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A Finite Difference Analogue to the Problem of Zofia Szmydt

Analogon problemu Zofii Szmydt w metodzie różnic skończonych

Аналог проблемы Софии Шмыдт в методе конечных разностей

In this paper we will consider a finite difference approximation of the well-known Z. Szmydt problem (cf. [3], e.g.). To do this we use an operator calculus found in [6]. This is another example for the possibility of an unified treatment of differential and difference problems proposed in [9].

1. Elements of an operator calculus. Let be given the linear spaces L^0 and L^1. According to [6] we fix the following definitions and statements.

Definition 1. An operator S belonging to $L(L^1, L^0)$ with $S(L^1) = L^0$ is called an (algebraic) derivative. Each operator $T \in L(L^0, L^1)$ which satisfies $ST = \mathrm{id}\,|_{L^0}$ is called an (algebraic) integral with respect to S. The operator $s = \mathrm{id}\,|_{L^1} - TS$ is then the boundary condition corresponding to S and T.

Theorem 1. *The differential equation problem*

$$Su = f,\ f \in L^0 \tag{1}$$
$$su = u_0,\ u_0 \in \operatorname{Ker} S$$

has the solution

$$u = u_0 + Tf. \tag{2}$$

We will omit examples to illustrate the notions above, they will be found in [6] or [5]. Suppose that there are given two isomorphic mappings $\psi_0 : L^0 \to \bar{L}^0$ and $\psi_1 : L^1 \to \bar{L}^1$, where $\bar{L}^0$ and $\bar{L}^1$ are linear spaces, too.

Definition 2. The operator $\bar{S} = \psi_0 S \psi_1^{-1}$, $\bar{T} = \psi_1 T \psi_0^{-1}$ and $\bar{s} = \psi_1 s \psi_1^{-1}$ will be called equivalent derivative, integral and boundary condition with respect to S, T and s.

Theorem 2. *Problem* (1) *is equivalent to*

$$\bar{S}\bar{u} = \bar{f}, \bar{f} = \psi_0 f \in \bar{L}^0 \tag{3}$$
$$\bar{s}\bar{u} = \bar{u}_0, \bar{u}_0 = \psi_1 \bar{u}_0 \in \operatorname{Ker} \bar{S}.$$

The image of (2) *with respect to* ψ_1 *is the unique solution of this problem.*

2. Preliminary definitions and results. Let us denote by $D(R^2)$ the space $C_0^\infty(R^2)$ with the usual topology. By $D'(R^2)$ we realize the space of linear and continuous functionals on $D(R^2)$. The elements of $D'(R^2)$ are called distributions. For details in notions and theorems see [7], e.g. We will consider the following problem: Find a distribution $u^{(h)}$ satisfying

$$\bar{\partial}_\xi \bar{\partial}_\eta u^{(h)} = f(\xi, \eta), f \in D'(R^2), \tag{4}$$

where the distribution $\bar{\partial}_\xi \bar{\partial}_\eta u^{(h)}$ is defined by

$$(\bar{\partial}_\xi \bar{\partial}_\eta u^{(h)}, \phi) = (u^{(h)}, \partial_\xi \partial_\eta \phi), \phi \in D(R^2),$$

with $(\partial_\xi \partial_\eta \phi)(\xi, \eta) = (\phi_h^h(\xi, \eta) - \phi_h(\xi, \eta) - \phi^h(\xi, \eta) + \phi(\xi, \eta)) / h^2$, $h > 0$ is a discretization parameter. We used the abbreviation $\phi_{kh}^{jh}(\xi, \eta) = \phi(\xi + kh, \eta + jh)$, j and k are integers.

Problem (4) is an approximation of

$$u''_{\xi\eta} = f(\xi, \eta) \tag{5}$$

and must be interpreted as a suitable extension of the usual numerical problem

$$u_{k,j}^{(h)} - u_{k-1,j}^{(h)} - u_{k,j-1}^{(h)} + u_{k-1,j-1}^{(h)} = h^2 f(kh, jh),$$

which is a possible discretization of (5) by backward finite difference formulaes using h as a discretization parameter for both directions. We are interested in boundary conditions assuring existence and uniqueness of solutions of (4).

Definition 3. The distribution $E^{(h)}$ will be called a fundamental solution of (4) if it satisfies the equation

$$\bar{\partial}_\xi \bar{\partial}_\eta E^{(h)} = \delta,$$

where δ is the Dirac distribution.

The following fact is easy to show.

Theorem 3. *The distribution*

$$E^{(h)}(\xi, \eta) = h^2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \delta(\xi - kh, \eta - jh)$$

is a fundamental solution of (4).

In chapter 6 we will show how fundamental solutions for the operator $\bar{\partial}_\xi \bar{\partial}_\eta + \lambda^2$ can be constructed.

Theorem 4. *If* $E^{(h)}$ *is a fundamental solution of* (4) *and if the convolution* $u^{(h)} = E^{(h)} * f$ *exists, then* $u^{(h)}$ *is a solution of* (4).

Proof. Let the convolution $u * v$ exist. Then $\bar{\partial}_\xi(u * v) = u * \bar{\partial}_\xi v = (\bar{\partial}_\xi u) * v$. The same equality holds if we replace the differences with respect to ξ by such ones with respect to η. Therefore,

$$\bar{\partial}_\xi \bar{\partial}_\eta (E^{(h)} * f) = (\bar{\partial}_\xi \bar{\partial}_\eta E^{(h)}) * f = f.$$

If $E^{(h)}$ is the fundamental solution given in Theorem 3 we get

$$u^{(h)}(\xi, \eta) = (E^{(h)} * f)(\xi, \eta) = h^2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f(\xi - kh, \eta - jh).$$

Let $\alpha = \alpha(\xi)$ and $\beta = \beta(\eta)$ be two curves in the (ξ, η)-plane satisfying the condition

$$\bigwedge_{\nu \in \Gamma} \bigvee_{\mu, \omega \in \Gamma} \alpha(\nu h) = \mu h, \ \beta(\nu h) = \omega h$$

and let for fixed (x, y) the region

$$\Omega^0_{(x,y)} = \{(\xi, \eta) : \alpha(x) + h/2 < \eta \leqslant y + h/2, \beta(\eta) + h/2 < \xi \leqslant x + h/2\}$$

be not empty. For the sake of simplicity we assume that $(x, y) = (Kh, Jh)$, where J and K are integers.

To solve our problem we define

$$L^0 = \{f \in D'(R^2) : \operatorname{supp} f \subset \Omega^0_{(x,y)}, f \in \mathcal{L}(\Omega^0_{(x,y)})\}$$

$$L^1 = \{u \in D'(R^2) : \operatorname{supp} u \subset \Omega^1_{(x,y)}, u \in \mathcal{L}(\Omega^1_{(x,y)})\},$$

where $\Omega^1_{(x,y)} = (\xi, \eta) : \alpha(x) - h/2 < \eta \leqslant y + h/2, \beta(\eta) - h/2 < \xi \leqslant x + h/2$.

Further, Theorem 4 suggests the following definitions:

$$S: L^1 \to L^0, S: u \to (\bar{\partial}_\xi \bar{\partial}_\eta u) \mid \Omega^{\circ}_{(x,y)}$$

(6)

$$T: L^0 \to L^1, T: f \to E^{(h)} * f.$$

For $f \in L^0$ we have

$$STf = (\bar{\partial}_\xi \bar{\partial}_\eta (E^{(h)} * f)) \mid \Omega^{\circ}_{(x,y)} = f \mid \Omega^{\circ}_{(x,y)} = f.$$

Similar like in [3] we derive a fundamental formulae.

Theorem 5. *For $u \in D'(R^2) \cap \mathcal{L}(\Omega^1_{(x,y)})$ the following identity holds:*

(7)
$$\iint_{\Omega^1_{(x,y)}} u\, \partial_\xi \partial_\eta \phi\, d\xi\, d\eta - \iint_{\Omega^0_{(x,y)}} \bar{\partial}_\xi \bar{\partial}_\eta u \phi\, d\xi\, d\eta =$$

$$= -(1/h) \int_{\alpha(x)+h/2}^{y+h/2} \int_{x-h/2}^{x+h/2} \phi_h \bar{\partial}_\xi u d\xi d\eta - (1/h) \int_{\alpha(x)-h/2}^{\alpha(x)+h/2} \int_{\beta(\eta)-h/2}^{x+h/2} u \partial_\xi \phi\, d\xi\, d\eta +$$

$$+ (1/h) \int_{y-h/2}^{y+h/2} \int_{\beta(\eta)-h/2}^{x+h/2} u \partial_\xi \phi^h\, d\xi\, d\eta - (1/h) \int_{\alpha(x)+h/2}^{y+h/2} \int_{\beta(\eta-h)+h/2}^{\beta(\eta)+h/2} \phi \bar{\partial}_\xi u^{-h}\, d\xi\, d\eta +$$

$$+ (1/h) \int_{\alpha(x)+h/2}^{y+h/2} \int_{\beta(\eta)-h/2}^{\beta(\eta)+h/2} \phi\, \bar{\partial}_\eta u d\xi\, d\eta +$$

$$+ (1/h^2) \int_{\alpha(x)+h/2}^{y+h/2} \Big(\int_{\beta(\eta)-h/2}^{\beta(\eta)+h/2} u^{-h} \phi\, d\xi - \int_{\beta(\eta-h)-h/2}^{\beta(\eta-h)+h/2} u^{-h} \phi\, d\xi \Big)\, d\eta.$$

By means of distributions (7) *reads*

(8)
$$\bar{\partial}_\xi \bar{\partial}_\eta u \mid \Omega^1_{(x,y)} - (\bar{\partial}_\xi \bar{\partial}_\eta u) \mid \Omega^{\circ}_{(x,y)} = V(u),$$

where $V(u)$ is a distribution defined by informations of u on the 'boundary' of $\Omega^0_{(x,y)}$ and given at the right-hand side of (7). *If h tends to zero and if u is sufficiently smooth then* (7) *turns over to the fundamental formulae known from Riemann's method* (cf. [3]).

Theorem 6. *Each distribution $u \in D'(R^2) \cap \mathcal{L}(\Omega^1_{(x,y)})$ can be represented as*

(9)
$$u \mid \Omega^1_{(x,y)} = E^{(h)} * ((\bar{\partial}_\xi \bar{\partial}_\eta u) \mid \Omega^{\circ}_{(x,y)} + V(u)),$$

where $E^{(h)}$ is the fundamental solution defined in Theorem 3.

Proof. We have

$$u_{|\Omega^1_{(x,y)}} = \delta * u_{|\Omega^1_{(x,y)}} = \bar{\delta}_\xi \bar{\delta}_\eta E^{(h)} * u_{|\Omega^1_{(x,y)}} = E^{(h)} * \bar{\delta}_\xi \bar{\delta}_\eta u_{|\Omega^1_{(x,y)}}.$$

Identity (8) finishes the proof.

3. The problem of Z. Szmydt – existence, uniqueness and convergence results. Following Theorem 1 and our definitions (6) we have to define the boundary condition

$$su = u - TSu,\ u \in L^1.$$

In view of (9) we get

$$su = E^{(h)} * V(u).$$

The question now is: Which values of u on the „boundary" are necessary to describe su at the point (x, y)? To this aim we assume that there is given a function $\phi \in D(R^2)$ with supp $\phi \subset U(x, y)$, where U is a sufficiently small neighbourhood of (x, y). Via (7) we calculate

$$(su, \phi) = (E^{(h)} * V(u), \phi) =$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \Bigg\{ \int_{\alpha(x)-h/2+jh}^{\alpha(x)+h/2+jh} \int_{\beta(\eta-jh)-h/2+kh}^{x+h/2+kh} u_{-kh}^{-jh} \phi \, d\xi \, d\eta +$$

$$+ \int_{\alpha(x)+h/2+jh}^{y+h/2+jh} \int_{\beta(\eta-jh)-h/2+kh}^{\beta(\eta-jh)+h/2+kh} u_{-kh}^{-jh} \phi \, d\xi \, d\eta -$$

$$- \int_{\alpha(x)+h/2+jh}^{y+h/2+jh} \int_{\beta(\eta-h-jh)-h/2+kh}^{\beta(\eta-jh)+h/2+kh} u_{-kh}^{-h-jh} \phi \, d\xi \, d\eta -$$

$$- \int_{\alpha(x)-h/2+jh}^{\alpha(x)+h/2+jh} \int_{\beta(\eta-jh)+h/2+kh}^{x+3h/2+kh} u_{-h-kh}^{-jh} \phi \, d\xi \, d\eta +$$

$$+ \int_{\alpha(x)+h/2+jh}^{y+h/2+jh} \int_{\beta(\eta-jh-h)+h/2+kh}^{\beta(\eta-jh)+h/2+kh} u_{-h-kh}^{-h-jh} \phi \, d\xi \, d\eta \Bigg\}.$$

Therefore,

$$(su)(x,y) = (su)(Kh, Jh) = \sum_{k=0}^{K-\beta(\alpha(x))/h} u(x-kh, \alpha(x)) +$$

$$+ \sum_{j=0}^{J-1-\alpha(x)/h} u(\beta(y-jh), y-jh) -$$

$$- \sum_{j=0}^{J-1-\alpha(x)/h} \sum_{k=K-\beta(y-jh)/h}^{K-\beta(y-jh-h)/h} u(x-kh, y-jh-h) -$$

$$- \sum_{k=0}^{K-1-\beta(\alpha(x))/h} u(x-kh-h, \alpha(x)) +$$

$$+ \sum_{j=0}^{J-1-\alpha(x)/h} \sum_{k=K-\beta(y-jh)/h}^{K-1-\beta(y-jh-h)/h} u(x-kh-h, y-jh-h),$$

that is

$$(su)(x,y) = u(x, \alpha(x)) + h \sum_{j=0}^{J-1-\alpha(x)/h} \bar{\partial}_\eta u(\beta(y-jh), y-jh). \tag{10}$$

We are able now to formulate the problem of Z. Szmydt for the difference equation (4) and give its solution: Find a function $u^{(h)}$ satisfying the functional equation

$$\bar{\partial}_\xi \bar{\partial}_\eta u = f(\xi,\eta), (\xi,\eta) \in \Omega, \operatorname{supp} f \subset \Omega \tag{11}$$

and the boundary conditions

$$\begin{aligned} u(\xi,\eta) &= g_0(\xi), \alpha(\xi) - h/2 < \eta \leq \alpha(\xi) + h/2 \\ \bar{\partial}_\eta u(\xi,\eta) &= g_1(\eta), \beta(\eta) - h/2 < \xi \leq \beta(\eta) + h/2. \end{aligned} \tag{12}$$

For the sake of simplicity we assume that the given f, g_0 and g_1 are at least continuous and that for every point $(\xi,\eta) \in \Omega$ the domain $\Omega^0_{(\xi,\eta)}$ is not empty.

Theorem 7. *There exists a unique solution of* (11), (12). *Its values at the gridpoints* $(\xi,\eta) = (Kh, Jh)$ *are given by*

$$u^{(h)}(\xi,\eta) = g_0(\xi) + h \sum_{j=0}^{J-1-\alpha(\xi)/h} g_1(\eta - jh) + \tag{13}$$

$$+ h^2 \sum_{j=0}^{J-1-\alpha(\xi)/h} \; \sum_{k=0}^{K-1-\beta(\eta-jh)/h} f(\xi - kh, \eta - jh).$$

Similar formulae we have for the other points of Ω.

The proof follows immediately from Theorem 1, Chapter 2 and the considerations above.

Like in the „continuous" case we can formulate some other problems resulting from the Z. Szmydt problem, namely

i) the Darboux problem

Here we set $\alpha(\xi) = \eta_0 = \alpha h$, $\beta(\eta) = \xi_0 = \beta h$ and the boundary conditions are

$$u(\xi,\eta) = g_0(\xi), \eta_0 - h/2 < \eta \leqslant \eta_0 + h/2$$

$$u(\xi,\eta) = g_1(\eta), \xi_0 - h/2 < \xi \leqslant \xi_0 + h/2.$$

From (10) we calculate

$$(su)(\xi,\eta) = u(\xi, \alpha(\xi)) + \sum_{j=0}^{J-1-\alpha} (u(\beta h, \eta - jh) - u(\beta h, \eta - jh - h)) =$$

$$= u(\xi, \eta_0) + u(\xi_0, \eta) - u(\xi_0, \eta_0).$$

ii) the Cauchy problem

Let us assume that $\eta = \alpha(\xi) \Longleftrightarrow \xi = \beta(\eta)$ holds. Then

$$(su)(\xi,\eta) = u(\xi, \alpha(\xi)) + h \sum_{j=0}^{J-1-\alpha(\xi)/h} \bar{\partial}_\eta u(\xi_j, \alpha(\xi_j)),$$

where $\xi_j = \beta(\eta - jh)$ and we get the initial conditions

$$u(\xi,\eta) = g_0(\xi), \alpha(\xi) - h/2 < \eta \leqslant \alpha(\xi) + h/2$$

$$\bar{\partial}_\eta u(\xi,\eta) = g_1(\xi), \alpha(\xi) - h/2 < \eta \leqslant \alpha(\xi) + h/2.$$

iii) the Picard problem

We set $\beta(\eta) = 0$ and obtain

$$(su)(\xi, \eta) = u(\xi, \alpha(\xi)) + u(0, \eta) - u(0, \alpha(\xi)),$$

that means the boundary conditions are

$$u(\xi, \eta) = g_0(\xi), \alpha(\xi) - h/2 < \eta \leq \alpha(\xi) + h/2$$

$$u(\xi, \eta) = g_1(\eta), -h/2 < \xi \leq h/2.$$

In all three cases we can state a result like in Theorem 7.

If h tends to zero (13) gives the well-known solution formulae of the „continuous" problem. The corresponding operators S, T and s are the same like in [6], namely

$$u(\xi, \eta) = u(\xi, \alpha(\xi)) + \int_{\alpha(\xi)}^{\eta} u'_\eta(\beta(\rho), \rho)\, d\rho + \int_{\alpha(\xi)}^{\eta} \int_{\beta(\rho)}^{\xi} f(\sigma, \rho)\, d\sigma\, d\rho.$$

The conditions for the curves α and β were only restricted by the definition of $\Omega^0_{(x,y)}$, where, in addition, we can change the directions of the occuring inequalities. Let us consider an example of an ill-posed problem.

We are looking for a solution of

$$\bar{\partial}_\xi \bar{\partial}_\eta u = 0 \tag{14}$$

$$u(\xi, \eta) = g_0(\xi), (\partial_\eta u)(\xi, \eta) = g_1(\xi), -h/2 < \eta \leq h/2. \tag{15}$$

Solving (14) at the gridpoints we have among others

$$u(\xi + h, h) = -u(\xi, 0) + u(\xi, h) + u(\xi + h, 0) = hg_1(\xi) + g_0(\xi + h),$$

that means we must have $g_1(\xi + h) = g_1(\xi)$ because of

$$g_1(\xi + h) = (u(\xi + h, h) - u(\xi + h, 0)) / h =$$

$$= (hg_1(\xi) + g_0(\xi + h) - g_0(\xi + h)) / h = g_1(\xi).$$

Therefore, let g_1 be a constant. It is easy to verify that

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

is a solution of (14) for arbitrary F and G. It we set $F(\xi) = g_0(\xi)$ we must have $G(0) = 0$ and in view of

$$g_1(\xi) = g_1 = (G(h) - G(0)) / h$$

we get $G(h) = hg_1$. For example, we can set $G(\eta) = g_1 \eta + H(\eta)$, where $H(0) = H(h) = 0$. Finally, for each function H with $H(0) = (H(0) - H(0)) / h = 0$ the function

$$u(\xi, \eta) = g_0(\xi) + g_1 \eta + H(\eta)$$

is a solution of (14) satisfying the initial conditions (15). If h tends to zero we arrive at the well-known incorrect problem

$$u''_{\xi\eta} = 0, u(\xi, 0) = g_0(\xi), u'_\eta(\xi, 0) = g_1$$

with the solutions

$$u(\xi, \eta) = g_0(\xi) + g_1 \eta + H(\eta),$$

where H is an arbitrary function satisfying $H(0) = H'(0) = 0$.

4. A nonlinear problem. Using explicite finite difference formulae to calculate an approximative solution for the (nonlinear) problem

$$u''_{\xi\eta} = f(u)$$

$$u(\xi, \alpha(\xi)) = g_0(\xi) \tag{16}$$

$$u'_\eta(\beta(\eta), \eta) = g_1(\eta)$$

we come to

$$\partial_\xi \partial_\eta u = f(u), (\xi, \eta) \in \Omega \tag{17}$$

$$\begin{aligned} u(\xi, \eta) &= g_0(\xi), \alpha(\xi) - h/2 < \eta \leqslant \alpha(\xi) + h/2 \\ \partial_\eta u(\xi, \eta) &= g_1(\eta), \beta(\eta) - h/2 < \xi \leqslant \beta(\eta) + h/2. \end{aligned} \tag{18}$$

Similar like in the preceding chapters we derive a solution formulae, namely

$$\begin{aligned} u^{(h)}(\xi, \eta) &= g_0(\xi) + h \sum_{j=\alpha(\xi)/h}^{J-1} g_1(jh) + \\ &+ h^2 \sum_{j=\alpha(\xi)/h}^{J-1} \sum_{k=\beta(jh)/h}^{K-1} f(u^{(h)}(kh, jh)), \end{aligned} \tag{19}$$

where $(\xi, \eta) = (Kh, Jh)$. We note that the solution of (17), (18) exists and is unique if we demand assumptions about α and β closed to that one in Chapter 2 and if Ω is such a domain that the values of $u^{(h)}$ occuring on the right-hand side of (19) can be obtained from (17) by formulae of type (19). For our further considerations we will identify $u^{(h)}$ with the function which arises using linear interpolation over the gridpoint values $u^{(h)}(\xi;\eta)$ given by (19).

Theorem 8. *Let f and g_1 be bounded and continuous, g_0 and α continuously differentiable. Then the sequence $\{u^{(h)}\}$, $h \to 0$, is compact in the space of continuous functions (with respect to any bounded subset of Ω), the limits of the converging subsequences are solutions of* (16).

Proof. We have for sufficiently small h

$$|u^{(h)}(\xi_1,\eta_1) - u^{(h)}(\xi_2,\eta_2)| \leqslant$$

$$\leqslant |g_0(x_1) + h \sum_{j=\alpha(x_1)/h}^{J_1-1} g_1(jh) + h^2 \sum_{j=\alpha(x_1)/h}^{J_1-1} \sum_{k=\beta(jh)/h}^{K_1-1} f(u^{(h)}(kh, jh)) -$$

$$- g_0(x_2) - h \sum_{j=\alpha(x_2)/h}^{J_2-1} g_1(jh) - h^2 \sum_{j=\alpha(x_2)/h}^{J_2-1} \sum_{k=\beta(jh)/h}^{K_2-1} f(u^{(h)}(kh, jh))| \leqslant$$

$$\leqslant C(\alpha,\beta,g_0,g_1,f) \max(|\xi_1 - \xi_2|, |\eta_1 - \eta_2|),$$

where $(x_i, y_i) = (K_i h, J_i h)$ are suitable gridpoints in the neighbourhood of (ξ_i, η_i), $i = 1,2$. Using Arzela's Theorem we have the compactness of our sequence in the maximum-norm, because the $u^{(h)}$ are equi-bounded. Taking a suitable subsequence $h \to 0$ we obtain

$$u^{(h)}(\xi,\eta) \to u(\xi,\eta) = g_0(\xi) + \int_{\alpha(\xi)}^{\eta} g_1(\sigma)\,d\sigma + \int_{\alpha(\xi)}^{\eta} \int_{\beta(\sigma)}^{\xi} f(u(\rho,\sigma))\,d\rho\,d\sigma.$$

Of course, u is a solution of (16), moreover, if f is Lipschitz continuous then u is locally unique.

Using fixed-point techniques we can discuss further questions connected with the Z. Szmydt problem for finite differences.

5. Connection with the one-dimensional wave equation. A possible discretization of

(20) $$\bar{u}''_{tt}(x,t) - a^2 \bar{u}''_{xx}(x,t) = \bar{f}(x,t), \; t > 0$$

may be

(21) $$(\bar{S}\bar{u})(x,t) = \partial_t \bar{\partial}_t \bar{u} - a^2 \partial_x \bar{\partial}_x \bar{u} = \bar{f}(x,t), \; t > -\tau/2,$$

where $(\partial_t \bar{\partial}_t \bar{u})(x, t) = (\bar{u}(x, t + \tau) - 2\bar{u}(x, t) + \bar{u}(x, t - \tau))/\tau^2$ and $(\partial_x \bar{\partial}_x \bar{u})(x, t) = (\bar{u}(x + h, t) - 2\bar{u}(x, t) + \bar{u}(x - h, t))/h^2$.

We will assume that $h = a\tau$. To construct suitable initial conditions for (21) we define the isomorphic mappings ψ_0, ψ_1 acting as follows:

$$f(\xi, \eta) = (\psi_0^{-1} \bar{f})(\xi, \eta) = \bar{f}((\eta - \xi)/2, (\eta + \xi)/2a),$$

$$\bar{f}(x, t) = (\psi_0 f)(x, t) = f(-x + at, x + at)$$

and

$$u(\xi, \eta) = (\psi_1^{-1} \bar{u})(\xi, \eta) = 4a^2 \bar{u}((\eta - \xi)/2, (\eta + \xi)/2a + \tau)),$$

$$\bar{u}(x, t) = (\psi_1 u)(x, t) = (1/4a^2) u(-x + at - h, x + at - h).$$

From Definition 2 we have for the equivalent derivative with respect to ψ_0, ψ_1

$$S = \psi_1 \bar{S} \psi_0^{-1}$$

and therefore (by suitable calculations)

$$(Su)(\xi, \eta) = \bar{\partial}_\xi \bar{\partial}_\eta u(\xi, \eta)$$

where the stepsize here is $2h$. We define the curves $\alpha(\xi) = -\xi - 2h$ and $\beta(\eta) = -\eta - 2h$. Knowing the integral T of S given by (6) and (13) at the gridpoints, namely

$$(Tf)(\xi, \eta) = (2h)^2 \sum_{j=0}^{(\xi+\eta)/2h} \; \sum_{k=0}^{(\xi+\eta)/2h - 1 - j} f(\xi - kh, \eta - jh)$$

we obtain the integral $\bar{T}$ of $\bar{S}$ according to

$$(\bar{T}\bar{f})(x, t) = (\psi_1 T \psi_0^{-1} \bar{f})(x, t) =$$

$$= \tau^2 \sum_{j=0}^{\bar{j}=1} \; \sum_{k=0}^{t/\tau - 1 - j} f(x - jh + kh, t - (j + 1)\tau - k\tau)$$

and changing the summation

(22) $$(\bar{T}\bar{f})(x, t) = (h\tau)/a \sum_{\sigma=0}^{t/\tau - 1} \sum_{\rho} \bar{f}(\rho h, \sigma\tau),$$

where ρ steps with stepsize 2 from $x/h - t/\tau + \sigma + 1$ to $x/h + t/\tau - \sigma - 1$. Here we took into consideration that (x, t) is a gridpoint iff (ξ, η) is a gridpoint.

If $h = a\tau$ tends to zero we have

$$(23)\qquad (\bar{T}\bar{f})(x,t) \to (1/2a)\int_0^t \int_{x-at+a\sigma}^{x+at-a\sigma} \bar{f}(\rho,\sigma)\,d\rho\,d\sigma.$$

For sufficiently smooth $\bar{f}$ the integral on the right-hand side is a solution of (20). To construct the boundary conditions for $\bar{S}$, $\bar{T}$ we use

$$(su)(\xi,\eta) = u(\xi, -\xi - 2h) + 2h \sum_{j=0}^{(\xi+\eta)/2h} (\bar{\partial}_\eta u)(-\eta + 2(j-1)h, \eta - 2jh)$$

given by (10). Elementary calculations show that

$$(24)\qquad (\bar{s}\bar{u})(x,t) = \bar{u}(x - at, 0) + h\sum_\rho (\partial_x \bar{u})(\rho h, 0) + \tau \sum_\rho (\bar{\partial}_t \bar{u})(\rho h, 0),$$

where ρ steps with stepsize 2 from $x/h - t/\tau + 1$ to $x/h + t/\tau - 1$. If h tends to zero we get now

$$(\bar{s}\bar{u})(x,t) \to \bar{u}(x - at, 0) + 1/2 \int_{x-at}^{x+at} \bar{u}'_x(\rho,0)\,d\rho + 1/2a \int_{x-at}^{x+at} \bar{u}'_t(\rho,0)\,d\rho =$$

$$(25)\qquad = (\bar{u}(x+at,0) + \bar{u}(x-at,0))/2 + 1/2a \int_{x-at}^{x+at} \bar{u}'_t(\rho,0)\,d\rho.$$

The last term is a solution of the homogenous equation (20), if the corresponding derivatives exist.

Collecting the preceding results we have proved the following statement (cf. [1]).

Theorem 9. *Equation (21) possesses in connection with the initial conditions*

$$\bar{u}(x,t) = \bar{g}_0(x), \ -\tau/2 < t \leq \tau/2$$

$$\bar{\partial}_t \bar{u}(x,t) = \bar{g}_1(x), \ -\tau/2 < t \leq \tau/2$$

a unique solution $\bar{u}^{(h)}$, $h = a\tau$, given at the gridpoints by (22), (24) and

$$\bar{u}^{(h)}(x,t) = (\bar{s}\bar{u}^{(h)})(x,t) + (\bar{T}\bar{f})(x,t).$$

If h approaches zero the sequence $\bar{u}^{(h)}(x,t)$ converges for each (x,t) to the solution of (20) given by (23), (25) with the initial conditions

$$\bar{u}(x, 0) = \bar{g}_0(x)$$

$$\bar{u}'_t(x, 0) = \bar{g}_1(x),$$

if the right-hand sides $\bar{f}$, $\bar{g}_0$ and $\bar{g}_1$ are sufficiently smooth (cf. [3]).

6. The construction of fundamental solutions for $\bar{\partial}_\xi \bar{\partial}_\eta + \lambda^2$. In connection with Definition 3 we are looking for a solution $E^{(h)}$ satisfying

$$\bar{\partial}_\xi \bar{\partial}_\eta E^{(h)} + \lambda^2 E^{(h)} = \delta. \tag{26}$$

Applying the Fourier transformation given by the formulae (cf. [2])

$$(FE^{(h)}, F\phi) = (2\pi)^2 (E^{(h)}, \phi)$$

$$(F\phi)(s, t) = \iint_{R^2} e^{i(s\xi + t\eta)} \phi(\xi, \eta)\, d\xi\, d\eta, \phi \in D(R^2),$$

on both sides of (26) we obtain the problem

$$P^{(h)}(s, t) FE^{(h)} = 1, \tag{27}$$

where

$$P^{(h)}(s, t) = \frac{(1 - e^{ihs})(1 - e^{iht})}{h^2} + \lambda^2.$$

Equation (27) possesses the formal solution $FE^{(h)} = 1/P^{(h)}$. Because $P^{(h)}$ vanishes for certain s, t we have to interpret $1/P^{(h)}$ in a suitable way. We will do this in an analoguous manner like in [9], the main idea can be found for example in [4], where fundamental solutions for differential operators are constructed.

We assume for a moment that λ is a positive real number.

Theorem 10. *The distribution $FE^{(h)}$ defined by*

$$(FE^{(h)}, F\phi) = \iint_{R^2} \frac{(F\phi)(s + i\sigma, t + i\sigma)}{P^{(h)}(s + i\sigma, t + i\sigma)}\, ds\, dt \tag{28}$$

with $\sigma = \sigma(h)$ from $e^{\sigma h} = 1 + \lambda h$, is a solution of (27) *and, therefore, $E^{(h)}$ is a solution of* (26). *If h tends to zero $E^{(h)}$ converges to a fundamental solution E of the corresponding differential operator $\partial^2 / \partial_\xi \partial_\eta + \lambda^2$.*

Proof. By suitable calculations we obtain

$$|P^{(h)}(s+i\sigma, t+i\sigma)| = \left| \frac{(1+\lambda h - e^{ihs})(1+\lambda h - e^{iht})}{(1+\lambda h)^2 h^2} + \lambda^2 \right| \geqslant \lambda^2 \left(1 + \frac{1}{(1+\lambda h)^2}\right),$$

that means, (28) defines a distribution acting on $F\phi$. Because $P^{(h)}(s+i\sigma, t+i\sigma)$ tends to $P(s+i\lambda, t+i\lambda) = -(s+i\lambda)(t+i\lambda)+\lambda^2$ if h approaches zero we can use Lebesgue's Theorem get

$$(FE^{(h)}, F\phi) \to \iint_{R^2} \frac{(F\phi)(s+i\lambda, t+i\lambda)}{P(s+i\lambda, t+i\lambda)}\, ds\, dt \overset{\text{def}}{=} (FE, F\phi).$$

In view of the continuity of the inverse Fourier transformation we have $E^{(h)} \to E$. Further,

$$(P^{(h)} FE^{(h)}, F\phi) = (FE^{(h)}, P^{(h)} F\phi) = \iint_{R^2} (F\phi)(s+i\sigma, t+i\sigma)\, ds\, dt = (1, F\phi),$$

that is (27). Similar considerations show that the distribution FE satisfies $(-is)(-it)FE + \lambda^2 FE = 1$. Therefore, $E^{(h)}$ and E are the desired fundamental solutions.

Theorem 13. *The fundamental solution $E^{(h)}$ given in* (28) *has the form*

$$E^{(h)}(\xi, \eta) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{kj}\, \delta(\xi - kh, \eta - jh)$$

where

(29) $$c_{kj} = h^2 \sum_{m=0}^{k} \binom{j}{m}\binom{k}{m} (-1)^m (\lambda h)^{2m} (1+\lambda^2 h^2)^{-j-k-1}.$$

If $h \to 0$ we have

$$E^{(h)}(\xi, \eta) \to E(\xi, \eta) = \sum_{m=0}^{\infty} 1/(m!)^2 (-1)^m (\lambda^2 \xi\eta)^m\, Y(\xi) \otimes Y(\eta)$$

(Y is the Heaviside unit function). E is the fundamental solution found in [8].

Proof. Formulae (28) shows that $FE^{(h)}$ is a periodic distribution with period $T = (2\pi/h, 2\pi/h)$. That means (cf. [7]) that

$$E^{(h)}(\xi, \eta) = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} c_{kj}\, \delta(\xi - kh, \eta - jh),$$

where the c_{kj} are the generalized Fourier coefficients of $1/P^{(h)}$ given by

$$c_{kj} = \frac{h^2}{(2\pi)^2} \int_0^{2\pi/h} \int_0^{2\pi/h} \frac{e^{-ikh(s+i\sigma)}\, e^{-ijh(t+i\sigma)}}{P^{(h)}(s+i\sigma,\, t+i\sigma)}\, ds\, dt.$$

The substitution $u = e^{-ih(s+i\sigma)}$, $v = e^{-ih(t+i\sigma)}$ leads to

$$c_{kj} = \frac{h^2}{(2\pi i)^2} \oint_{|v|=e^{h\sigma}} \oint_{|u|=e^{h\sigma}} \frac{u^k v^j}{(u - u_0(v))(v - v_0)(1 + \lambda^2 h^2)}\, du\, dv$$

with $u_0(v) = \dfrac{v - 1}{v(1 + \lambda^2 h^2) - 1}$ and $v_0 = \dfrac{1}{1 + \lambda^2 h^2}$. If $|v| = e^{\sigma h} = 1 + \lambda h$, then

(30) $$|u_0(v)| \leqslant 1 + \frac{\lambda^2 h^2 |v|}{|v|(1 + \lambda^2 h^2) - 1} < 1 + \lambda h = e^{\sigma h}.$$

In accordance to (30) we calculate for $k \geqslant 0$ and $j \geqslant 0$

$$c_{kj} = \frac{h^2}{2\pi i} \oint_{|v|=e^{\sigma h}} \frac{v^j (v-1)^k}{(v - v_0)^{k+1} (1 + \lambda^2 h^2)^{k+1}}\, dv =$$

$$= h^2 \frac{1}{k!} \frac{1}{(1 + \lambda^2 h^2)^{k+1}} \sum_{m=0}^{k} \binom{k}{m} \frac{d^m}{dv^m} v^j \Big|_{v=v_0} \frac{d^{k-m}}{dv^{k-m}} (v-1)^k \Big|_{v=v_0}.$$

This is already (29) because $c_{kj} = c_{jk}$ and

$$c_{kj} = \frac{h^2}{2\pi i} \oint_{|v|=e^{\sigma h}} \frac{v^j}{(v - v_0)(1 + \lambda^2 h^2)} \Big(\frac{1}{(-k-1)!} \frac{d^{-k-1}}{du^{-k-1}} \Big(\frac{1}{u - u_0(v)} \Big)_{|u=0} +$$

$$+ (u_0(v))^k \Big)\, dv = 0$$

for $k < 0$.
To show the covergence of $E^{(h)}$ to E we pay attention to

$$(E^{(h)}, \phi) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{kj}\, \phi(kh, jh) \to \iint_{R^2} E\phi\, d\xi\, d\eta = (E, \phi).$$

We notice that (29) defines a fundamental solution of $\bar{\partial}_\xi\, \bar{\partial}_\eta + \lambda^2$ for every complex number λ such that $1 + \lambda^2 h^2$ does not vanish.

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STRESZCZENIE

W pracy tej znajduje się przybliżone rozwiązanie problemu Z. Szmydt przy pomocy równań różnicowych. W tym celu stosuje się formalizm użyty w monografii Bittnera [6]. Jest to dalszy przykład analogii pomiędzy równaniami różnicowymi i różniczkowymi.

РЕЗЮМЕ

В этой работе дается приближенное решение задачи С. Шмыдт при использовании разностных уравнений. Для этой цели использован формализм из монографии Р. Биттнера [6]. Это служит очередным примером аналогии между разностными и дифференциальными уравнениями.