ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA

VOL.XXXIX.20 SECTIO A 1985

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The Radius of α-starlikeness in the Family of Close-to-convex Functions

Promień α-gwiaździstości funkcji prawie wypukłych

Радиус α-звездообразности семейства функций близких к выпуклым

1 Introduction. Let $H(\Delta)$ denote the class of all functions holomorphic in the unit disk $\Delta = \{z \in C : |z| < 1\}$. The function $f \in H(\Delta)$ is said to be close-to-convex [3] if $f(0) = f'(0) - 1 = 0$ and $\operatorname{Re} \frac{f'(z)}{g'(z)} > 0$, $z \in \Delta$, for a certain univalent function $g \in H(\Delta)$ that maps the disk Δ onto a convex domain. The class of close-to-convex functions will be denoted by K. According to the results of papers [1,4-6] the class K is identical with the class L of linearly accessible functions [2], that is of such univalent functions $f \in H(\Delta)$, $f(0) = f'(0) - 1 = 0$ that $C - f(\Delta)$ is the union of closed half-lines such that the corresponding open half-lines have no points in common. Moreover, Z. Lewandowski in [5] has proved that for $|z| < 4\sqrt{2} - 5$ we have $\operatorname{Re} \frac{zf'(z)}{f(z)} > 0$ for any $f \in L$ and that this result is sharp. This means that $4\sqrt{2} - 5$ is the radius of starlikeness of the class L. The result of Z.Lewandowski was established by making use of the following theorem of M.Biernacki [2]:

$$\left\{ \frac{zf'(z)}{f(z)} : f \in L \right\} = \left\{ \frac{(1+s)^2}{\left(1 + \frac{s+t}{2}\right)(1-r^2)} : |s| \le r, |t| \le r \right\} \tag{1}$$

where $r = |z|$.

In this paper we shall determine $\min\left\{ \operatorname{Re} \frac{zf'(z)}{f(z)} : f \in L \right\}$ whence we shall obtain the radius of α-starlikeness of the class $L = K$. **2. Main result.** Making use of (1) we shall prove:

Theorem. *If $z \in \Delta$ and $|z| = r$, then for any $f \in L$ the following estimation holds true:*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \begin{cases} \dfrac{1-r}{1+r} & \text{for } 0 \leq r \leq 2\sqrt{3}-3 \\ \dfrac{2}{\sqrt{(1-r)}} - \dfrac{1}{1-r} - \dfrac{1}{2} & \text{for } 2\sqrt{3}-3 \leq r \leq 1 . \end{cases} \tag{2}$$

The result is sharp.

Proof. By (1) we have:

$$\min\left\{ (1-r)^2 \operatorname{Re} \frac{zf'(z)}{f(z)} : f \in L \right\} =$$

$$= \min\left\{ \frac{\operatorname{Re}\left((1+s)^2(2+\bar{s})\right) - |s||1+s|^2}{2(1+\operatorname{Re} s)} : |s| = r \right\} =$$

$$= \min\{w(x) : -1 \leq x \leq 1\} ,$$

where

$$w(x) = \frac{1-r^2}{2} + 2rx + \frac{(1+r)(1-r^2)}{2(1+rx)} ,$$

$rx = \operatorname{Re} s$, $-1 \leq x \leq 1$.

If

$$x_o = \left(1 + \frac{1}{r}\right) \frac{\sqrt{1-r}}{2} - \frac{1}{r} \leq -1 ,$$

then $w'(x) \geq 0$ for $x \in \langle -1, 1 \rangle$, that is $w(x) \geq w(-1) = (1-r)^2$.
For any $r \in (0,1)$ we have $x_o < 0$. If $x_o \geq -1$ then

$$w(x) \geq w(x_o) = \frac{(1-r)(4\sqrt{1-r} + r - 3)}{2}$$

It suffices now to solve the inequalities $x_o \leq -1$ and $x_o \geq -1$.

Definition. The number:

$$r(\alpha) = \inf_{f \in L} \sup \left\{ r : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \text{ for } |z| < r \right\}$$

is called the radius of α–starlikeness of the class L.

From (2) we directly obtain:

Corollary.

$$r(\alpha) = \begin{cases} 1 - \dfrac{1}{\left(1 + \sqrt{\dfrac{1}{2} - \alpha}\right)^2} & \text{for } \alpha \leq \dfrac{\sqrt{3}-1}{2} , \\ \dfrac{1-\alpha}{1+\alpha} & \text{for } \dfrac{\sqrt{3}-1}{2} \leq \alpha < 1 . \end{cases}$$

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STRESZCZENIE

W pracy znaleziono dokładne oszacowanie od dołu funkcjonału $\operatorname{Re}\dfrac{zf'(z)}{f(z)}$ w klasie L funkcji prawie wypukłych i wyznaczono promień α-gwiaździstości tej klasy.

РЕЗЮМЕ

В работе получено точную нижнюю оценку функционала $\operatorname{Re}\dfrac{zf'(z)}{f(z)}$ в классе функций близких к выпуклым и определено радиус α-звездообразности этого класса.