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The Degree Theory for Local Condensing Mappings

Teoria stopnia topologicznego dla odwzorowań wielowartościowych, lokalnie ściągających

Теория индекса для многозначных локально сжимающих отображений

The present paper is a continuation of [4]. We define the topological degree for the new class of multivalued local condensing mappings and show the fixed point and odd mapping theorems.

Let G be an open subset of a Banach space x.

Definition 1. An USC mapping $T: \bar{G} \to 2^X$ (see [4]) such that $T(\bar{G})$ is bounded is called local condensing if for each $x \in \bar{G}$ there exists an open neighbourhood U_x of x such that $T/_{\bar{U}_x \cap \bar{G}}$, the restriction of T on $\bar{U}_x \cap \bar{G}$, is condensing and $T(x)$ is convex and closed.

Lemma 1. *If $T: \bar{G} \to 2^X$ is local condensing mapping and*

(1) $$\sigma_T = \{x \in G : x \in T(x)\}$$

is a compact subset of G then there exists an open bounded subset $V \subset G$ such that $\sigma_T \subset V$ and $T/_{\bar{V}}$ is condensing.

Definition 2. For local condensing mapping $T: \bar{G} \to 2^X$ such that σ_T is compact and $0 \notin (I - T)(\partial G)$ we define

(2) $$\deg(I - T, G, 0) = \deg(I - T, V, 0)$$

where T is condensing on $\bar{V}$.

Remark. If T is a condensing mapping then T is 1-set contraction. The right hand side of (2) denotes the degree in the sense of [4].

Lemma 2. *This degree* deg $(I - T, G, 0)$ *is independent of the choise of* V.

Theorem 1. *Let* $T: \bar{G} \to 2^X$ *be a local condensing mapping. Suppose that* σ_T *is compact and* $x \bar{\in} T(x)$ *for* $x \in \partial G$. *Then the above defined degree has the following properties:*

a) *if* $I - T$ *is closed mapping and* deg $(I - T, G, 0) \leqslant 0$ *then there exists* $x \in G$ *such that* $x \in T(x)$.

b) *if* G_1, G_2 *are open subsets of* G *such that* $\bar{G}_1 \cup \bar{G}_2 = \bar{G}$, $G_1 \cap G_2 = \emptyset$ *and* $0 \bar{\in} (I - T)(\partial G_i)$, $i = 1, 2$, *then* deg $(I - T, G, 0) =$ deg $(I - T, G_1, 0) +$ deg $(I - T, G_2, 0)$.

Theorem 2 (Homotopy property). *Let* $H: \bar{G} \times [0, 1] \to 2^X$ *be a mapping satisfying the following conditions:*

(i) *the set* $\sigma_H = \{x \in G: x \in H(x, t), t \in [0, 1]\}$ *is compact and* $x \bar{\in} H(x, t)$ *for all* $(x, t) \in \partial G \times [0, 1]$,

(ii) *the mapping* $t \to H(., t)$ *is continuous in the sense that for each* $t \in [0, 1]$ *and* $\epsilon > 0$ *there exists* $\delta > 0$, *such that* $\sup_{x \in \bar{G}} d^*(H(x, t), H(x, t')) < \epsilon$ *for all* $t \in [0, 1]$ *satisfying* $|t - t'| < \delta$,

(iii) H *is „local uniformly condensing" (as the mapping* $t \to H(., t)$) *i.e. for each* $(x, t) \in \bar{G} \times [0, 1]$ *there exist an open neighbourhood* $U_x \subset X$ *of* x *and an open neighbourhood* $J_t \subset R$ *of* t *such that*

$$\alpha(H(A x (J_t \cap [0, 1]))) < \alpha(A)$$

for every $A \subset U_x \cap \bar{G}$ *with* $\alpha(A) > 0$.

Then

$$\deg (I - H(., t), G, 0) = \text{const } (t).$$

(α is the measure of noncompactness, see [2]. Condition (ii) compare to d) in Theorem 2, [4]).

Remark. Condition (iii) implies, in particular, that for every $t \in [0, 1]$ mapping $H(., t)$ is a locally condensing map.

Proof of Theorem 2. First we verify that deg $(I - H(., t), G, 0)$ is constant in sufficiently small neighbourhood of any $t_0 \in [0, 1]$.

Let $x \in \bar{G}$. Choose U_x, $J_{t_0, x}$ for (x, t_0), as in (iii). We have $\bigcup_{x \in \sigma_H} U_x \supset \sigma_H$ and from compactness of σ_H there exist $U_{x_1}, \ldots, U_{x_n}$ such that $U = \bigcup_{i=1}^{n} U_{x_i} \cap \bar{G} \supset \sigma_H$. Let J_{t_0} be equal to $\bigcap_i J_{t_0, x_i} \cap [0, 1]$. For the restriction of H on $\bar{U} \times J_{t_0}$ is condensing (and so

1-set concentration) we obtain $\deg(I - H(.\,,t), G, 0) = \deg(I - H(.\,,t), U, 0) = \mathrm{const}(t)$ for $t \in J_{t_0}$, (see [4]). It gives that the degree is constant on whole interval $[0, 1]$.

Remark. If $H: \bar{G} \times [0, 1] \to 2^X$ is a such that for each $t \in [0, 1]$ mapping $H(.\,, t)$ is condensing and mapping $t \to H(.\,, t)$ is continuous in the sense of (ii) then condition (iii) is satisfying.

Corollary 1. *If $H: \bar{G} \times [0, 1] \to 2^X$ satisfies* (i), (ii) *and* (iv) *for each $x \in \bar{G}$ there exists $U_x \subset X$ such that*

$$\alpha(H(A \times [0, 1])) < \alpha(A)$$

for $A \subset U_x \cap \bar{G}$ with $\alpha(A) > 0$, then $\deg(I - H(.\,, t), G, 0)$ *is constant on* $[0, 1]$.

Corollary 2. *Let $H: \bar{G} \times [0, 1] \to 2^X$ be continuous in t uniformly in the sense that for any $\epsilon > 0$ there exists $\delta > 0$ such that $\| H(x, t) - H(x', t) \| < \epsilon$ for $x, x' \in \bar{G}$ with $\| x - x' \| < \delta$ and all $t \in [0, 1]$. Suppose that $H(.\,, t)$ is local condensing mapping for $t \in [0, 1]$ and that H satisfies condition* (i) *of Theorem 2.*

Then we have

$$\deg(I - H(.\,, t), G, 0) = \mathrm{const}\,(t).$$

Applications. Lemma 3. *Let X be a Banach space, $J = [0, 1]$ and A be a bounded subset of X. Then*

$$\alpha(J \cdot A) = \alpha(A)$$

where $J \cdot A = ta: t \in J, a \in A$.

Proof. We have $A \subset J \cdot A$ and hence $\alpha(A) \leqslant \alpha(J \cdot A)$. Set $\epsilon > 0$. There exist subsets $B_1, \dots, B_n$ of X such that

(3) $\quad A \subset \bigcup_{j=1}^{n} B_j$ and $\delta(B_j) < \alpha(A) + \epsilon/2, \;\; j = 1, \dots, n,$

where $\delta(B_j) = \sup_{x, y \in B_j} \| x - y \|$.

We claim that for every $t_0 \in J$ there is an open neighbourhood J_{t_0} of t_0 such that

(4) $\quad \delta(J_{t_0} \cdot B_j) \leqslant \alpha(A) + \epsilon, \;\; j = 1, \dots, n.$

Indeed, let J_{t_0} satisfies $\delta(J_{t_0}) \leqslant \epsilon/(4M)$ where $0 < M = \sup_{x \in \cup B_j} \|x\|$. Then by (3) we have

$$\delta(J_{t_0} \cdot B_j) = \sup_{t, t' \in J_{t_0}, b, b' \in B_j} \|tb - t'b'\| \leqslant \epsilon/2 + \delta(B_j) \leqslant \alpha(A) + \epsilon .$$

Now let $J_{t_1}, \dots, J_{t_m}$ be a finite subcover of J chosen from cover J_{t_0}, $t_0 \in [0, 1]$. By (4) applied to J_{t_i}, $i = 1, \dots, m$, we obtain

$$\alpha(J \cdot A) \leqslant \alpha(A) + \epsilon$$

since family $J_{t_i} \cdot B_j$, $i = 1, \dots, m$, $j = 1, \dots, n$, is an open cover of $J \cdot A$. So statement $\alpha(J \cdot A) \leqslant \alpha(A)$ follows from arbitrality of ϵ.

Corollary 3. *Let $T: \bar{G} \to 2^X$ be a local condensing mapping and $x_0 \in G$. If $H(x, t) = tT(x) + (1 - t)x_0$, $x \in \bar{G}$, $t \in J$, then for any $x \in \bar{G}$ there exists an open neighbourhood U_x of x such that $\alpha(H(A \times J)) < \alpha(A)$ for $A \subset U_x \cap G$, $\alpha(A) > 0$, i.e. for segment homotopy H condition* (iv) *is valid.*

Definition 3. For mapping $T: D \to 2^X$, where $D \subset X$, and $K \subset X$ we define

$$T^{\urcorner}(K) = \{x \in D: T(x) \cap K \neq \emptyset\}.$$

For example, if $T = f: D \to X$ then $T^{\urcorner}(K) = f^{-1}(K) = \{x \in D: f(x) \in K\}$ (we identify f and $T(x) = \{f(x)\}$).

Definition 4. A mapping $T: D \to 2^X$ is called proper if set $T^{\urcorner}(K)$ is compact for every compact subset K of X.

Lemma 4.

1. *If T is a proper mapping then it is closed.*
2. *If T is proper then for each sequence $\{x_n\} \subset D$ and $\{y_n\} \subset X$, $y_n \in T(x_n)$ such that $y_n \to y_0 \in X$ there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_0 \in D$ with $x_{n_k} \to x_0$.*

Theorem 3 (the fixed point theorem). *Let G be an open subset of a Banach space X. Let $T: \bar{G} \to 2^X$ be a local condensing mapping. Suppose that σ_T (the fixed point set of T) is compact, possible empty, tT is proper for all $t \in [0, 1]$ and there exists $w \in G$ such that $m(x - w) \bar{\in} T(x) - w$ for $x \in \partial G$, $m > 1$.*
Then there exists $x \in \bar{G}$ such that $x \in T(x)$.

Proof. If T has a fixed point on ∂G then the theorem is true. Suppose that $x \bar{\in} T(x)$ for $x \in \partial G$. Consider the mapping

$$h(x, t) = tT(x) + (1 - t)w.$$

By Corollary 3 h satisfies the homotopy conditions. Hence, in view of Theorem 2, deg $(I - T, G, 0)$ = deg $(I - w, G, 0) = 1$ and so T has a fixed point, by Lemma 4 and Theorem 1.

Theorem 4. (**The odd mapping theorem**). *Let G be an open bounded subset of a Banach space X, symmetric about the origin, and $0 \in G$. Let $T: \bar{G} \to 2^X$ be a local condensing mapping. Suppose that σ_T is compact, $0 \notin (I - T)(\partial G)$ and $T(-x) = T(x)$ for all $x \in \bar{G}$. Then* deg $(I - T, G, 0)$ *is an odd number.*

Proof. There exists a neighbourhood V of σ_T such that $T/\bar{V}$ is condensing and deg $(I - T, G, 0)$ = deg $(I - T, V, 0)$. Set $W = V \cap (-V)$. W is symmetric about 0 and $\sigma_T \subset W$. Let $T_1 = T/\bar{W}$. T_1 is USC and condensing, satisfies $0 \notin (I - T)(\partial W)$, $T_1(-x) = -T_1(x)$ for $x \in \bar{W}$. T_1 being condensing is 1-set contraction. Hence for $\bar{T} = tT_1$, where $1 - t > 0$ is sufficiently small, we obtain

$$\deg (I - \bar{T}, W, 0) = \deg (I - T_1, W, 0).$$

Now, from the Approximation Theorem for set contractions (see [8]) there exists a single valued compact mapping $g: W' \to X$, where W' is open bounded set symmetric about the origin, such that

$$\deg (I - \bar{T}, W, 0) = \deg (I - g, W', 0).$$

We see that $f(x) = (1/2)\, g(x) - (1/2)\, g(-x)$ is an odd compact mapping. It is an approximation of T since $T(x) = (1/2)\, T(x) - (1/2)\, T(-x)$. Hence

$$\deg (I - \bar{T}, W, 0) = \deg (I - f, W', 0)$$

and the statement follows from the Odd Mapping Theorem (see [3]).

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STRESZCZENIE

W pracy rozszerzono teorię stopnia topologicznego dla odwzorowań wielowartościowych i lokalnie ściągających podając także pewne zastosowania.

РЕЗЮМЕ

Расширяется применимость теории топологического индекса на локально сжимающие отображения и приводятся некоторые применения.