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A Generalization of Chebyshev's Inequality

Pewne uogólnienie nierówności Czebyszewa

Некоторые обобщение неравенства Чебышева

Let a_{ij} and p_i $(i,j=1,\ldots,n)$ be real numbers. Let us adopt the notations:

$$\underset{1}{\Delta} a_{ij} = a_{i+1,j} - a_{ij} ,$$

$$\underset{2}{\Delta} a_{ij} = a_{i,j+1} - a_{ij} ,$$

$$\underset{1}{\Delta}\underset{2}{\Delta} a_{ij} = \underset{1}{\Delta}(\underset{2}{\Delta} a_{ij}) = a_{i+1,j+1} - a_{i,j+1} - a_{i+1,j} + a_{ij} ,$$

$$P_k = \sum_{i=1}^{k} p_i , \quad \bar{P}_k = \sum_{i=k}^{n} p_i , \quad (k=1,\ldots,n) .$$

Then the following theorem is valid:

Theorem 1. (a) *Let n-tuple p satisfy the conditions* $0 \le p_k \le p_n$ $(1 \le k \le n)$. *If* $\underset{1}{\Delta}\underset{2}{\Delta} a_{ij} \ge 0$ $(i,j=1,\ldots,n-1)$, *then*

$$D(a;p) \ge 0 \tag{1}$$

where

$$D(a;p) = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j a_{jj} - \sum_{i,j=1}^{n} p_i p_j a_{ij} .$$

If $\underset{1}{\Delta}\underset{2}{\Delta}a_{ij} \leq 0$ $(i,j = 1,\ldots,n-1)$, *then the reverse inequality in* (1) *is valid.*

(b) *Let be either* $0 \leq p_n \leq \bar{p}_k$ $(k = 1,\ldots,n)$ *or* $0 \leq p_n \leq \bar{p}_k$ $(k = 1,\ldots,n)$. *If* $\underset{1}{\Delta}\underset{2}{\Delta}a_{ij} \leq 0$ $(i,j = 1,\ldots,n-1)$, *then* (1) *holds; and if* $\underset{1}{\Delta}\underset{2}{\Delta}a_{ij} \geq 0$ $(i,j = 1,\ldots,n-1)$, *then the reverse inequality in* (1) *is valid.*

Proof. As identity $\sum_{j=1}^{n} p_j(a_{jj} - a_{ij}) = \alpha_i + \beta_i$ is valid, where

$$\alpha_i = \sum_{j=1}^{i-1} p_j(a_{jj}-a_{ij}-a_{j+1,j+1}+a_{i,j+1}), \quad \beta_i = \sum_{j=i+1}^{n} \bar{p}_j(a_{jj}-a_{ij}-a_{j-1,j-1}+a_{i,j-1}),$$

then, because $\alpha_1 = 0$ and $\beta_n = 0$, we obtain:

$$\begin{aligned} D(a,p) &= \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{n} p_j(a_{jj} - a_{ij}) \right) = \sum_{i=1}^{n} p_i\alpha_i + \sum_{i=1}^{n} p_i\beta_i = \\ &= \sum_{i=2}^{n} \bar{P}_i(\alpha_i - \alpha_{i-1}) + \sum_{i=1}^{n-1} P_i(\beta_i - \beta_{i+1}) = \\ &= \sum_{i=2}^{n} \bar{P}_i \left(\sum_{j=1}^{i-2} P_j \underset{1}{\Delta}\underset{2}{\Delta} a_{i-1,j} - P_{i-1} \underset{1}{\Delta} a_{i-1,i-1} \right) + \\ &+ \sum_{i=1}^{n} P_i \left(\sum_{j=i+2}^{n} \bar{P}_j \underset{1}{\Delta}\underset{2}{\Delta} a_{i,j-1} + \bar{P}_{i+1} \underset{1}{\Delta} a_{i,i+1} \right) \end{aligned}$$

i.e.

$$D(a;p) = \sum_{i=2}^{n}\sum_{j=1}^{i-1} \bar{P}_i P_j \underset{1}{\Delta}\underset{2}{\Delta} a_{i-1,j} + \sum_{i=1}^{n-1}\sum_{j=i+2}^{n} P_i \bar{P}_j \underset{1}{\Delta}\underset{2}{\Delta} a_{i,j-1}. \qquad (2)$$

Using (2), we can easily prove Theorem 1.

Remarks: 1° We can also easily show that the conditions of Theorem 1 are necessary and sufficient.

2° If a and b are monotonous n-tuples, then from Theorem 1, for $a_{ij} = a_i b_j$, we can get the well-known Chebyshev's inequality.

Theorem 2. *Let* p *be positive* n*-tuple such that*

$$M = \max_{1\leq k\leq n-1} (P_k \bar{P}_{k+1}).$$

If either $\underset{1}{\Delta}\underset{2}{\Delta}a_{ij} \geq 0$ $(i,j = 1,\ldots,n-1)$ *or* $\underset{1}{\Delta}\underset{2}{\Delta}a_{ij} \leq 0$ $(i,j = 1,\ldots,n-1)$ *then*

$$|D(a;p)| \leq M |a_{nn} - a_{n1} - a_{1n} + a_{1,1}| .$$

Proof. From (2) we have:

$$|D(a;p)| \le M \left| \sum_{i=2}^{n} \sum_{j=1}^{i-1} \underset{1}{\Delta}\underset{2}{\Delta} a_{i-1,j} + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \underset{1}{\Delta}\underset{2}{\Delta} a_{i,j-1} \right| =$$

$$= M|a_{nn} - a_{n1} - a_{1n} + a_{1,1}| .$$

Corollary 1. If either $\underset{1}{\Delta}\underset{2}{\Delta} a_{ij} \ge 0 \quad (i,j = 1,\dots,n-1)$ or $\underset{1}{\Delta}\underset{2}{\Delta} a_{ij} \le 0 \quad (i,j = 1,\dots,n-1)$ then

$$\left| n \sum_{i=1}^{n} a_{ii} - \sum_{i,j=1}^{n} a_{ij} \right| \le \left[\frac{n}{2}\right] \left(n - \left[\frac{n}{2}\right]\right) |a_{nn} - a_{n1} - a_{1n} + a_{1,1}| .$$

Analogously we can prove the corresponding integral analogue of Theorems 1 and 2.

Theorem 3. *Let the function $\lambda : [a,b] \mapsto R$ be either continuous or of bounded variation and let the function $f : [a,b] \times [a,b] \mapsto R$ have partial derivative $\frac{\partial^2 f}{\partial x \partial y}$ for every $x, y \in [a,b]$.*

(a) *Let $\lambda(a) \le \lambda(x) \le \lambda(b)$, for all $x \in [a,b]$.*

If $\frac{\partial^2 f}{\partial x \partial y} \ge 0$ (for all $x, y \in [a,b]$ then

$$T(f;\lambda) \ge 0 \tag{3}$$

where

$$T(f;\lambda) = \int_a^b d\lambda(y) \int_a^b f(x,x) d\lambda(x) - \int_a^b \int_a^b f(x,y) d\lambda(x) d\lambda(y) .$$

If $\frac{\partial^2 f}{\partial x \partial y} \le 0$ (for all $x, y \in [a,b]$) then the reverse inequality in (3) is valid.

(b) *Let be either $\lambda(a) \le \lambda(b) \le \lambda(x)$ for all $x \in [a,b]$ or $\lambda(b) \ge \lambda(a) \ge \lambda(x)$ for all $x \in [a,b]$. If $\frac{\partial^2 f}{\partial x \partial y} \le 0$ (for all $x, y \in [a,b]$) then (3) holds, and if $\frac{\partial^2 f}{\partial x \partial y} \ge 0$ (for all $x, y \in [a,b]$) then the reverse inequality in (3) holds.*

Theorem 4. *Let λ be nondecreasing function on $[a,b]$ such that $M = \max_{x \in [a,b]}((\lambda(x) - \lambda(a))(\lambda(b) - \lambda(x)))$. If either $\frac{\partial^2 f}{\partial x \partial y} \ge 0$ (for all $x, y \in [a,b]$) or $\frac{\partial^2 f}{\partial x \partial y} \le 0$ (for all $x, y \in [a,b]$), then*

$$|T(f;\lambda)| \le M |f(b,b) - f(a,b) - f(b,a) + f(a,a)| .$$

Remark 8. If λ is continuous nondecreasing function then

$$M = \frac{1}{4}(\lambda(b) - \lambda(a))^2.$$

The previous results are generalizations of some results from [1], [2], [3] and [4].

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STRESZCZENIE

W pracy tej zostało podane uogólnienie dobrze znanej nierówności Czebyszewa. Podano również uogólnienie pewnej nierówności M.Biernackiego, H.Pidek i C.Rylla-Nardzewskiego.

РЕЗЮМЕ

В данной работе представлено обобщение известного неравенства Чебышева. Представлено также обобщение некоторого неравенства Бернацкого, Пидек и Рилл-Нардзевского.