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Generalized Bielecki Theorem

Uogólnione twierdzenie Bieleckiego

Обобщенная теорема Белецкого

1. Introduction. The following theorem has been proved by Bielecki [1]:

Suppose that $N(t, s, x)$ is a bounded real function defined for $0 \leqslant t, s \leqslant T$ and $x \in \mathbb{R}$ satysfying the Lipschnitz condition with respect to x:

$$|N(t, s, x) - N(t, s, y)| \leqslant L(t)\,|x - y| \text{ for all } x, y \in \mathbb{R} \tag{1.1}$$

where $L(t)$ is a non-negative locally integrable function over the interval $0 \leqslant t \leqslant T$. Write for $x \in C[0, T]$, $p \in \mathbb{R}$:

$$\|x\|_p = \max_{0<t<T} \left\{\exp\left[-p\int_0^t L(s)\,ds\right] |x(t)|\right\} \tag{1.2}$$

Then the equation

$$x = G(x) + y, \text{ where } G(x)(t) = \int_0^t N(t, s, x)\,ds,\ y \in C[0, T] \tag{1.3}$$

has a unique solution, which is the limit of a uniformly convergent sequence of successive approximations:

$$x(t) = \lim_{n\to 0} x_n(t), \text{ where } x_0 = y$$

and

$$x_n(t) = y(t) + \int_0^t N(t, s, x_{n-1}(s))\,ds \quad (n = 1, 2, ..)$$

The proof is based on the fact that for $p > 1$ we have

(1.4) $$\| G(x) - G(y) \|_p \leqslant (1/p) \| x - y \|_p \text{ for } x, y \in C[0, T]$$

(and could be found, for instance, in [2]).

Therefore this method makes it possible to apply the Banach fixed-point theorem without restrictions on the modulus of the function $N(t, s, x)$ of the type „if $N(t, s, x)$ is small enough...".

Inequality (4) shows that by taking p greater we obtain a faster approximation.

This theorem could be also formulated for $T = +\infty$, in which case, instead of the space $C[0, T]$ we consider

$$X_p = \left\{ x : \exp \left[-p \int_0^t L(s)\, ds \right] | x(t) | < \text{const} \right\}$$

for a $p > 1$, provided that the function L is locally intergrable for $t > 0$.

In the present paper we shall show that the Bielecki theorem can be extended for a class of non-linear operators acting in a Banach space. This extension will be done in two steps: 1° we shall generalize the Bielecki theorem for functions of real variable with values in a Banach space; 2° we shall apply the obtained theorem and properties of shifts introduced by the present author for a general case. Examples of applications to hyperbolic equation and equations with transformed argument will be also given.

2. Bielecki Theorem for functions of real variable. Let E be a Banach space with the norm $\| \quad \|_E$. Let $X = C([a, b], E)$ be the Banach space of all functions determined for $a \leqslant t \leqslant b$ and with values in E equipped with the norm

(2.1) $$\| x \| = \sup_{0 \leqslant t \leqslant b} \| x(t) \|_E \text{ for } x \in X$$

Theorem 2.1. *Suppose that*

1° *the function $N(t, s, u)$ determined and continuous for $0 \leqslant a \leqslant t, s \leqslant b, x \in E$ and with values in E satisfies the Lipschitz condition:*

(2.3) $$\| N(t, s, u) - N(t, s, v) \|_E \leqslant L(t) \| u - v \|_E \textit{ for all } t, s \in [a, b], u, v \in E$$

where L is a locally intergrable non-negative function.

2° *the function $h \in C[a, b]$ satisfies the conditions:*

(2.4) $$h(a) = a \textit{ and } a \leqslant h(t) \leqslant t \textit{ for } t \in [a, b]$$

Write:

(2.5) $$\| x \|_p = \sup_{a \leqslant t \leqslant b} \left\{ \exp \left[\left(-p \int_a^{h(t)} L(s)\, ds \right] \| x(t) \|_E \right\}$$

for all $x \in X$ and $p \in \mathbb{R}_+$.

Then the equation

(2.6) $$x(t) = \int_a^{h(t)} N(t, s, x(s))\, ds + y(t), y \in X$$

has a unique solution which is a limit in the norm $\| \quad \|_p$ *of the sequence of successive approximations:*

$$x = \lim_{n \to \infty} x_n, \text{ where } x_0 = y,$$

$$x_{n+1}(t) = \int_a^{h(t)} N(t, s, x_n(s))\, ds + y(t) \ (n = 0, 1, 2, \ldots)$$

Proof is going on the same lines as the original Bielecki's proof. Observe that for $p = 0$ $\| x \|_0 = \| x \|$ and all norms $\| \quad \|_p$ for $p \geq 0$ are equivalent. The mapping G defined by means of the equality:

(2.7) $$G(u)(t) = \int_a^{h(t)} N(t, s, u(s))\, ds + y(t), u \in X$$

maps the space X into itself.

We shall show that

(2.8) $$\| G(u) - G(v) \|_p \leq (1/p) \| u - v \|_p \text{ for } u, v \in X, p > 1.$$

Indeed, observe that the function

(2.9) $$L_1(t) = \int_a^t L(s)\, ds$$

is non-negative. Hence for $p > 0$ we have

$$\exp\left[p \int_a^{h(t)} L(s)\, ds\right] = \exp\left[pL_1(h(t))\right] \geq 1$$

and for all $u, v \in X$

$$\| u(t) - v(t) \|_E \leq \exp\left[pL_1(h(t))\right] \| u - v \|_p$$

Since $L_1'(t) = L(t), L_1(h(a)) = L_1(a) = 0$, and $1 - e^{-u} \leq 1$ for $u \geq 0$, we find

$$\exp\left[-pL_1(h(t))\right] \| G(u) - G(v) \|_E =$$

$$= \exp\left[-pL_1(h(t))\right] \left\| \int_a^{h(t)} [N(t, s, u(s))\, ds - N(t, s, v(s))\, ds] \right\|_E \leq$$

$$\leqslant \exp[-pL_1(h(t))] \| \int_a^{h(t)} L(s) \| u(s) - v(s) \|_E ds \leqslant$$

$$\leqslant \exp[-pL_1(h(t))] \int_a^{h(t)} L(s) \exp[pL_1(s)] \| u - v \|_p ds \leqslant$$

$$\leqslant \exp[-pL_1(h(t))] \int_a^{h(t)} L_1'(s) \exp[pL_1(s)] ds \| u - v \| =$$

$$= (1/p) \exp[-pL_1(h(t))] \exp[pL_1(s)] \Big|_a^{h(t)} \| u - v \|_E =$$

$$= (1/p) \exp[-pL_1(h(t)] [\exp pL_1(h(t)) - 1] \| u - v \|_E =$$

$$= (1/p) \| u - v \| \quad 1 - \exp[-pL_1(h(t))] \quad \leqslant (1/p) \| u - v \|_p.$$

Therefore for $p > 1$ the mapping G has a unique fixed point which is a limit in the norm $\| \quad \|_p$ of the sequence of successive approximations. But all norms $\| \quad \|_p$ for $p \in \mathcal{R}_+$ are equivalent. This finishes the proof of our theorem.

In the same manner we can consider Equation (2.6) in the spaces $C(\mathcal{R}, E)$, $C(\mathcal{R}_+, E)$ etc. We have only to assume that the function $h(t) \leqslant t$ on $\mathcal{R}$ (or $\mathcal{R}_+$, respectively).

Example 2.1. Suppose that $h \in C^1$ $[a, b]$, h maps the interval $[a, b]$ onto itself, $h(a) = a$, $0 \leqslant a \leqslant h(t)\, t$ and $h'(t) > 0$ for $t \in [a, b]$. Suppose that the $\mathcal{R}^n$-valued function $N(t, s, x)$ is determined and continuous for $t, s \in [a, b]$, $x \in \mathcal{R}^n$ and satisfies the Lipshitz condition:

(2.10) $$\| N(t, s, u) - N(t, s, v) \|_{\mathcal{R}^n} \leqslant L(t) \| u - v \|_{\mathcal{R}^n} \text{ for } u, v \in \mathcal{R}^n$$

where L is a function such that the function

$$\tilde{L}(t) = L(t) / h'(h^{-1}(t))$$

is a non-negative function intergrable over $[a, b]$, where h^{-1} denotes the inverse function. Consider a differential equation in $\mathcal{R}^n$ with transformed argument:

(2.11) $$x(t) = N[t, x(h(t))]$$

with the initial condition

(2.12) $$x(a) = x_0$$

The system (2.11) – (2.12) is equivalent to an integral equation:

(2.13) $$x(t) = \int_a^t N(s, h(s))\, ds + x_0$$

If we change variable $s \to h^{-1}(u)$ and we write:

$$\tilde{N}(t,x) = N(h^{-1}(t),x)\,/\,h'(h^{-1}(t))$$

we can rewrite the equation (2.13) in the form:

$$x(t) = \int_a^{h(t)} N(u, x(u))\,du + x_0 \tag{2.14}$$

The functions h, $\tilde{L}$, $\tilde{N}$ satisfy all assumptions of Theorem 2.1. We therefore conclude that the equation (2.14), hence the initial problem (2.11) – (2.12), has a unique solution which is a limit of the sequence of successive approximations (in the norms $\|\ \ \|_p$, $p > 1$, $p = 0$):

$$x = \lim_{n\to\infty} x_n, \text{ where}$$

$$x_{n+1}(t) = \int_a^{h(t)} \tilde{N}(u, x_n(u))\,du + x_0 \text{ for } n = 0, 1, 2, \ldots$$

3. **Bielecki Theorem for right invertible operators.** Let X be a linear space (over $\mathcal{R}$ or C). Let D be a linear right invertible operator defined on a linear subset $\mathrm{dom}\, D \subset X$ and with the range in X such that $\ker D \neq \{0\}$. Let R be an arbitrarily fixed right inverse of D, i.e. $DR = I$ (we assume that $\mathrm{dom}\, R = X$) and let F be an initial operator for D corresponding to R, i.e. a projection onto $\ker D$ such that $FR = 0$. By definition,

$$F = I - RD \text{ on } \mathrm{dom}\, D. \tag{3.1}$$

Let $\{S_h\}_{h\in\mathcal{R}}$ be a family of induced R-shifts (cf. the author, [3]), i.e. a family of linear operators defined on X with the property:

$$\begin{gathered} S_0 = I \\ \forall_{k\in \mathbb{N}\cup\{0\}}\ \forall_{h\in\mathcal{R}}\ S_h R^k F = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k-j)!} h^{k-j} R^j F \end{gathered} \tag{3.2}$$

This family is an Abelian group (with respect to superposition of operators as a structure operation), moreover, preserves constants, i.e.

$$S_h z = z \text{ for all } z \in \ker D,\ h \in \mathcal{R} \tag{3.3}$$

If R is a Volterra right inverse (i.e. the operators $I - \lambda R$ are invertible for all $\lambda \in$ C) then we can define another family of shifts, so-called D-shifts, in the following way:

$$S_0 = I$$

(3.4) $$\forall_{\lambda \in C} \ \forall_{h \in R} \ S_h (I - \lambda R)^{-1} F = e^{-\lambda h} (I - \lambda R)^{-1} F$$

which has the same properties, as R-shifts (cf. the author, [3]). In general, these two families do not coincide. However, if, for instance, X is a Banach space, R is quasi-nilpotent then R-shifts and D-shifts coincide.

In [3] (Theorems 5.1, 5.2, 5.5, 5.7) we have proved the following facts:

Let X be a complete linear metric locally convex space. Let D be a closed right investible operator, let F be a continuous initial operator for D corresponding to a continuous right inverse. Let $P(R)$ be the set of all generalized polynomials, i.e.

(3.5) $$P(R) = \mathrm{lin}\left\{R^k z : z \in \ker D,\ k \in \mathrm{N} \cup \{0\}\right\}$$

(resp. R is Volterra and $E(R) = \mathrm{lin}\left\{(I - \lambda R)^{-1} z : z \in \ker D,\ \lambda \in \mathrm{C}\right\}$ be the set of all generalized exponentials).

The sets $P(R)$ and $E(R)$ are independent of the choice of a right inverse R. Assume that $\overline{P(R)} = X$ (resp. $\overline{E(R)} = X$) and the $\{S_h\}_{h \in \mathcal{R}}$ is a strongly continuous group of R-shifts (resp. D-shifts). Then

1° D is an infinitesimal generator for $\{S_n\}_{n \in \mathcal{R}}$, $\overline{\mathrm{dom}\, D} = X$ and $S_h D = DS_h$ on dom D for all $h \in \mathcal{R}$;

2° the canonical mapping $\kappa = FS_h$ which transforms elements of the space x into ker D-valued functions $\kappa x = \hat{x}(h)$ of a real variable h separates points, i.e.

(3.6) $$\hat{x} = \hat{y} \text{ if and only if } x = y, \text{ where } \hat{x}(h) = FS_h x$$

3° The following equalities hold:

(3.7) $$\kappa D = (d/dt)\,\kappa,\ \kappa R = \int_0^t \kappa,\ (\kappa F x)(t) = (\kappa x)(0),$$
$$(S_h \kappa x)(t) = (\kappa x)(t - h)$$

for all $x \in X$, h, $t \in \mathcal{R}$.
This means that

(3.8) $$(\widehat{Dx})(t) = \hat{x}'(t),\ (\widehat{Rx})(t) = \int_0^t \hat{x}(s)\, ds,$$
$$(\widehat{Fx})(t) = \hat{x}(0),\ (\widehat{S_h x})(t) = \hat{x}(t - h)$$

for all $x \in X$, h, $t \in \mathcal{R}$.

Theorem 3.1. *Suppose that X is a Banach space, D is a closed right invertible operator, F is a bounded initial operator for D corresponding to a bounded right inverse R,*

$\overline{P(R)} = X$ (resp. R is Volterra and $\overline{E(R)} = X$) and S_h $_{h\in R}$ is a strongly continuous group of R-shifts (resp. D-shifts). Suppose, moreover, that $G : X \to X$ is a non-linear mapping satysfying the following conditions:

(3.9) $$G(FS_t x) = FS_t G(x) \text{ for all } t \in R, x \in X$$

(3.10) $$\| G(x) - G(y) \| \leq M \| x - y \| \text{ for all } x, y \in X$$

Then the problem

(3.11) $$Dx = G(x), Fx = x_0, x_0 \in D$$

has a unique solution, which is the limit (in norm) of sequence of successive approximations:

(3.12) $$x = \lim_{x\to\infty} x_n, x_{n+1} = RG(x_n) + x_0 \ (n = 0, 1, 2, \ldots).$$

Proof. By our assumptions properties 1°, 2°, 3° holds, also we have $\widehat{G(x)} = FS_t G(x) = G(FS_t x) = G(\hat{x})$. Moreover, since.

(3.13) $$\| S_h x \| \leq Ce^{|h|} \| x \| \text{ for all } h \in R, x \in X$$

(cf. [3], Theorem 5.8), we find

(3.14) $$\| \kappa G(x) - \kappa G(y) \| = \| G(\hat{x}) - G(\hat{y}) \| \leq CM \| F \| e^{|t|} \| x - y \|$$

for $x, y \in X$

Indeed, $\| \kappa G(x) - \kappa G(y) \| = \| G(\hat{x}) - G(\hat{y}) \| = \| G(FS_t x) - G(FS_t y) \| \leq M \| FS_t x - FS_t y \| \leq CM \| F \| e^{|t|} \| x - y \|$.

Observe that the function

(3.15) $$L(t) = CM \| F \| e^{|t|} \ (t \in R)$$

is a non-negative locally integrable function of real variable.

On the other hand the problem (3.11) is equivalent to the equation

(3.16) $$x = RG(x) + x_0, x_0 \in \ker D$$

Apply to both sides of Equation (3.16) the canonical mapping κ. Then by our assumptions and Formulae (3.7), (3.8), (3.14) we get

$$\hat{x}(t) = FS_t RG(x) + FS_t x_0 = \int_0^t FS_\tau G(x) d\tau + Fx_0 =$$

$$= \int_0^t G(FS_\tau x) d\tau + x_0 = \int_0^t G(\hat{x}(\tau)) d\tau + x_0$$

where $x_0 = \hat{x}(0)$. All assumptions of Theorem (2.1) (with $h(t) \equiv t$ and $a = 0$, $N = \kappa G$ are satisfied. We therefore conclude that the equation

(3.17) $$\hat{x}(t) = \int_0^t (\hat{x}(\tau))\, d\tau + x_0$$

has a unique solution which is the limit (in norm) of the sequence of successive approximations:

(3.18) $$\hat{x} = \lim_{n \to \infty} \hat{x}_n, \text{ where } \hat{x}_{n+1}(t) = \int_0^t (\hat{x}_n(\tau))\, d\tau + x_0 \; (n = 0, 1, 2, \ldots)$$

for $t \in [0, T]$, where $T > 0$ is arbitrarily fixed.

But the canonical mapping separates points. This means that Equation (3.16), hence also the problem (3.11), we started with, has a unique solution, which is the limit (in norm) of a sequence of successive approximations:

(3.19) $$x = \lim_{n \to \infty} x_n, \text{ where } x_{n+1} = RG(x_n) + x_0 \; (n = 0, 1, 2, \ldots).$$

Example 3.1. Consider a non-linear problem of the Darboux type:

(3.20) $$\frac{\partial^2 x(t, s)}{\partial t\, \partial s} = G(t, s, x(t, s)) \text{ in } \Omega = [0, a] \times [0, b]$$

(3.21) $$x(t, 0) = \sigma(t), x(0, s) = \omega(s) \text{ for } t \in [0, a], s \in [0, b]$$

where the function $G(t, s, x)$ determined for $t, s \in \Omega$, x belonging to a Banach space E satisfies the Lipschitz condition:

(3.22) $$\| G(t, s, x) - G(t, s, y) \|_E \leqslant L(t) \| x - y \|_E \text{ for } x, y \in E,$$

L is a non-negative, locally integrable function, $\sigma \in C([0, a], E)$, $\omega \in C([0, b], E)$ and $\sigma(0) = \omega(0) = 0$.

The operator $D = \partial^2 / \partial t\, \partial s$ is right invertible and closed in the space $C(\Omega)$. The conditions (3.21) induce an initial operator F of the form

(3.23) $$(Fx)(t, s) = x(t, 0) + x(0, s) - x(0, 0)$$

corresponding to a Volterra right inverse $R = \int_0^t \int_0^s$. Since $C(\Omega)$ is a Banach space and R is quasi-nilpotent we can consider only a family of R-shifts, which is a strongly continuous group and

$$(3.24) \qquad \| S_h x \| \leqslant Ce^{|h|} \| x \| \text{ for } x \in X, h \in \mathbb{R}.$$

(cf. Theorem 5.8 in [3]).

We may write (cf. Example 4.7 in [3]):

$$(3.25) \qquad S_h F = \exp\left(t \int_0^s\right) F_1 + \exp\left(s \int_0^t\right) \int_0^t F_2$$

where $(F_1 x)(t, s) = x(t, 0), (F_2 x)(t, s) = x(0, s)$, for $x \in C(\Omega)$.

All assumption of Theorem 3.1 are satisfied. We therefore conclude that the problem (3.20) – (3.21) has a unique solution which is a limit of a sequence of successive approximations:

$$x = \lim_{n \to \infty} x_n, \text{ where } x_0(t, s) = \sigma(t) + \omega(s)$$

$$x_{n+1}(t, s) = \int_0^t \int_0^s G(u, v, x_n(u, v))\, dv\, du + x_0, \text{ for } n = 0, 1, 2, \ldots$$

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STRESZCZENIE

W pracy tej podano pewne uogólnienie klasycznego już twierdzenia Bieleckiego z 1956 r., znacznie rozszerzającego zakres stosowalności metody Banacha-Caccioppoli-Tichonowa. Podano również zastosowanie uogólnionego twierdzenia Bieleckiego do równań hiperbolicznych oraz równań z przesuniętym argumentem.

РЕЗЮМЕ

В работе дается некоторое обобщение, классической уже теоремы А. Белецкого из 1956 г., значительно расширяющие область применимости метода Банаха-Качиопполи-Тихонова. Одновременно приводятся некоторые применения обобщенной теоремы к гиперболическим уравнениям и уравнениям с отклоняющимся аргументом.