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On Harmonic Univalent Mappings

O odwzorowaniach harmonicznych jednolistnych

Abstract. The authors deal with two classes of univalent harmonic mappings of the unit disk satisfying the conditions (1.3) and (1.4). The image domains show to be starlike, or convex, resp. and the convolution of any univalent, harmonic and convex function in the unit disk with a function satisfying (1.4) is a starlike function.

1. Preliminary remarks. A few years ago Clunie and Sheil-Small have initiated studies of harmonic univalent functions in the unit disk Δ. Such mappings $h(z)$ can be written in the form $h(z) = f(z) + \overline{g(z)}$ where $f(z)$ and $g(z)$ are functions analytic in Δ. Imposing the normalization conditions $f(0) = f'(0) - 1 = g'(0) = 0$ they distinguished the class S_H^0 of sense-preserving, harmonic univalent functions. They proved that S_H^0 is a compact and normal family and they obtained many other fundamental properties of S_H^0 and some of its subclasses. The most striking facts are the failures of the Osgood-Taylor-Carathéodory theorem and the Carathéodory Convergence Theorem. Unlike in the analytic case, the convexity of $h(\Delta)$ does not imply convexity of $h(|z| < r)$, $r < 1$. These facts which are both a blessing and a curse, make the study of harmonic mappings interesting and difficult.

In this article we address ourselves to two special subclasses of univalent harmonic mappings. They are closer to analytic functions, they constitute a harmonic counterpart of classes introduced many years ago by Goodman [2] and they proved to be useful in studying questions of neighborhoods (Ruscheweyh [4]) and in constructing explicit k-q.c. extensions (Fait, Krzyż and Zygmunt [3]). We study geometric properties of our mappings, we obtain some preliminary results concerning neighborhoods and the problem of convolution multipliers raised in [1].

Notations and definitions. Let $\Delta_r = \{z : |z| < r\}$, $0 < r < 1$ and let $\Delta_1 = \Delta$. Suppose that the functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1.1}$$

are analytic in Δ and that

$$h(z) = f(z) + \overline{g(z)}\ . \tag{1.2}$$

Denote by HS the class of all functions of the form (1.1) that satisfy the condition

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le 1 - |b_1|\ , \quad 0 \le |b_1| < 1\ . \tag{1.3}$$

and by HC the subclass of HS that consists of all functions subject to the condition

$$\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \le 1 - |b_1|\ , \quad 0 \le |b_1| < 1\ . \tag{1.4}$$

The corresponding subclasses of HS and HC with $b_1 = 0$ will be denoted by HS^0, HC^0, resp. Let us notice that, if $|b_1| = 1$ and (1.3) is satisfied, then the mappings $z + b_1\bar{z}$ are not univalent in Δ and of no interest.

If f, g, F, G are of the form (1.1) and if

$$h(z) = f(z) + \overline{g(z)}\quad ,\quad H(z) = F(z) + \overline{G(z)}$$

then the convolution of h and H is defined to be the function

$$h * H(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n} \tag{1.5}$$

while the integral convolution is defined by

$$h \diamond H(z) = z + \sum_{n=2}^{\infty} \frac{a_n A_n}{n} z^n + \overline{\sum_{n=1}^{\infty} \frac{b_n B_n}{n} z^n}\ . \tag{1.6}$$

Following St. Ruscheweyh [4] we call the δ–neighborhood of h the set

$$N_\delta(h) = \left\{ H : \sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + |b_1 - B_1| \le \delta \right\}\ . \tag{1.7}$$

2. Main results. We start the presentation of our results with showing univalence and starlikeness of functions in HS and HS^0, resp.

Theorem 1. *The class HS consists of univalent sense-preserving harmonic mappings.*

Proof. For h in HS and for $|z_1| \le |z_2| < 1$ we have

$$|f(z_1) - f(z_2)| \ge |z_1 - z_2| \left(1 - |z_2| \sum_{n=2}^{\infty} n|a_n|\right)$$

$$|g(z_1) - g(z_2)| \le |z_1 - z_2| \left(|b_1| + |z_2| \sum_{n=2}^{\infty} n|b_n|\right)\ .$$

Hence

$$|h(z_1)-h(z_2)| \geq |f(z_1)-f(z_2)| - |g(z_1)-g(z_2)|$$
$$\geq |z_1-z_2|(1-|b_1|)(1-|z_2|) > 0 .$$

If $J(h)$ stands for the Jacobian of $h(z)$ then

$$J(h) = |f'(z)|^2 - |g'(z)|^2$$

and some calculus similar to that above yields the inequality

$$J(h) \geq (|f'(z)| + |g'(z)|)(1-|b_1|)(1-|z|) > 0 .$$

Remarks. (i) The functions $h_n(z) = z + \frac{n}{n+1}\,\bar{z}$ are in HS and the sequence converges uniformly to $z+\bar{z}$. Thus the class HS is not compact.
(ii) If $h \in HS$, then for each r, $0<r<1$, $r^{-1}h(rz)$ is also in HS.

(iii) If $h \in HS$ and $h_0(z) = \dfrac{h(z) - b_1\overline{h(z)}}{1-|b_1|^2}$, then $h_0 \in HS^0$, but $h(z) = h_0(z) + \overline{b_1 h_0(z)}$ may not be in HS.

Theorem 2. *Each member of HS^0 maps Δ onto a domain starlike w.r.t. the origin.*

Proof. Let r, $0<r<1$ be a fixed number and let $C_r = h(|z|=r)$, $h \in HS^0$. Thus C_r is a simple closed regular curve. In order to show that $h(\Delta_r)$ is a starshaped domain it suffices to prove that

$$(*) \qquad \frac{d}{d\theta}\arg h(re^{i\theta}) > 0 \quad \text{for } 0 \leq \theta \leq 2\pi .$$

In view of (1.1) the condition $(*)$ takes the form

$$(**) \qquad 1 + \mathrm{Re}\,\frac{\sum_{n=2}^{\infty}\left[(n-1)a_n r^{n-1}e^{i(n-1)\theta} - (n+1)\bar{b}_n r^{n-1}e^{-i(n+1)\theta}\right]}{1+\sum_{n=2}^{\infty}\left(a_n r^{n-1}e^{i(n-1)\theta} + \bar{b}_n r^{n-1}e^{-i(n+1)\theta}\right)} > 0 .$$

We now set

$$A_n = a_n r^{n-1}, \quad B_n = \bar{b}_n r^{n-1}, \quad z_n = e^{i(n-1)\theta}, \quad \zeta_n = e^{-i(n+1)\theta}$$

and we find that $(**)$ is equivalent to

$$\left|1+\sum_{n=2}^{\infty}(A_n z_n + B_n\zeta_n)\right|^2 +$$
$$+\,\mathrm{Re}\left\{\left(1+\sum_{n=2}^{\infty}(A_n z_n + B_n\zeta_n)\right)\overline{\sum_{n=2}^{\infty}\left[(n-1)A_n z_n - (n+1)B_n\zeta_n\right]}\right\} > 0$$

which can be written as

$$\left|1+\sum_{n=2}^{\infty}(A_n z_n + B_n\zeta_n)+\frac{1}{2}\sum_{n=2}^{\infty}\left[(n-1)A_n z_n-(n_1)B_n\zeta_n\right]\right|^2$$
$$-\frac{1}{4}\left|\sum_{n=2}^{\infty}\left[(n-1)A_n z_n-(n+1)B_n\zeta_n\right]\right|^2>0 .$$

We can now estimate the first term from below and the second from above by making use of (1.3) and the triangle inequalities. After some calculus we obtain

$$\frac{d\arg h(re^{i\theta})}{d\theta}\geq A\Big(1-\sum_{n=2}^{\infty}(|A_n|+|B_n|)\Big)\geq A(1-r)>0$$

where A is a positive constant.

This proves starlikeness of C_r and the remaining part follows from the formula

$$h(\Delta)=\bigcup_r h(\Delta_r).$$

Theorem 3. *Functions of the class HC^0 map Δ_r onto convex domains.*

Proof. It is possible to give a justification along the line of Th.2 but we want to give another one based on a result of Clunie and Sheil-Small.

Let us notice that, if $h(z)=f(z)+\overline{g(z)}\in HC^0$, then for each real φ the analytic functions $F=f+e^{i\varphi}g$ satisfy the condition

$$\sum_{n=2}^{\infty}n^2|a_n+e^{i\varphi}b_n|\leq\sum_{n=2}^{\infty}n^2(|a_n|+|b_n|)\leq 1 .$$

It follows from a result of Goodman [2] that $F(z)$ is convex and univalent. Now, a result of Clunie and Sheil-Small [1, Th.5.7] implies the convexity of $h(z)$.

Remark. If $h=f+\bar{g}\in HS^0$ then the function

$$G(z)=\int_0^1\frac{h(tz)}{t}\,dt=\int_0^z\frac{f(u)}{u}\,du+\overline{\int_0^z\frac{g(u)}{u}\,du}$$

satisfies (1.4) with $b_1=0$, hence $G(z)$ is a convex harmonic mapping. This resembles the well-known analytic case. Convexity of $G(z)$, however, does not imply starlikeness of $h(z)$ (or even univalence) in a general situation.

Theorem 4. *Each function in the class HS^0 maps disks Δ_r, $r<\frac{1}{2}$, onto convex domains. The constant $\frac{1}{2}$ is best possible.*

Proof. We give a justification based on a trick. Let $h\in HS^0$ and let r, $0<r<1$, be fixed. Then $r^{-1}h(rz)\in HS^0$ and we have

$$\sum_{n=2}^{\infty}n^2(|a_n|+|b_n|)r^{n-1}=\sum_{n=2}^{\infty}n(|a_n|+|b_n|)(nr^{n-1})\leq\sum_{n=2}^{\infty}n(|a_n|+|b_n|)\leq 1$$

provided $nr^{n-1} \leq 1$ which is true if $r \leq \frac{1}{2}$.

We conclude with some geometric considerations and a distortion theorem.

Theorem 5. *If $h \in HS$, then*

(i) $$|h(z)| \leq |z|(1+|b_1|) + \frac{1-|b_1|}{2}|z|^2$$

(ii) $$(1-|b_1|)\Big(|z| - \frac{|z|^2}{2}\Big) \leq |h(z)|\,.$$

Equalities are rendered by the functions

$$h_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{1-|b_1|}{2}\bar{z}^2\,,$$

for properly chosen real θ.

Proof. We shall justify the case (i) only. We have

$$|h(z)| \leq |z|(1+|b_1|) + |z|^2 \sum_{n=2}^{\infty}(|a_n| + |b_n|)\,.$$

But (1.3) gives

$$\sum_{n=2}^{\infty}(|a_n| + |b_n|) \leq \frac{1-|b_1|}{2} - \frac{1}{2}\sum_{n=3}^{\infty}(n-2)(|a_n| + |b_n|) \leq \frac{1-|b_1|}{2}$$

with all coefficients but a_2, b_2 vanishing.

It follows that the class HS is uniformly bounded, hence, it is normal. The class HS^0 is also compact and convex. HS^0 has a non-empty set of extreme points. We recall that a function $h(z) \in HS^0$ is said to be an extreme point of HS^0 if it cannot be written as a proper convex combination of functions in the class.

Theorem 6. *The extreme points of HS^0 are only the functions of the form: $z + a_n z^n$ or $z + \overline{b_m z^m}$, with $|a_n| = \frac{1}{n}$, $|b_m| = \frac{1}{m}$.*

Proof. Suppose that $h(z) = z + \sum_{n=2}^{\infty}(a_n z^n + b_n \bar{z}^n)$ is such that $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) < 1$, $a_k > 0$. Then, if $\lambda > 0$ is small enough we can replace a_k by $a_k - \lambda$, $a_k + \lambda$ and we obtain two functions $h_1(z)$, $h_2(z)$ that satisfy the same condition and for which one gets $h(z) = \frac{1}{2}[h_1(z) + h_2(z)]$. Let now $h(z)$ be such that

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) = 1 \quad , \quad a_k \neq 0\,,\ k_l \neq 0\,. \tag{2.1}$$

If $\lambda > 0$ is small enough and if x, y, $|x| = |y| = 1$ are properly chosen complex numbers, then leaving all but a_k, b_l coefficients of $h(z)$ unchanged and replacing a_k, b_l by

$$a_k + \frac{\lambda x}{k} \quad , \quad b_l - \frac{\lambda y}{l} \tag{2.2}$$

$$a_k - \frac{\lambda x}{k} \quad , \quad b_l + \frac{\lambda y}{l} \tag{2.3}$$

we obtain functions $h_1(z)$, $h_2(z)$ that satisfy (2.1) and such that $h(z) = \frac{1}{2}(h_1(z) + h_2(z))$.

We conclude our considerations with some statements about convolutions and neighborhoods.

Let K_H^0 denote the class of harmonic univalent functions of the form (1.2) with $b_1 = 0$ that map Δ onto convex domains. It is known [1, Th.5.10] that the sharp inequalities

$$2|A_n| \leq n+1 \quad , \quad 2|B_n| \leq n-1$$

are true.

If $H(z)$, $G(z)$ are in K_H^0 then $H * G$ (or $H \diamond G$) may not be convex, but it may be univalent or even convex if one of the functions satisfies some additional conditions (see [1], Th.5.14). In this direction we have

Theorem 7. *Suppose that $H(z) = z + \sum_{n=2}^{\infty}(A_n z^n + B_n \bar{z}^n)$ is in K_H^0. Then*
(i) *If $h(z) \in HC^0$, then $h * H$ is starlike univalent and $h \diamond H$ is convex.*

(ii) *If $h(z)$ satisfies the condition $\sum_{n=2}^{\infty} n^3(|a_n| + |b_n|) \leq 1$ then $h * H$ is convex univalent.*

Proof. We justify the case (i). If $h(z) = z + \sum_{n=2}^{\infty}(a_n z^n + b_n \bar{z}^n)$, then for $h * H$ we obtain

$$\sum_{n=2}^{\infty} n(|a_n A_n| + |b_n B_n|) \leq \sum_{n=2}^{\infty} n^2\Big(|a_n|\,\Big|\frac{A_n}{n}\Big| + |b_n|\,\frac{|B_n|}{n}\Big) \tag{2.4}$$

$$\leq \sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leq 1 .$$

Hence, $h * H \in HS^0$. The transformation $\int_0^1 \frac{h*H(tz)}{t}\,dt = h \diamond H(z)$ now shows that $h \diamond H(z) \in HC^0$.

Remark. If $h \in HS^0$ and $H \in K_H^0$, then $h * H$ need not be univalent. To see that we take $h(z) = \frac{1}{n}z^n + z$ and as $H(z)$ the extremal function for the coefficient problem. It gives $h * H(z) = z + \frac{n+1}{2n}z^n$. But $h \diamond H(z)$ is univalent and starlike. In terms of $N_\delta(h)$ the condition (2.4) reads that $N_1(z + b_1\bar{z}) \subset HS^0$.

We now give an extension of this remark

Theorem 8. *Assume that $h(z) = z + b_1\bar{z} + \sum_{n=2}^{\infty}(a_n z^n + b_n \bar{z}^n)$ belongs to HC. If $\delta \leq \frac{1}{2}(1 - |b_1|)$, then $N_\delta(h) \subset HS$.*

Proof. Let $H(z) = z + B_1\bar{z} + \sum_{n=2}^{\infty}(A_n z^n + B_n \bar{z}^n)$ belong to $N_\delta(h)$. We have

$$
\begin{aligned}
|B_1| + \sum_{n=2}^{\infty} n(|A_n| + |B_n|) &\le |B_1 - b_1| + \\
&+ \sum_{n=2}^{\infty} n(|A_n - a_n| + |B_n - b_n|) + |b_1| + \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \\
&\le \delta + |b_1| + \frac{1}{2}\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \\
&\le \delta + \frac{1}{2}(1 + |b_1|) \le 1 .
\end{aligned}
$$

Hence, $H(z) \in HS$.

By considering the mappings

$$
h(z) = z + b_1\bar{z} + \frac{1}{4}(1 - |b_1|)\bar{z}^2 \quad ; \quad H(z) = z + b_1\bar{z} + \frac{1}{2}(1 - |b_1|)\bar{z}^2
$$

we conclude that the result cannot be improved. Problems of determining neighborhoods for other harmonic mappings will be considered elsewhere.

REFERENCES

[1] Clunie, T., Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. 19 (1984), 3–25.

[2] Goodman, A. W., *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. 8 (1957), 588–601.

[3] Fait, M., Krzyż, J., Zygmunt, J., *Explicit quasiconformal extensions for some classes of univalent functions*, Comment. Math. Helv. 51 (1976), 279–285.

[4] Ruscheweyh, St., *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. 81 (1981), 521–528.

STRESZCZENIE

Rozważane są dwie klasy funkcji harmonicznych i jednolistnych w kole jednostkowym, wyróżnione warunkami (1.3) i (1.4). Dowodzi się, że odwzorowują one koła na obszary gwiaździste i wypukłe oraz że splot każdej funkcji jednolistnej harmonicznej i wypukłej w kole z funkcją spełniającą warunek (1.4) jest funkcją gwiaździstą.

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