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An Extension of Bielecki's Method of Proving of Global Existence and Uniqueness Results for Functional Equations

Pewne uogólnienie metody Bieleckiego dowodzenia twierdzeń o globalnym istnieniu i jednoznaczności rozwiązań równań funkcyjnych

Некоторые обощение метода Белецкого устанавливания теорем глобалного существования и однозначности решений функциональных уравнений

Since 1956 when A. Bielecki published his note *Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des equationes differentielles ordinaires* [1] the method of weighted norm has been used very frequently to establish global existence and uniqueness results for wide classes of differential, differential-delay, differential-integral, integral, integral-functional and other functional equations. There is a huge number of papers which make use of Bielecki's method. Among them is a number of papers due to C. Corduneanu and his students which may be found in the review paper of Corduneanu [3]. Bielecki's students. J. Błaż, T. Dłotko and K. Zima used his method extensively in the early sixties to establish existence and uniqueness results for differential equations with deviated arguments. There are many others, including the author of the present paper who have employed Bielecki's method. It is not the aim of the present paper to give a review of the results obtained by the method mentioned above but to present a general result obtained in the spirit of Bielecki's method. This will be an application of an abstract result formulated in [5].

1. Let $C(I, B)$ denote the space of continuous functions defined on the interval $I = [0, a)$, $0 < a \leq +\infty$, with values in a Banach space B; $\|\cdot\|$ will denote the norm in B. Let an operator $F : C(I, B) \to C(I, B)$ be given. Consider the equation

$$x(t) = (Fx)(t), \qquad t \in I. \tag{1}$$

We are interested in establishing of the existence and uniqueness of solution of the equation (1). In order to do this take some $x_0 \in C(I, B)$ and $u_0 \in C(I, R_+)$, $R_+ = [0, +\infty)$, and define

$$V(u_0) = \{u : u \in C(I, R_+), 0 \leq u(t) \leq c u_0(t), c \geq 0\},$$

$$D(x_0, u_0) = \{x : x \in C(I, B), \|x(t) - x_0(t)\| \leq c u_0(t), c \geq 0\},$$

with the usual partial order in $V(u_0)$,

$$\text{i.e. } u \leq v \iff u(t) \leq v(t), \qquad t \in I.$$

We will use the following:

Assumption A_1. Assume that

(i) there is a nondecreasing operator $\Omega : V(u_0) \to V(u_0)$ such that

$$\| (Fx)(t) - (Fy)(t) \| \leq \Omega(\|x - y\|)(t), \qquad t \in I,$$

for any $x, y \in D(x_0, u_0)$.

(ii) there is a function $\varphi : R_+ \to R_+$, which is upper semicontinuous from the right, having the properties:

$$\phi(0) = 0, \ \phi(s) < s, \ s > 0, \text{ for which}$$

$$\Omega(s u_0)(t) \leq \phi(s) u_0(t), \ t \in I, \ s \geq 0,$$

(iii) there is $p > 0$ such that

$$\| x_0(t) - (F x_0)(t) \| := q(t) \leq p u_0(t), \ t \in I.$$

Now we are in position to formulate.

Theorem 1. *If Assumption A_1 is fulfilled then there exists in $D(x_0, u_0)$ a unique solution of equation (1), say x^*. This solution is a limit of the sequence of iterations of x_0 by F, i.e. $F^n x_0 \to x^*$, and the convergence is uniform in any compact subset of I.*

Proof. First we define in $D(x_0, u_0)$ a metric by putting

$$d(x, y) = \sup_{t \in I_0} \frac{\|x(t) - y(t)\|}{u_0(t)} = \inf_{c > 0} \{c : \|x(t) - y(t)\| \leq c u_0(t), c \geq 0\}$$

where

$$I_0 = \{t : t \in I, u_0(t) \neq 0, \ x, y \in D(x_0, u_0)\}.$$

It is clear that $d(x, y)$ is finite and it satisfies the metric axioms, so $(D(x_0, u_0), d)$ is a metric space. It is also easy to prove that this is a complete metric space. Next we observe that $F(D(x_0, u_0)) \subset D(x_0, u_0)$. Indeed, for any $x \in D(x_0, u_0)$ we get

$$\|(Fx)(t) - x_0(t)\| \leqslant \|(Fx)(t) - (Fx_0)(t)\| + \|(Fx_0)(t) - x_0(t)\| \leqslant \Omega(\|x - x_0\|)(t) + q(t) \leqslant$$

$$\leqslant \Omega(cu_0)(t) + pu_0(t) \leqslant \phi(c)u_0(t) + pu_0(t) = (\phi(c) + p)\, u_0(t)$$

for some $c > 0$, this is proving our assertion.

Now we show that F is a nonlinear contraction in $D(x_0, u_0)$. In fact for any $x, y \in D(x_0, u_0)$ we have

$$\|(Fx)(t) - (Fy)(t)\| \leqslant \Omega(\|x - y\|)(t) \leqslant \Omega(d(x, y)u_0)(t) \leqslant \varphi(d(x, y))u_0(t),$$

$t \in I$,

but this means that

$$d(Fx, Fy) \leqslant \varphi(d(x, y)).$$

Now the assertion of Theorem 1 is implied by the **Boyd-Wong's** result of [2].

Remark 1. Observe that in our considerations we did not use the fact that I is an interval in R_+. One can take for I any topological space T.

Remark 2. The theory developed above works fairly well if we replace the Banach space B by a locally convex topological space with the family of seminorms $\|\cdot\|_\tau$, $\tau \in \Theta$. In this case $C(I, R_+)$ should be replaced by $C(I, R_+^\Theta)$ and $u_0(t)$ by $u_0(t, \tau)$, $\tau \in \Theta$. For abstract considerations consult [5].

Note that we have not assumed that F is defined on the whole space $C(I, B)$, sometimes it happens that F is defined only on $C(x_0, u_0)$. In applications x_0 is usually taken as $x_0(t) \equiv 0$.

2. Let us now discuss brifly the Assumption A_1. First of all we observe that the condition (ii) of this assumption holds if the operator Ω has the properties:

(ii') $\Omega(su_0) \leqslant s\Omega(u_0)$, $s \geqslant 0$, $\Omega(u_0) \leqslant \alpha u_0$ for some $0 \leqslant \alpha < 1$.

In this case (ii) holds with $\varphi(s) = \alpha s$. Usually there is a problem how to find the function $u_0 \in C(I, R_+)$ for which the conditions (ii) and (iii) of Assumption A_1 hold. To solve this problem we introduce.

Assumption A_2. Assume that

(i) there exists a nondecreasing operator $\Omega : C(I, R_+) \to C(I, R_+)$ such that $\|(Fx)(t) - (Fy)(t)\| \leqslant \Omega(\|x - y\|)(t)$, $t \in I$, for any $x, y \in C(I, B)$,

(ii) $\Omega(su) \leqslant s\Omega(u)$, $s \geqslant 0$, $u \in C(I, R_+)$,

(iii) there exists $u_0 \in C(I, R_+)$ and $\lambda > 1$ such that

$$u_0(t) \geqslant \lambda \Omega(u_0)(t) + q(t), \quad t \in I. \tag{2}$$

It is quite clear that Assumption A_2 implies the Assumption A_1 with $\varphi(s) = \alpha s$, $\alpha = 1/\lambda$, $p = 1$.

As a collorary of Theorem 1 we get.

Theorem 2. *If Assumption A_2 holds then there exists in $D(x_0, u_0)$ a unique solution of equation* (1) *and it is the limit of the iterations of x_0 by F.*

Now it is natural to ask when does there exists $u_0 \in C(I, R_+)$ and $\lambda > 1$ for which (2) holds. The answer to this question gives us the following:

Lemma 1. *If the condition* (ii) *of Assumption A_2 holds and there exists $0 \leqslant \nu < 1$ such that*

$$\Omega(q)(t) \leqslant \nu q(t), \qquad t \in I, \tag{3}$$

then (2) *holds for*

$$u_0(t) = \frac{q(t)}{1 - \lambda\nu}, \quad t \in I,$$

and $\lambda > 1$ such that $\lambda\nu < 1$.

Proof. For the λ mentioned we get

$$q(t) = u_0(t)(1 - \lambda\nu) = u_0(t) - \lambda\nu u_0(t) = u_0(t) - \frac{\lambda}{1 - \lambda\nu} \cdot \nu q(t) \leqslant$$

$$\leqslant u_0(t) - \lambda \frac{\Omega(q)(t)}{1 - \lambda\nu} \leqslant u_0(t) - \lambda\Omega(u_0)(t)$$

what gives (2).

Remark 3. We note that $D(x_0, u_0) = D(x_0, v_0)$ if $u_0 = c v_0$ for some $c > 0$, so in the case of Lemma 1 we get $D(x_0, u_0) = D(x_0, q)$.

Note the following obvious observation: if for some $u_0 \in C(I, R_+)$ the inequality

$$u_0(t) \geqslant \bar{\lambda}(\bar{\Omega} u_0)(t) + \bar{q}(t), \qquad t \in I,$$

holds for some $\bar{\Omega}$, $\bar{q}$, $\bar{\lambda}$ such that $\bar{\Omega} u \geqslant \Omega u$, $\bar{q} \geqslant q$, $\bar{\lambda} \geqslant \lambda > 1$ then (2) holds for this u_0.

If we assume that q is bounded, say $q(t) \leqslant Q$, $t \in I$, and

$$\sup_{t \in I} \Omega(1)(t) = \nu < 1,$$

then we can take for u_0

$$u_0(t) \equiv \frac{Q}{1 - \lambda\nu}, \quad t \in I.$$

for which (2) holds.

For the case when Ω is a linear operator we get.

Lemma 2. *If the series*

$$(4) \qquad \sum_{n=0}^{\infty} \lambda^n (\Omega^n q)(t), \quad t \in I,$$

converges to the continuous function $u_0 \in C(I, R_+)$ *then the inequality* (2) *holds; here we mean* $(\Omega^0 q)(t) = q(t), (\Omega^{n+1} q)(t) = \Omega(\Omega^n q)(t), n = 0, 1, \ldots, t \in I.$

It is easy to see that the series (4) converges to a continuous function if condition (3) holds and $1 < \lambda < 1/\nu$. Indeed in this case we have

$$(\Omega^n q)(t) \leqslant \nu^n q(t), \qquad t \in I, n = 0, 1, \ldots, .$$

3. Let us now ask the question what can be said about the case $\lambda = 1$ in Assumption A_2. Unfortunately in this case the Bielecki's method will not work because what we will only be able to show is that F is non-expanding in the metric space $D(x_0, u_0)$. However the assertion of Theorem 2 will hold true but this is due to the comparison method (see [4] – [7]).

Let us quote briefly the result we mean. Take

Assumption A_2'. Assume that Assumption A_2 holds with the following changes:

a) the space $C(I, R_+)$ is replaced by the space of upper semicontinuous functions $C_0(I, R_+)$, $\Omega u \in C(I, R_+)$ if $u \in C(I, R_+)$,

b) condition (iii) holds for $\lambda = 1$ and $u_0 \in C(I, R_+)$,

c) the operator Ω is continuous in the following sense: $u_n(t) \searrow u(t) \in C_0(I, R_+)$ implies $(\Omega u_n)(t) \searrow (\Omega u)(t)$, in pointwise sense,

d) $u(t) \equiv 0$, $t \in I$, is the only upper semicontinuous solution of the equation

$$u(t) = (\Omega u)(t), \qquad t \in I$$

satisfying the condition $0 \leqslant u(t) \leqslant u_0(t), t \in I$. Now we can formulate

Theorem 3. *If Assumption* A_2' *holds then there exists in* $D(x_0, u_0)$ *a unique solution of equation* (1) *and it is the limit of the iterates of* x_0 *by* F.

For the proof of this theorem see [6].

4. Let us now discuss some applications of the general result.

a) Consider the functional equation

$$(1a) \qquad x(t) = F(t, x(\beta(t))), \qquad t \in I,$$

where $F \in C(I \times B, B), \beta \in (I, I)$. Assume that there exists $l \in C(I, R_+)$ such that

$$(ia) \qquad \| F(t, x) - F(t, y) \| \leqslant l(t) \| x - y \|, \qquad t \in I, x, y \in B.$$

Let $q(t) \geq \| F(t, x_0(\beta(t))) - x_0(t) \|$ for some $x_0 \in C(I, B)$. Now $(\Omega u)(t) = l(t) \cdot u(\beta(t))$ and the inequality (2) has the form

(2a) $$u_0(t) \geq \lambda l(t) u_0(\beta(t)) + q(t), \qquad t \in I.$$

A continuous solution to this inequality exists if the series

(4a) $$\sum_{n=0}^{\infty} \lambda^n l_n(t)\, q(\beta^n(t)),$$

converges, where $l_{n+1}(t) = l(t) l_n(\beta(t)), l_0(t) = 1, \beta^{n+1}(t) = \beta(\beta^n(t)), \beta^0(t) = t$. For some $\lambda > 1$ this series certainly converges to a continuous function if the inequality

(3a) $$l(t)\, q(\beta(t)) \leq \alpha q(t), \qquad t \in I,$$

holds for some $\alpha < 1$.

Take $l(t) \equiv l \geq 0, q(t) = Qt^p \exp(\sigma t), p \geq 0, Q \geq 0, \sigma \in R$ and suppose that

$$0 \leq \inf_{t \in I} \frac{\beta(t)}{t} = \underline{\beta} \leq \overline{\beta} = \sup_{t \in I} \frac{\beta(t)}{t} < +\infty .$$

Now we see that (3a) holds if: $l\overline{\beta}^p < 1$, and $\overline{\beta} \leq 1$, for $\sigma \geq 0$ or $l\overline{\beta}^p < 1$ and $\underline{\beta} \geq 1$ for $\sigma < 0$. These are useful sufficient conditions for the existence and uniqueness of solution of equation (1a) in the space $D(x_0, q)$. In the same way can consider the equation

(1a′) $$x(t) = F(t, x(\beta_1(t)), \ldots, x(\beta_r(t))), \qquad t \in I,$$

if we assume the continuity of given functions β_i, F and the Lipschitz condition of the form

$$\| F(t, x_1, \ldots, x_r) - F(t, y_1, \ldots, y_r) \| \leq \sum_{i=1}^{\nu} (t) \| x_i - y_i \|, \quad t \in I, x_i, y_i \in B.$$

Now

$$(\Omega u)(t) = \sum_{i=1}^{\nu} k_i(t)\, u(\beta_i(t))$$

and (3a) should be replaced by the condition

(3a′) $$\sum_{i=1}^{\nu} k_i(t)\, q(\beta_i(t)) \leq \alpha q(t), \; t \in I,$$

for some $0 \leq \alpha < 1$ and

$$q(t) \geq \| F(t, x_0(\beta_1(t)), \ldots, x_0(\beta_r(t)) - x_0(t) \| .$$

If $q(t) = Qt^p, p \geq 0, k_i(t) \leq \overline{k}_i < +\infty$, then (3a′) holds if

$$\sum_{i=1}^{\nu} k_i \overline{\beta}_i^p < 1$$

for

$$\bar{\beta}_l = \sup_{t \in I} \frac{\beta_l(t)}{t} < +\infty .$$

This is an effective sufficient condition for the existence and uniqueness of solution of equation (1a′) in the space $D(x_0, q)$.

b) Let us consider now the integral equation

$$x(t) = \int_0^{\alpha(t)} f(t, s, x(s))\, ds + h(t), \quad t \in I, \tag{1b}$$

where the functions $f \in C(I^2 \times B, B)$, $h \in C(I, B)$, and $\alpha \in C(I, I)$ are given. Assume there is $L \in C(I^2, R_+)$ such that

$$\| f(t, s, x) - f(t, s, y) \| \leqslant L(t, s) \| x - y \| , \qquad t \in I, x, y \in B. \tag{ib}$$

Now the inequalities (2) and (3) take the form

$$u_0(t) \geqslant \int_0^{\alpha(t)} L(t, s)\, u_0(s)\, ds + q(t), \ \lambda > 1, \ t \in I. \tag{2b}$$

$$\int_0^{\alpha(t)} L(t, s)\, q(s)\, ds \leqslant \alpha q(t), \ t \in I, \ \alpha < 1, \tag{3b}$$

and

$$q(t) \geqslant \| x_0(t) - \int_0^{\alpha(t)} f(t, s, x_0(s))\, ds - h(t) \| .$$

In the general case it is not easy to find a continuous function u_0 for which (2b) holds. Clearly we can get for u_0 the continuous sum of the series (4) with Ωq defined by left hand side of (3b) but we need investigate the convergence of this series. There is no problem if $L(t, s) = L(s)$, $\alpha(t) \leqslant t$ and q is taken as a nondecreasing function. In this case it easy to check that we can take

$$u_0(t) = q(t) \exp (\lambda \int_0^t L(s)\, ds), \lambda > 1, t \in I .$$

Moreover if we know that

$$q(t) = Q \exp (p \int_0^t L(s)\, ds) \ \ Q \geqslant 0, \ p > 1,$$

then (3b) holds for $\alpha = 1/p < 1$ and we can take

$$u_0(t) = \frac{Q}{1 - \lambda\alpha} \exp (p \int_0^t L(s)\, ds), \ t \in I,$$

for some $\lambda > 1$ such that $\lambda\alpha < 1$.

Now the existence and uniqueness result for equation (1b) hold in the space $D(x_0, u_0)$

with u_0 defined above. This is just the case that appeared in Bielecki's note of 1956. There is no problem also if we assume $L(t, s) = K(t) L(s)$ and $\alpha(t) \leq t$. In this case the explicite formula for u_0 can be written down easily.

c) Finally let us consider the integral-functional equation of the form

(1c) $$x(t) = f(t, \int_0^{\alpha(t)} g(t, s, x(s))\, ds,\ x(\beta(t))),\ t \in I,$$

where $$f \in C(I \times B^2, B),\ g \in C(I^2 \times B, B),\ \alpha, \beta \in C(I, I).$$

Assume

$$\| f(t, x, y) - f(t, \bar{x}, \bar{y}) \| \leq k_1(t) \| x - \bar{x} \| + l(t) \| y - \bar{y} \|,$$

$$\| g(t, s, x) - g(t, s, y) \| \leq k_2(t, s) \| x - y \|,$$

for some continuous functions $k_1, l \in C(I, R_+)$, $k_2 \in C(I^2, R_+)$.
Take

$$\bar{q}(t) \geq \| f(t, \int_0^{\alpha(t)} g(t, s, x(s)\, ds, x_0(\beta(t))) - x_0(t) \|$$

for some $x_0 \in C(I, B)$.

Now the operator Ω related to the equation (1c) has the form

$$(\Omega u)(t) = k_1(t) \int_0^{\alpha(t)} k_2(t, s) u(s)\, ds + l(t) u(\beta(t)),\ t \in I.$$

According to the general theory on the existence and uniqueness of solution of equation (1c) it is enough to find $u_0 \in C(I, R_+)$ for which (2) holds. In order to do this it is enough to find a function $q \in C(I, R_+)$, $q(t) \geq \bar{q}(t)$, $t \in I$, for which (3) holds. In the general case it is not easy to find such q but we are to do this under some additional assumptions. We assume that k_1 is bounded, k_2 does not depend on t, $\alpha(t) \leq t$, $\beta(t) \leq t$, $t \in I$, $\bar{q}$ is nondecreasing and the inequality

(3c) $$l(t) \bar{q}(\beta(t)) \leq \alpha \bar{q}(t),\qquad t \in I,\ 0 \leq \alpha < 1,$$

holds. Under this assumption we can conclude that (3) holds for the operator Ω defined above and for the function

$$q(t) = \bar{q}(t) \exp(\mu \int_0^t k_2(s)\, ds),\ t \in I,$$

for such μ that the inequality $\nu = K/\mu + \alpha < 1$ is fulfilled; here $k_1(t) \leq K$, $t \in I$. Indeed, we get

$$(\Omega q)(t) = k_1(t) \int_0^{\alpha(t)} k_2(s) \overline{q}(s) \exp(\mu \int_0^s k_2(\tau)\, d\tau)\, ds +$$

$$+ l(t)\, \overline{q}(\beta(t)) \exp(\int_0^{\beta(t)} \mu k_2(s) ds) \leq k_1(t) \overline{q}(t) \int_0^t k_2(s) \exp(\mu \int_0^s k_2(\tau) d\tau) ds +$$

$$+ \alpha \overline{q}(t) \exp(\int_0^t \mu k_2(s) ds) \leq (\frac{K}{\mu} + \alpha) \overline{q}(t) \exp(\mu \int_0^t k_2(\tau) d\tau) = \nu q(t).$$

Now we see that the condition (3c) is crucial for the existence and uniqueness of solutions of equation (1c) of Volterra type in the space $D(x_0, q)$. Note that all considerations given above hold true in the case $I = [0, a]$, $0 < a < +\infty$. In this case we get $D(x_0, q) = D(x_0, \overline{q})$.

The same considerations can be carried out for more general integral-functional equations of the form

$$x(t) = f(t, \int_0^{\alpha_1(t)} g_1(t, s, x(s)) ds, \ldots, \int_0^{\alpha_p(t)} g_p(t, s, x(s)) ds, x(\beta_1(t)), \ldots, x(\beta_q(t))), \; t \in I,$$

as well as for functional equations of the form

$$x(t) = F(t, x_{\beta_1(t)}, x_{\beta_2(t)}, \ldots, x_{\beta_\gamma(t)}), \; t \in I,$$

where $F: I \times [C(I_\tau, B)]^\gamma \to B$, $\beta_i: I \to I$, $I_\tau = [-\tau, 0]$, $\tau > 0$, $x_t = x(t+s)$, $s \in I_\tau$.

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STRESZCZENIE

W pracy dyskutowany jest problem istnienia rozwiązań abstrakcyjnego równania funkcyjnego $x(t) = (Fx)(t)$ w postaci $C(I, B)$. Metoda Bieleckiego użyta jest do otrzymania twierdzeń egzystencjalnych.

РЕЗЮМЕ

В работе рассматривается вопрос существования решений абстрактного, функционального уравнения $x(t) = (Fx)(t)$ в пространстве $C(J, B)$. Использован метод А. Белецкого для установления теорем о существовании.