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Local Order Function for Homogeneous Rotation Invariant Distribution and Their Multiplication

Lokalna funkcja rzędu dystrybucji dla dystrybucji jednorodnych niezmienniczych ze względu na obrót i dla ich iloczynów

Локальная функция ранга для однородных обобщенных функций, инвариантных относительно вращений, и для произведний таких функций

Consider a set R^* consisting of two points t^+ and t^- for every $t \in R$ and of point ∞ with the ordering:

$$\text{if } s < t \text{ then } s^- < s^+ < t^- < t^+ < \infty .$$

R^* is equipped with the topology induced by this ordering. Addition in R^* is defined so that

$$s^- + t^- = s^- + t^+ \stackrel{\text{df}}{=} (s+t)^-$$

$$s^+ + t^+ \stackrel{\text{df}}{=} (s+t)^+, \; t^- + \infty = t^+ + \infty \stackrel{\text{df}}{=} \infty .$$

A particular example of R^* which we shall use below is given by

$$t^- = \{ s \in R : s < t \}, \; t^+ = \{ s \in R : s \leq t \}, \; \infty = R.$$

According to Ambrose [1] we introduce the local order function $O_U(x, l)$ of a distribution $U \in D'(R^n)$ as follows:

$$\text{let } x \in R^n \text{ and } l \in S^{n-1} \text{ then}$$

$O_U(x, l) = \{s \in R$: there exist neighbourhoods Q of the point x and E of l in S^{n-1} such that for all $\omega \in D(Q)$ we have

$$\int_{\Gamma_E} |(\omega U)^\wedge(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty\}$$

where $\wedge$ denotes the Fourier transform in R^n and Γ_E is the cone $\{y \in R^n : y/|y| \in E\}$.

In this paper we shall compute the order function for homogeneous rotation invariant distribution in R^n. Such distributions will be denoted by $|x|^\lambda$ and are defined as follows [1]:

For $\lambda \in C$, Re $\lambda > -n$ we set

$$|x|^\lambda [\varphi] = \int_{R^n} |x|^\lambda \varphi(x)\, dx \text{ for } \varphi \in D(R^n).$$

The function $\lambda \to |x|^\lambda \in D'(R^n)$ for Re $\lambda > -n$ admits meromorphic extension to the whole complex plans with simple poles at the points $-n, -n-2, -n-4, \ldots$. According to Gelfand, Shilov [2] we denote for $k, m = 0, 1, 2, \ldots$ [2].

$$|x|^\lambda \ln^m |x| \overset{\text{df}}{=} d^m/d\lambda^m\, |x|^\lambda \text{ for } \lambda \neq -n, -n-2, -n-4, \ldots,$$

(2) $$\delta^{(2k)}(|x|) \overset{\text{df}}{=} (2k)!\, \text{Res}_{\lambda=-n-2k} |x|^\lambda,$$

$$|x|^{-n-2k} \ln^m |x| \overset{\text{df}}{=} \lim_{\lambda \to -2k-n} \frac{d^m}{d\lambda^m}\left(|x|^\lambda - \frac{\delta^{(2k)}(|x|)}{(2k)!\,(\lambda + 2k + n)}\right).$$

Proposition 1 ([2] p. 222). *For* $k, m = 0, 1, 2, \ldots$

$$(|x|^\lambda \ln^m |x|)^\wedge(\xi) = \sum_{i=0}^{m} c_{im}(\lambda) |\xi|^{-n-\lambda} \ln^i |\xi| \text{ for } \lambda \neq -n, -n-2, \ldots$$

$$(\delta^{(2k)}(|x|))^\wedge(\xi) = c_k |\xi|^{2k},$$

$$(|x|^{-2k-n} \ln^m |x|)^\wedge(\xi) = \sum_{i=0}^{m+1} d_{imk} |\xi|^{2k} \ln^i |\xi|,$$

where c_{im}, c_k, d_{imk} *are some constants.*

Lemma. *Let* Δ *be the Laplace operator in* R^n. *We have*

$$\Delta(|x|^{-n+2} \ln^m |x|) = \begin{cases} (-n+2)\,\delta(|x|) & \text{for } n \geqslant 3, m = 0, \\ -\delta(|x|) & \text{for } n = 2, m = 1, \end{cases}$$

and

$$\Delta (|x|^{\lambda} \ln^{m} |x|) = \lambda (\lambda + n - 2) |x|^{\lambda - 2} \ln^{m} |x| +$$
$$+ m (2\lambda + n - 2) |x|^{\lambda-2} \ln^{m-1} |x| + m (m-1) |x|^{\lambda-2} \ln^{m-2} |x|$$

otherwise.

Proof. It follows by differentiating m times with respect to λ the identity

$$\Delta |x|^{\lambda} = \lambda (\lambda + n - 2) |x|^{\lambda - 2}$$

and computing residua at singular values of λ.

Proposition 2. *Let $U \in D'(R^n)$ and $l \in S^{n-1}$. Then $O_U(0, l) = s^-$ if and only if $O_{\Delta U}(0, l) = (s-2)^-$.*

This proposition is only a reformulation of the microlocal version (for a conical neighbourhood) of the regularity theorem for elliptic operators in Sobolev spaces (see [4] Theorem 7.2 p. 61. Also cf. [3], Theorem 2).

Theorem. *For every $l \in S^{n-1}$*

(3) $$O_{|x|2k}(0, l) = +\infty \text{ for } k = 0, 1, 2, \dots,$$

(4) $$O_{\delta^{(2k)}(|x|)}(0, l) = [-2k, -n/2]^- \text{ for } k = 0, 1, 2, \dots,$$

(5) $$O_{|x|^{\lambda}\ln^{m}|x|}(0, l) = [\operatorname{Re}\lambda + n/2]^-$$

for all $\lambda \in C$ if $m \geqslant 1$ and for $\lambda \neq 0, 2, 4, \dots,$ if $m = 0$.

Proof. Directly from the definition of the local order function, it follows that every $l \in S^{n-1}$

$$O_f(0, l) = +\infty \text{ if } f \in C^{\infty}$$

in some neighbourhood of 0,

(6) $$O_{\delta(|x|)}(0, l) = [-n/2]^-.$$

Hence follow formulas (3) and the first one of (4). To consider the remaining cases denote by V_λ the distribution $|x|^{\lambda} \ln^{m} |x|$ (for $\lambda \in C$, $m = 0, 1, 2, \dots$) or the distribution $\delta^{(-\lambda-n)}(|x|)$ if $\lambda = -2k - n$ ($k = 1, 2, \dots$). By Proposition 1 we have

$$(V_\lambda)^\wedge(\xi) = \sum_{i=0}^{\widetilde{m}} e_i \,|\xi|^{-\lambda-n} \ln^i|\xi|$$

for some $\widetilde{m}$ and some constants e_i. Suppose now that Re $\lambda < -n/2$. Then (7) is a locally square integrable function and we have

$$\int_{R^n} |(V_\lambda)^\wedge(\xi)|^2 (1+|\xi|^2)^s d\xi = \sum_{i=0}^{2\widetilde{m}} e_i' \int_{R^n} |\xi|^{-2\mathrm{Re}\lambda-2n} \ln^i|\xi| (1+|\xi|^2)^s d\xi =$$

$$= \sum_{i=0}^{2\widetilde{m}} e_i'' \int_0^\infty r^{-2\mathrm{Re}\lambda-n-1} \ln^i r \,(1+r^2)^s dr,$$

with suitable constants e_i', e_i''. Hence $V_\lambda \in H^s$ if $s < \mathrm{Re}\,\lambda + n/2$. Since Sobolev H^s spaces are closed under multiplication by functions in $D(R^n)$, it follows that $O_{V\lambda}(0,l) \geqslant [\mathrm{Re}\,\lambda + n/2]^-$ for every l. To prove the equality suppose that there exists $l \in S^{n-1}$ such that $O_{V\lambda}(0,l) > [\mathrm{Re}\,\lambda + n/2]^-$. Then for some cone Γ_E, $(|E|>0)$ and some function $\omega \in D(R^n)$, $\omega = 1$ in a neighbourhood of zero we would have

$$\int_{\Gamma_E} |(\omega V_\lambda)^\wedge(\xi)|^2 (1+|\xi|^2)^{\mathrm{Re}\lambda + n/2} d\xi < +\infty .$$

Since $(1-\omega)V_\lambda$ is integrable for Re $\lambda < -n$, its Fourier transform is bounded. This together with (9) gives that the integral

$$\int_{\Gamma_E} |(V_\lambda)^\wedge(\xi)|^2 (1+|\xi|^2)^{\mathrm{Re}\lambda+n/2} d\xi$$

is convergent. On the contrary a calculus analogous to (8) proves that this integral a divergent. Thus we have proved that for every $l \in S^{n-1}$

$$O_{V_\lambda}(0,l) = [\mathrm{Re}\,\lambda + n/2]^- \text{ if Re } \lambda < -n.$$

Therefore $O_{\delta(2k)}(0,l) = [-2k - n/2]^-$ for $k = 1, 2, \ldots$, and

(10) $$O_{|x|^\lambda \ln^m |x|}(0,l) = [\mathrm{Re}\,\lambda + n/2]^- \text{ for Re } \lambda < -n.$$

So we have proved all formulas (4) and some of (5). Denote by W_λ any distribution of the norm

(11) $$\sum_{i=0}^{m} \alpha_i |x|^\lambda \ln^i|x|, \alpha_i \in C, \sum_{i=0}^{m} |\alpha_i|^2 > 0.$$

By (10) $O_{W_\lambda}(0,l) \geq [\mathrm{Re}\,\lambda + n/2]^-$ and, as before for V_λ, we prove that

(12) $$O_{W_\lambda}(0,l) = [\mathrm{Re}\,\lambda + n/2]^- \text{ for every } l \in S^{n-1} \text{ and Re } \lambda < -n.$$

To prove the remaining formulas (5) it suffices to prove that for all λ such that $\mathrm{Re}\,\lambda \geqslant -n$

(13) $$O_{W_\lambda}(0, l) = [\mathrm{Re}\,\lambda + n/2]^- \text{ for every } l \in S^{n-1}.$$

We show first that (13) holds for $-n \leqslant \mathrm{Re}\,\lambda < -n + 2$. To this end observe that Lemma and formulas (12) we get for $-n \leqslant \mathrm{Re}\,\lambda < -n + 2$

(14) $$O_{\Delta W_\lambda}(0, l) = [\mathrm{Re}\,\lambda - 2 + n/2]^- \text{ for every } l \in S^{n-1}.$$

Hence by Proposition 2 we obtain formulas (13) for $-n \leqslant \mathrm{Re}\,\lambda < n + 2$. In the next step we consider the belt $-n + 2 \leqslant \mathrm{Re}\,\lambda < -n + 4$. By Lemma, formulas (13) valid for $\mathrm{Re}\,\lambda < -n + 2$ and by (6) we get (14) for $-n + 2 \leqslant \mathrm{Re}\,\lambda < -n + 4$. Therefore by Proposition 2 follow formulas (13) for $-n + 2 \leqslant \mathrm{Re}\,\lambda < -n + 4$. To finish the proof by induction take $k \geqslant 2$ and suppose that the relations (13) are true for $\mathrm{Re}\,\lambda < -n + 2k$. Then by Lemma we get (14) for $-n + 2k \leqslant \mathrm{Re}\,\lambda < -n + 2k + 2$ and hence by Proposition 2 follow formulas (13) for $-n + 2k \leqslant \mathrm{Re}\,\lambda < -n + 2k + 2$.

Remark 1 (see [2] and [5]). Both some fundamental solution E and its Fourier transform $E^\wedge$ for an arbitrary operator $P(\Delta)$, P – a polynomial in one variable, are series of distributions of the form (2).

Remark 2 (see [1]). If we know the local orders of two distributions U and V we can multiply them under the condition that for every $x \in R^n$ and $l \in S^{n-1}$

$$O_U(x, l) + O_V(x, -l) \geqslant 0^+.$$

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STRESZCZENIE

W pracy wyznaczono lokalną funkcję rzędu dystrybucji w sensie W. Ambrose'a dla jednorodnych dystrybucji w R^n niezmienniczych ze względu na obroty.

РЕЗЮМЕ

В работе приводим локальную функцию ранга (в смысл В. Амброза) для однородных обобщенных функций в R^n инвариантных относительно вращений.