|\ln b|.$$

Thus F_2 is maximized at $x = 2b|\ln b|$. Because $b - \frac{\lambda}{4} \le 0$ G is maximized at $\sigma_o = 1$. Hence $|a_3| + \lambda|a_2|$ reaches the maximum at $x = 2(1-b)$ i.e. for the radial slit–mapping.

c$_2$) $0 \le b < e^{-1}$, $0 < \lambda < 4b$; $\lambda \ge 4b$; $\lambda < 0$

Now $1 + 1/\ln b > 0$. Take first $0 < \lambda$ which implies $x_o < 0$. Therefore F_2 is maximized at $x = 2b|\ln b|$. For the first λ-interval is $4(b - e^{-1}) < 0 < \lambda < 4b$. Hence G and also $|a_3| + \lambda|a_2|$ is maximized at σ_o i.e. for the symmetric 1:2–function.

If we take $\lambda \ge 4b$ similar reasoning shows that $|a_3| + \lambda|a_2|$ is maximized by the radial slit–mapping.

Let finally $\lambda < 0$. Now $x_o > 0$ and therefore $F_2(0) = 1 - b^2$ might yield the maximum for $|a_3| + \lambda|a_2|$. At least for $4(b - e^{-1}) < \lambda < 0$ there exists $\sigma_o \in [e^{-1}, 1]$ which maximizes G. It maximizes also $|a_3| + \lambda|a_2|$ if

$$G(\sigma_o) \ge F_2(0) = 1 - b^2.$$

Consider the equality case. Because σ_o is the root of $(\ )_o = 0$ we have for it $\lambda = 4(\sigma \ln \sigma + b)$, $G = 1 - b^2 + \lambda(\sigma - \frac{\lambda}{4}) + 2(\sigma - b)^2$. Hence, in the equality case $G(\sigma_o) = F_2(0) \Leftrightarrow \lambda(\sigma - \frac{\lambda}{4}) + 2(\sigma - b)^2 = 0$. For σ_o and λ belonging to the limit case we thus have

$$(8) \qquad \begin{cases} \lambda = 4(\sigma \ln \sigma + b), \\ (\sigma - b)^2 + 2(\sigma \ln \sigma + b)(\sigma - \sigma \ln \sigma - b) = 0. \end{cases}$$

In the following Table there are numerical values determining some points of the limiting curve $\lambda = \bar{\lambda}(b)$

Tab. 1.

b	σ_o	$\lambda = \bar{\lambda}(b)$
e^{-1}	e^{-1}	0
0.35	0.446˙036˙140	−0.040˙438˙564
0.3	0.515˙348˙288	−0.166˙522˙916
0.25	0.558˙798˙102	−0.300˙808˙328
0.2	0.593˙010˙158	−0.439˙495˙008
0.15	0.622˙159˙601	−0.581˙004˙821
0.1	0.648˙045˙493	−0.724˙473˙971
0.05	0.671˙627˙813	−0.869˙368˙333
10^{-2}	0.693˙062˙266	−1.012˙404˙738
10^{-9}	0.693˙485˙183	−1.015˙332˙776

For $\bar{\lambda} < \lambda < 4b$ the symmetric 1:2–mapping is the extremal one. For $\lambda < \bar{\lambda}$ the symmetric 2:2–radial slit–mapping is the maximizing one. On $\lambda = \bar{\lambda}$ itself both of those types hold simultaneously.

In the terminal case $b = 0$ (8) yields for $\sigma \neq 0$:

$$\sigma_o = e^{\frac{1-\sqrt{3}}{2}} = 0.693˙485˙184,$$

$$\bar{\lambda} = -2(\sqrt{3}-1)e^{\frac{1-\sqrt{3}}{2}} = -1.015˙332˙778.$$

The maximal $|a_3| + \lambda|a_2| = 1$ and is attained also by the symmetric 2:2–radial slit–mapping. The results are schematically illustrated in Figure 2.

By the comparison consider the neighboring point $b = 0$, $\lambda = -1$ in which $\max(|a_3| - |a_2|) = G(\sigma_o) = \frac{3}{4} + \sigma_o(2\sigma_o - 1) = 1.029$, where $4\sigma_o \ln \sigma_o + 1 = 0$. This agrees with the result of [1], p. 114.

d) $0.5 < b < e^{-1/2}$; G

Now move on to consider the troublesome interval d) where also unsymmetric extremal domains for $|a_3|$ exist. Here we must rely upon the results of [6], pp. 306–311. According to this (7) remains to hold for the maximum so far as

$$2b|\ln b| \leq |a_2| \leq \bar{\sigma}(b) \tag{9}$$

where $\bar{\sigma}(b)$ is the root of

$$\sigma \ln \sigma - \frac{\sigma}{2} + b = 0. \tag{10}$$

The maximal $|a_3| + \lambda|a_2|$ is found from maximal $G = G(\sigma_o)$, where σ_o is the root of

(11) $$\sigma \ln \sigma + b - \frac{\lambda}{4} = 0.$$

The extremal domain is of symmetric 1:2-type. The largest λ for which the above remains to hold is the smallest λ of next Section. This λ appears to be

$$\lambda = 2\bar{\sigma}(b),$$

as will be seen in Section 5, e).

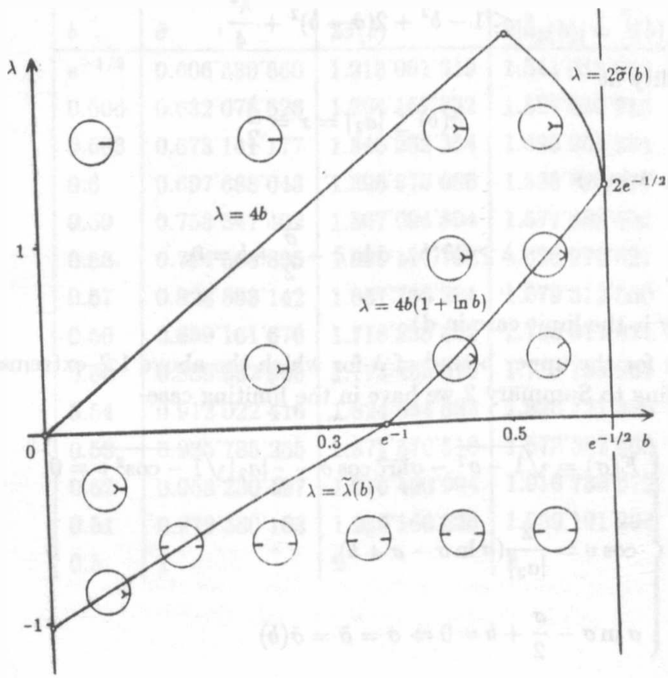

Fig. 2.

5. Maximizing $|a_3| + \lambda|a_2|$ by nonsymmetric extremal functions.

e) $0.5 < b < e^{-1/2}$; $2\bar{\sigma}(b) < \lambda < 2|\tilde{a}_2(b)| = \tilde{\lambda}(b)$

According to [6], p. 310, the smallest $|a_2|$ for which $|a_3|$ is still maximized by non-symmetric 1:2-mappings is

$$|a_2| = \tilde{\sigma}(b).$$

the root of (10). The maximal $|a_3|$ is

$$|a_3| = 1 - b^2 + 2(\tilde{\sigma} - b)^2 - |a_2|^2.$$

Thus

$$\begin{aligned} |a_3| + \lambda |a_2| &\le -x^2 + \lambda x + 1 - b^2 + 2(\tilde{\sigma} - b)^2 = F_3(x) \\ &= 1 - b^2 + 2(\tilde{\sigma} - b)^2 + \frac{\lambda^2}{4} - \left(x - \frac{\lambda}{2}\right)^2 \\ &\le 1 - b^2 + 2(\tilde{\sigma} - b)^2 + \frac{\lambda^2}{4}, \end{aligned}$$

with the equality at

$$\tilde{\sigma}(b) \le |a_2| = x = \frac{\lambda}{2}.$$

This yields for λ in e)

$$\lambda \ge 2\tilde{\sigma}(b), \ \tilde{\sigma} \ln \tilde{\sigma} - \frac{\tilde{\sigma}}{2} + b = 0, \tag{12}$$

where equality is the limit case in d).

Next, ask for the upper bound of λ for which the above 1:2 extremal type still holds. According to Summary 2 we have in the limiting case

$$\begin{cases} E(\sigma) = \sqrt{1 - \sigma^2} - \sigma \overline{\text{arc}} \cos \sigma - \frac{1}{2}|a_2|\sqrt{1 - \cos^2 v} = 0, \\ \cos v = \frac{2}{|a_2|}(\sigma \ln \sigma - \sigma + b), \\ \sigma \ln \sigma - \frac{\sigma}{2} + b = 0 \Rightarrow \sigma = \tilde{\sigma} = \tilde{\sigma}(b) \end{cases}$$

$\Rightarrow$

$$\cos v = -\frac{\sigma}{|a_2|};$$

$$\sqrt{1 - \sigma^2} - \sigma \ \overline{\text{arc}} \cos \sigma - \frac{1}{2}|a_2|\sqrt{1 - \frac{\sigma^2}{|a_2|^2}} = 0,$$

$\Rightarrow$

$$|a_2|^2 = |\tilde{\tilde{a}}_2|^2 = \tilde{\sigma}^2 + (2\sqrt{1 - \tilde{\sigma}^2} - \tilde{\sigma} \overline{\text{arc}} \cos \tilde{\sigma})^2.$$

The largest $|a_2| = |\tilde{\tilde{a}}_2|$ yields the largest $\lambda = \tilde{\tilde{\lambda}}$ so that $|\tilde{\tilde{a}}_2| = \frac{\tilde{\tilde{\lambda}}}{2}$;

$$\lambda \le \tilde{\tilde{\lambda}}(b) = 2|\tilde{\tilde{a}}_2| = 2\sqrt{\tilde{\sigma}^2 - (2\sqrt{1 - \tilde{\sigma}^2} - \tilde{\sigma} \overline{\text{arc}} \cos \tilde{\sigma})^2}. \tag{13}$$

In Table 2 there are some numerical values for the limits (12) and (13). In Figure 4 is the region of unsymmetric 1:2 extremal cases for the interval

$$2\tilde{\sigma}(b) < \lambda < \tilde{\tilde{\lambda}}(b).$$

Tab. 2.

b	$\tilde{\sigma}$	$2\tilde{\sigma}(b)$	$2\lvert\tilde{\tilde{a}}_2(b)\rvert = \tilde{\tilde{\lambda}}(b)$
$e^{-1/2}$	0.606'530'660	1.213'061'319	1.541'015'982
0.606	0.632'078'626	1.264'157'252	1.527'886'918
0.603	0.673'141'177	1.346'282'354	1.525'263'534
0.6	0.697'688'043	1.395'376'086	1.533'825'577
0.59	0.753'547'402	1.507'094'804	1.577'888'895
0.58	0.794'568'895	1.589'137'790	1.628'279'621
0.57	0.828'893'142	1.657'786'284	1.679'512'550
0.56	0.859'161'676	1.718'323'352	1.730'011'824
0.55	0.886'632'605	1.773'265'210	1.779'165'964
0.54	0.912'022'416	1.824'044'832	1.826'721'330
0.53	0.935'785'255	1.871'570'510	1.872'582'862
0.52	0.958'230'497	1.916'460'994	1.916'732'519
0.51	0.979'580'163	1.959'160'326	1.959'191'294
0.5	1	2	2

f) $0.5 < b < e^{-1/2}$; $\lambda \geq \tilde{\tilde{\lambda}}(b)$

Until now we have been dealing with the extremal domains of the type 2:2 or 1:2 and their limit cases. On the strip f) left these rather simple types are no more valid. The maximum will be reached by extremal functions of the type 1:1. The parametric presentation of the boundary arc III is described in Summary 3. According to it the boundary domain of the curved 1:1–type is determined by a point $(\alpha, \omega) \in T$ (Figure 3). In order to understand how the extremal curved 1:1–type is shifted to a radial slit–mapping, we may consider certain niveau–lines $|a_2|$ = *constant* and determine the points (α, ω) maximizing $|a_3|$. In Figure 3 there are examples of two main cases, $b < 0.6$ and $b > 0.6$. If $b < 0.6$, the extremal point (α, ω) tends to the origin and if $b > 0.6$ to the point $(\pi/2, \pi/2)$. Consider the first case more closely.

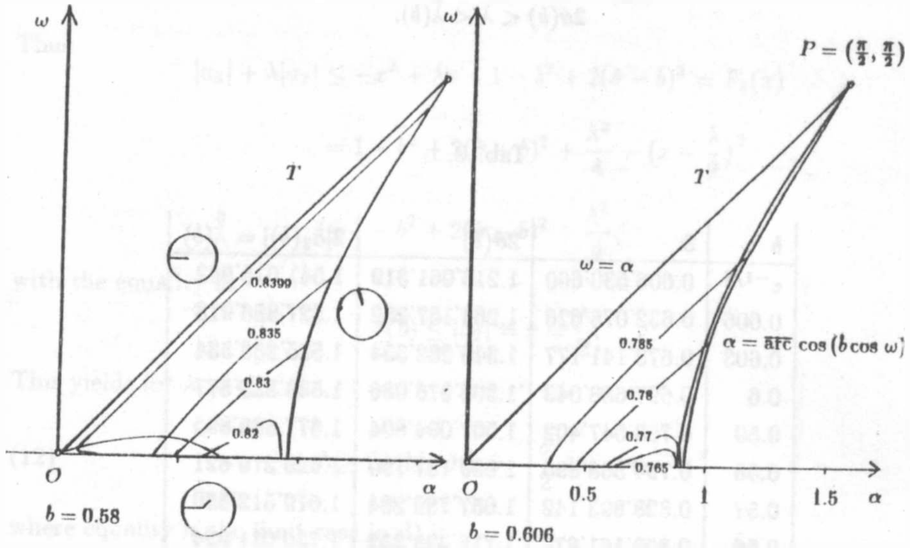

Fig. 3.

In the first case the point (α,ω) tends to the origin along an arc (Figure 3) and hence finally along the tangent of this arc. Hence we may put $\omega = k\alpha$ $(0 < k < 1)$ and will find the following developments by using the formulae of Summary 3:

$$|a_2| = 2(1-b) + M_1\alpha^2 + M_2\alpha^4 + \dots,$$
$$|a_3| = |R_3| + N_1\alpha^2 + N_2\alpha^4 + \dots;$$
$$|a_3| + \lambda|a_2| = |R_3| + \lambda\cdot 2(1-b) + (N_1+\lambda M_1)\alpha^2 + (N_2+\lambda M_2)\alpha^4 + \dots .$$

The radial slit-mapping is the extremal one provided

$$N_1 + \lambda M_1 \le 0,\ N_2 + \lambda M_2 \le 0.$$

The equality requires for the limiting λ and k:

$$(15) \qquad -\lambda = \frac{N_1}{M_1} = \frac{N_2}{M_2}.$$

The numbers M_i, N_i and R_3 are determined by the following expressions:

$$M_1 = (1-b)\left[\frac{k^2\ln^2 k}{(1-k)^2} - k\right],$$

$$M_2 = -\left\{\frac{1}{12}\left[3-(4+2b)k-5(1-b)k^2+(2+4b)k^3-3bk^4\right]\right.$$

$$+\frac{1}{3}\frac{k\ln k}{1-k}\left[2+(1-b)k+(1-b)k^2-2bk^2-2bk^3)+\frac{k\ln k}{1-k}(3+2(1-b)k-3bk^2)\right]$$

$$\left.+\frac{1}{4}(1-b)\left[\frac{k^2\ln^2 k}{(1-k)^2}-k\right]^2\right\};$$

$$N_1 = \frac{R_3Q_1+\frac{1}{2}R_1^2}{R_3},$$

$$N_2 = \frac{\frac{1}{2}Q_1^2+R_3Q_2+R_1R_2}{R_3};$$

$$Q_1 = \frac{4(1-b)^2k^2}{(1-k)^2}(\ln k-\ln^2 k)+\frac{4(1-b)(1-bk)k}{1-k}-4(1-b)^2k,$$

$$Q_2 = \frac{1}{3(1-k)}\left[3b-(2+5b+2b^2)k+(-4+4b-6b^2)k^2+(-8+20b-6b^2)k^3\right.$$

$$\left.+(2+5b-16b^2)k^4+(-3b+6b^2)k^5\right]$$

$$+\frac{2}{3}\frac{(1-b)k}{1-k}\left\{(2\ln k-1)\left[2+(1-b)k+(1-b)k^2-2bk^3\right]\right.$$

$$\left.+\frac{2(\ln k-1)k\ln k}{1-k}\left[3+2(1-b)k-3bk^2\right]\right\};$$

$$R_1 = \frac{4(1-b)k}{1-k}\left[\left(2(1-b)-\frac{1-bk}{1-k}\right)\ln k-(1-b)\right],$$

$$R_2 = \frac{2}{3(1-k)}\left[-2+4b+(6-6b-2b^2)k+(6-12b+6b^2)k^2+(2-6b+2b^2)k^3\right.$$

$$\left.+(-4b+6b^2)k^4\right]+\frac{2}{3}\frac{k\ln k}{(1-k)^2}\left[-3b+(b+2b^2)k+(12-13b+4b^2)k^2+(-9b+6b^2)k^3\right],$$

$$R_3 = (1-b)(3-5b).$$

In Table 3 are examples of solutions of the system (15). The solutions disappear on certain b-interval, the endpoints of which are connected with double roots of (15). However, as will be seen, all the solutions of (15) are not necessarily connected with the boundary curve in question, on which we write $\lambda = \bar{\lambda}(b)$.

Tab. 3.

b	k	$\bar{\lambda}(b)$
0.51	0.000˙000˙000˙034	2.040˙000˙000˙041
0.52	0.000˙010˙005	2.080˙010˙313
0.53	0.000˙912˙958	2.120˙504˙746
0.532˙259˙525	0.002˙800˙001	2.128˙848˙489
0.556˙861˙138	0.017˙873˙790	2.281˙895˙555
0.56	0.036˙032˙445	2.318˙413˙422
0.57	0.101˙969˙439	2.479˙066˙567
0.58	0.228˙182˙498	2.815˙194˙744
0.59	0.478˙278˙419	3.861˙230˙351

At $b = 0.6$ the system (15) yields no solution. Actually at this point the curved 1:1–mapping remains to hold when $\lambda \to +\infty$, i.e. the boundary curve $\lambda = \bar{\lambda}(b)$ has a straight line $b = 0.6$ as a vertical asymptot.

There exists another possibility for shifting from curved 1:1–extremal to the radial slit-case: The maximum dies out *inside* the triangle T leaving to the line segment OP the maximizing role. The b-interval where this happens appears to be, determined by PC–accuracy,

$$0.522 < b < 0.575. \tag{16}$$

Sharpening the endpoints by one decimal requires five more decimals in computations. In Table 4 there are examples of the boundary $\lambda = \hat{\lambda}(b)$, on which two simultaneous extremal domains exist, one is of curved 1:1–type obtained at the point $(\hat{\hat{\alpha}}, \hat{\hat{\beta}})$, and the second is the radial slit-mapping (Figure 4).

Tab. 4.

b	$\lambda = \hat{\lambda}(b)$	$\lambda = \hat{\hat{\lambda}}(b)$	$\hat{\hat{\alpha}}$	$\hat{\hat{\omega}}$
0.53	2.120˙505	2.120˙761	0.002˙126	0.000˙001
0.54	–	2.166˙601	0.001˙037	0.000˙004
0.55	–	2.226˙967	0.000˙819	0.000˙012
0.56	2.318˙413	2.318˙853	0.000˙628	0.000˙027
0.57	2.479˙067	2.479˙075	0.000˙533	0.000˙055

There remains the strip $0.6 < b < e^{-1/2}$, $\lambda > \tilde{\lambda}(b)$. Now passing to the limit of radial slit-mappings means that the maximal point (α, ω) converges to the point $P = (\pi/2, \pi/2)$ tangentially, i.e. along the straight line

$$\omega - \frac{\pi}{2} = k(\alpha - \frac{\pi}{2}).$$

The system (15) appears to be invariant for the alteration

$$\alpha' = \alpha - \frac{\pi}{2}, \quad \omega' = \omega - \frac{\pi}{2};$$

the numbers M_i and N_i are actually covariant i.e. only their signs are changed in this mapping. This implies that the numbers $k > b^{-1}$ and λ are obtained again from (15). Moreover, if

$$0.606\,499\,102 = b_o \leq b < e^{-1/2} = 0.606\,530\,659$$

then $k = b^{-1}$ and the expression of the quantity $\lambda = \hat{\lambda}(b)$ is simplified in this interval to the form

$$\lambda = \hat{\lambda}(b) = 4(1-b)\left\{1 + \frac{(1+2\ln b)[2(1-b) + (1+b)\ln b]}{(3-5b)[\ln^2 b - (1-b)^2/b]}\right\}.$$

In Table 5 there are examples of $\hat{\lambda}(b)$ in $0.6 < b < b_o$.

Tab. 5.

b	$\lambda = \hat{\lambda}(b)$
0.601	19.628˙799˙892
0.602	8.964˙265˙801
0.603	5.410˙841˙505
0.604	3.635˙192˙986
0.605	2.570˙652˙838
0.606	1.861˙664˙495
$b_o = 0.606\,499\,102$	1.589˙670˙589
0.606˙5	1.589˙219˙481
$e^{-1/2} = 0.606\,530\,659$	$1.573\,877\,364 = 4(1 - e^{-1/2})$

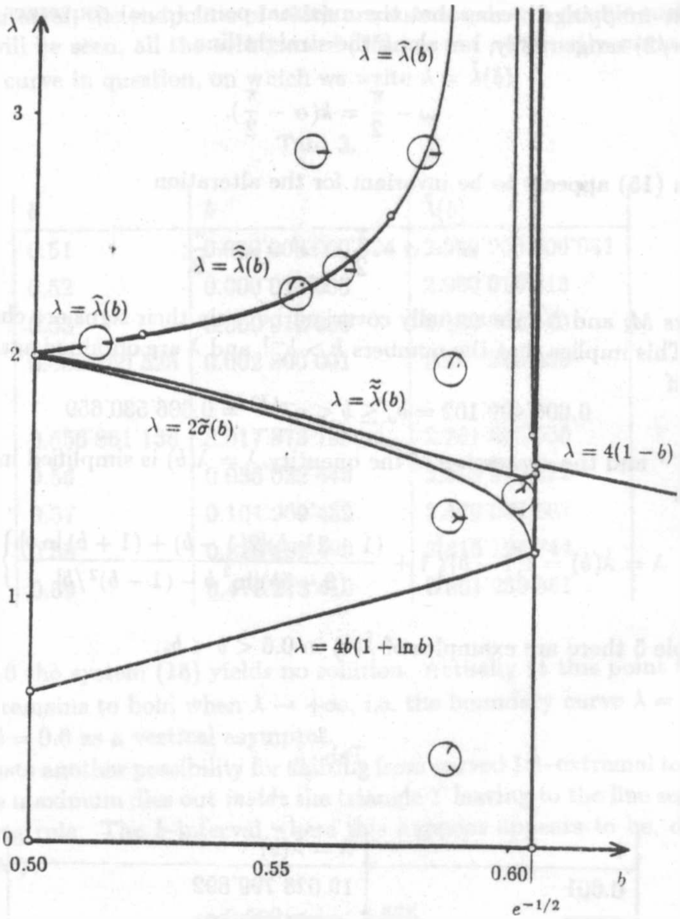

Fig. 4.

REFERENCES

[1] Duren, P., *Univalent functions*, Grundlehren der mathematischen Wissenschaften, Springer Verlag, Berlin-Heidelberg-Tokyo 1983.

[2] Siejka, H., Tammi, O., *On maximizing a homogeneous functional in the class of bounded univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math., Vol. 6, 1981, 273-288.

[3] Tammi, O., *On the maximalization of the coefficient a_3 of bounded schlicht functions*, Ann. Acad. Sci. Fenn. Ser. A I 149, 1953.

[4] Tammi, O., *Extremal Problems for Bounded Univalent Functions*, Lecture Notes in Mathematics, 646, Springer Verlag, Berlin-Heidelberg-New York 1978.

[5] T a m m i , O. , *Extremal Problems for Bounded Univalent Functions II*, ibid, 913, 1982.

[6] T a m m i , O. , *On the geometry of the coefficient body (a_2, a_3) for bounded univalent functions and some related coefficient problems*, Journal D'Anal. Math., Vol. 46, 1986, 304–317.

STRESZCZENIE

W pracy tej badano szczegółowo obszar zmienności (a_2, a_3) współczynników funkcji klasy $S(b)$ ograniczonych funkcji jednolistnych. Umożliwia to oszacowanie $|a_3|$ w terminach $|a_2|$. Zamiast rozpatrywać klasyczne wyrażenie $|a_3 + \lambda a_2|$ można oszacować od góry wyrażenie $|a_3| + \lambda|a_2|$. Ta drobna modyfikacja pozwala uzyskać skomplikowane oszacowania dla pewnych przedziałów zmienności parametru b przy pomocy komputera. Jednakże pewne efekty związane z zachowaniem się stycznych nie pozwalają na określenie z dostateczną dokładnością końców tych przedziałów.

(received June 20, 1991)