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The Modulus of Noncompact Convexity

Moduł niezwartej wypukłości

Модул некомпактной выпуклости

1. Introduction. The aim of this note is to introduce a new way of measuring convexity of balls in Banach spaces and to show its usefulness to the geometric theory of Banach spaces and to the theory of nonexpansive mappings. It is done by defining a new function $\Delta(\epsilon)$ which we call „the modulus of noncompact convexity". The function $\Delta(\epsilon)$ is defined with help of the so called „Kuratowski's measure of noncompactness". It measures the rotundity of the unit ball in similar way as the classical Clarcson's modulus of convexity $\delta(\epsilon)$ (see [4]) but „it neglects to notice flat compact spots laying close to the unit sphere".

2. Basic notations and definitions. Let $(X, \| \|)$ be an infinitely dimensional Banach space and let $B(x, r)$, $S(x, r)$ denote the ball and the sphere centered at x and of radius r. For any $A \subset X$, $\bar{A}$ and Conv A will denote the closure and the convex closed envelope of A respectively and for bounded A, $\alpha(A)$ will denote the Kuratowski's measure of noncompactnes:;

$\alpha(A) = \inf\ [d > 0$: A can be covered with a finite number of sets of diameter smaller then $d]$

We shall need only few basic properties of $\alpha(\cdot)$

a) $\alpha(A) = 0 \Leftrightarrow \bar{A}$ is compact
b) $\alpha(A) = \alpha(\bar{A})$
c) $A_1 \subset A_2 \Rightarrow \alpha(A_1) \leqslant \alpha(A_2)$
d) $\alpha(\text{Conv}\, A) = \alpha(A)$

e) If $(A_n)_{n=1,2,\ldots}$ is a decreasing sequence of nonempty closed sets with $\lim_{n\to\infty} \alpha(A_n) = 0$, then $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and compact.

f) $\alpha(\lambda A) = |\lambda| \alpha(A)$
g) $\alpha(A+B) \leqslant \alpha(A) + \alpha(B)$

The same properties has the Hausdorff measure of noncompactness $\chi(A)$ defined by

$\chi(A) = \inf [r > 0 : A$ can be covered with a finite number of balls of radius smaller then $r]$.

For any ball $\alpha(B(x, r)) = 2r$ and $\chi(B(x, r)) = r$. Both measures are „equivalent" i.e. $\chi(A) \leqslant \alpha(A) \leqslant 2\chi(A)$. For further properties of these and other measures see [1].

A bounded convex subset C of X is said to be diametral if for any $x \in C$

$$\sup [\, \|x-y\| : y \in C] = \operatorname{diam} C.$$

The space X has normal structure if it does not contain any diametral set consisting of more than one point [3], [16]. Observe that for any diametral set C, $\alpha(C) = \operatorname{diam} C$.

The modulus of convexity of the space X is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf [1 - \|(x+y)/2\| : x, y \in B(0,1), \;\; \|x-y\| \geqslant \epsilon]$$

and the coefficient of convexity of X is defined by

$$\epsilon_0(X) = \sup [\epsilon : \delta_X(\epsilon) = 0]$$

The space X is uniformly convex if $\epsilon_0 = 0$ and uniformly non-square if $\epsilon_0 < 2$. All uniformly non-square spaces are reflexive (even superreflexive) and all spaces having $\epsilon_0 < 1$ have normal structure (see [4], [9], [13]).

For H being a Hilbert space we have $\delta_H(\epsilon) = 1 - \sqrt{1-(\epsilon/2)^2}$ and for any Banach space X, $\delta_X(\epsilon) \leqslant \delta_H(\epsilon)$, [12]. For example

$$\delta_{l^p}(\epsilon) = 1 - (1-(\epsilon/2)^p)^{1/p} \quad \text{for } 2 \leqslant p < \infty$$

and

$$(1 - \delta_{l^p}(\epsilon) + \epsilon/2)^p + |1 - \delta_{l^p}(\epsilon) - \epsilon/2|^p = 2 \quad \text{for } 1 \leqslant p \leqslant 2$$

(see [12], [18]).

3. **Modulus of noncompact convexity.** Let us define the modulus of noncompact convexity of the space X as a function $\Delta_X : [0,2] \to [0, 1]$ given by the formula

$$\Delta_X(\epsilon) = \inf\, [1 - \inf_{x \in A} \| x \|]$$

where the first infimum is taken over all convex subsets A of the unit ball such that $\alpha(A) \geqslant \epsilon$. Obviously $\Delta(\epsilon)$ is nondecreasing and the following implication holds

$$\left.\begin{aligned} A &= \operatorname{Conv} A \\ A &\subset B(x, r) \\ \alpha(A) &\geqslant \epsilon r \end{aligned}\right\} \Longrightarrow \operatorname{dist}(x, A) \leqslant (1 - \Delta_X(\epsilon))r.$$

This is a counterpart of the classical implication

$$\left.\begin{aligned} \| x - y \| &\leqslant r \\ \| x - z \| &\leqslant r \\ \| y - z \| &\geqslant \epsilon r \end{aligned}\right\} \Longrightarrow \| x - (z + y)/2 \| \leqslant (1 - \delta_X(\epsilon))r$$

It is easy to see that for any space X, $\delta_X(\epsilon) \leqslant \Delta_X(\epsilon)$. Strong inequality may hold for some spaces. To show this let us recall certain characterization of Hausdorff measure of noncompactness in some spaces with bases [1].

Let X be a Banach space with Schauder bases $e_1, e_2, e_3, \ldots$. Denote by R_n n-th remainder operator;

$$R_n\Big(\sum_{i=1}^{\infty} \xi_i e_i\Big) = \sum_{i=n+1}^{\infty} \xi_i e_i \qquad i = n + 1$$

and assume that $||| R_n ||| = 1$ for $n = 1, 2, \ldots$. In this setting for any bounded set $A \subset X$

$$\chi(A) = \lim_{n \to \infty} \sup_{x \in A} \|R_n x\|$$

Standard bases in l^p spaces $1 \leqslant p < \infty$ has the above property. Let us start with evaluating Δ_{l^p} for $1 < p < \infty$. Suppose $A \subset B(0, 1)$ is a convex set with $\alpha(A) \geqslant \epsilon$. In view of $\chi(A) \geqslant 1/2 \cdot \alpha(A)$ we have

$$\lim_{n \to \infty} \sup_{x \in A} \| R_n x \| \geqslant \epsilon/2$$

Let us select a sequence $x_1, x_2, x_3, \ldots$ of elements of A satisfying $\| R_n x_n \| \geqslant \epsilon/2$ and weakly convergent to let us say z. It is always possible and $z \in A$ in view of reflexivity of l^p. Take any $\kappa > 0$ and find k big enough to satisfy $\| R_k z \| < \kappa$. Now for $n > k$ we have

$$1 \geqslant \| x_n \|^p = \|(I - R_k) x_n \|^p + \| R_k x_n \|^p \geqslant$$

$$\geqslant \|(I - R_k) x_n \|^p + (\epsilon/2)^p \longrightarrow \|(I - R_k) z \|^p + (\epsilon/2)^p \geqslant \|z\|^p - \kappa^p + (\epsilon/2)^p$$

and hence $\| z \| \leqslant (1-(\epsilon/2)^p)^{1/p}$ and consequently

$$\Delta_{l^p}(\epsilon) \geqslant 1-(1-(\epsilon/2)^p)^{1/p}$$

Considering the set

$$[x = \sum_{i=1}^{\infty} \xi_i e_i : \xi_1 \geqslant (1-(\epsilon/2)^p)^{1/p}$$

we actually see that

$$\Delta_{l^p}(\epsilon) = 1-(1-(\epsilon/2)^p)^{1/p}$$

for all $1 < p < +\infty$. Thus for $p \geqslant 2$, $\delta_{l^p}(\epsilon) = \Delta_{l^p}(\epsilon)$ but for $1 < p < 2$, $\delta_{l^p} < \Delta_{l^p}$. The above example shows also that Hilbert space is no longer the best space with respect to Δ. For $1 < p < 2$ we have $\Delta_{l^p} > \delta_{l^2}(\epsilon) = 1-\sqrt{1-(\epsilon/2)^2}$.

Observe also that $\lim_{p\to 1} \Delta_{l^p}(\epsilon) = \epsilon/2 \neq \Delta_{l^1}(\epsilon)$. Indeed the unit sphere in l^1 contains the set

$$A = \mathrm{Conv}\,[e_1, e_2; e_3, \ldots] = [x = (\xi_1, \xi_2, \ldots) : \xi_i \geqslant 0,$$

$$i = 1, 2, \ldots, \ \sum_{i=1}^{\infty} \xi_i = 1]$$

satisfying $\alpha(A) = 2$. Thus $\Delta_{l^1}(\epsilon) \equiv 0$. We will return to this observation later.

As we mentioned at the begining $\Delta(\epsilon)$ „neglects to notice...". To visualise what does this mean, let us consider the Day's space D. Consider the sequence of spaces $(\mathrm{R}^n, |\ |_n)$ where $|x|_n = |(\xi_1, \xi_2, \ldots, \xi_n)| = \max |\xi_i|$. Let

$$D = \ x = \ x_n \ : x_n \in \mathbb{R}^n, \ \sum_{n=1}^{\infty} |x_n|_n^2 < +\infty \ .$$

With the norm

$$\| x \| = (\sum_{n=1}^{\infty} |x_n|_n^2)^{1/2}$$

D is a reflexive space with $\delta_D(\epsilon) = 0$ for all ϵ. It is not superreflexive and in consequence it does not admit any uniformly non-square equivalent norm [5], [8]. However, similarly as above we can prove that $\Delta_D(\epsilon) = \delta_H(\epsilon) = 1-\sqrt{1-(\epsilon/2)^2}$.

Let us end this section with defining the coefficient of noncompact convexity

$$\epsilon_1(X) = \sup\,[\epsilon : \Delta_X(\epsilon) = 0]$$

Obviously $\epsilon_0(X) \geqslant \epsilon_1(X)$ and by analogy we shall call spaces with $\epsilon_1 = 0$, Δ – uniformly convex.

4. $\Delta_X(\epsilon)$ and reflexivity. R. C. James [14] proved that the space X is not reflexive if and only if for any $0 \leq t < 1$ there exist sequences x_n and f_n of elements of unit balls in X and X^* respectively such that for all $i, j = 1, 2, \ldots$

$$f_j(x_i) = \begin{cases} t & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}$$

Considering the set $A = \operatorname{Conv} x_1, x_2, x_3, \ldots$ we may observe that for any $z \in A$, $\| z \| = t$ and in view of $\| x_j - x_i \| \geq f_j(x_j - x_i) = t, j > i$ we have $\alpha(A) \geq t$. This implies $\Delta_X(t) \leq$ $\leq 1 - t$ and hence by monotonicity of Δ_X, $\epsilon_1(X) \geq 1$. We have just proved

Theorem 1. *If $\epsilon_1(X) < 1$ then X is reflexive.*

In consequence all Δ-uniformly convex spaces are reflexive. The condition $\epsilon_1(X) < 1$ is not necessary for reflexivity. For example l^2 space renormed by

$$\| x \|_\lambda = \| (x_1, x_2, \ldots) \|_\lambda = \max [\lambda | x_1 |, \| x \|_{l^2}]$$

for $\lambda \geq 1$ has $\epsilon_1 = 2(1 - \lambda^{-2})^{1/2}$ and $\epsilon_1 \to 2$ as $\lambda \to +\infty$.

5. $\Delta_X(\epsilon)$ and normal structure.

Theorem 2. *If $\epsilon_1(X) < 1$ then X has normal structure.*

Proof: Suppose the contrary and let C be a convex diametral subset of X consisting of more than one point. We may assume that diam $C = 1$. For any functional $f \in X^*$, $\| f \| = 1$ and any $0 \leq d < 1$ consider the set

$$U(f, d) = [x : f(x) > d]$$

Observe that

$$d \leq \inf [\|x\| : x \in U(f, d) \cap B(0, 1)] \leq 1 - \Delta_X(\alpha(U(f, d) \cap \bar{B}(0, 1))).$$

In view of $\epsilon_1(X) < 1$ there exists $\kappa > 0$ and $d < 1$ such that $\alpha(U(f, d) \cap \bar{B}(0, 1)) < 1 - \kappa$ for all such functionals f. Now consider the family of all sets

$$V(x, f) = (x + U(f, d)), \text{ where } x \in C.$$

All such sets are weakly open and since C is diametral it is contained in the union of all $V(x, f)$. Since C is weakly compact by reflexivity of X we can find a finite covering

$$C \subset \bigcup_{i=1}^{n} V(x_i, f_i)$$

and hence

$$\alpha(C) \leq \max_i \alpha(V(x_i, f_i) \cap C) \leq \max_i \alpha(V(x_i, f_i) \cap \bar{B}(x_i, 1)) \leq 1 - \kappa$$

and we have a contradiction with $\alpha(C) = \operatorname{diam} C = 1$.

This proves that for example Day's space D has normal structure, the fact that has been previously proven in [2], [11] via different methods.

6. Dual spaces and $\Delta^*(\epsilon)$. We already observed that $\lim_{p \to 1} \Delta_{l^p}(\epsilon) = \epsilon/2 \neq \Delta_{l^1}(\epsilon) = 0$. However l^1 is a dual space $l^1 = (c_0)^*$. For dual spaces $X = Y^*$ we may define another modified modulus $\Delta^*(\epsilon)$. Taking into account that in nonreflexive dual spaces not all closed convex subsets are weak-star closed, let us put

$$\Delta_X^*(\epsilon) = \inf \{1 - \inf_{x \in A} \|x\|\}$$

where the first infimum is taken with respect to all convex, weak-star closed subsets of $\bar{B}(0,1)$ with $\alpha(A) \geqslant \epsilon$. Analogously we may put

$$\epsilon_1^*(X) = \sup \{\epsilon : \Delta^*(\epsilon) = 0\}$$

This time using the same method as for l^p we get $\Delta_{l^1}^*(\epsilon) = \epsilon/2 > \Delta_{l^1}(\epsilon)$. Modifying the proof of Theorem 2 we obtain

Theorem 3. *If X is a dual space with $\epsilon_1^*(X) < 1$ then X has normal structure for weak-star compact sets.*

It means that X does not contain weak-star compact convex sets which are diametral of positive diameter.

For example l^1 has this property (see also [15], [17]).

7. $\Delta(\epsilon)$ and nonexpansive mappings. Let C be a closed bounded and convex subset of X. Recall that the mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$. C has the fixed point property for nonexpansive mappings (shortly f.p.p.) if any nonexpansive mapping $T : C \to C$ has a fixed point $x = Tx$. The basic fact in this theory is;

Kirk's Theorem. *Any weakly compact convex set having normal structure has f.p.p.* [16]

Thus according to Theorems 1, 2 all closed convex subset of a space X with $\epsilon_1(X) < 1$ have f.p.p.

One of the most elegant method of proving the fixed point theorems for nonexpansive mapping is based on the notion of asymptotic center of a sequence (see [6], [10]). Recall that if $\{x_n\}$ is a sequence of elements of C and $x \in C$ then the asymptotic radius $r(\{x_n\} X)$ of $\{x_n\}$ at x is defined by

$$r(\{x_n\}, x) = \limsup_{n \to \infty} \|x_n - x\|$$

Consecutively we put

$$r(\{x_n\}, C) = \inf_{x \in C} r(\{x_n\}, x)$$

and

$$A(\{x_n\}) = \{x : r(\{x_n\}, x) = r(\{x_n\}, C)\}$$

and call them asymptotic radius of $\{x_n\}$ in C and asymptotic center of $\{x_n\}$ in C respectively.

If $T: C \to C$ is nonexpansive and $\{x_n\}$ is a sequence of consecutive iterations $x_n = T^n x_0$ then $A(\{T^n x_0\}, C)$ is a closed convex T-invariant subset of C (possibly empty unless C is e.q. weakly compact). The same is true for $A(\{y_n\}, C)$ where $\{y_n\}$ is any sequence satisfying $\lim_{n\to\infty} y_n - Ty_n = 0$ (such sequence always exists in C).

In uniformly convex spaces, the asymptotic center of any bounded sequence consists of exactly one point and thus the asymptotic centers of sequences $\{T^n x_0\}$ or $\{y_n\}$ described above are fixed points of T, [7], [10].

The counterpart of this fact in our theory is the following Theorem 4 which one can proved utilizing and modifying the proof of Theorem 2.

Theorem 4. $\alpha(A(\{x_n\}, C)) \leqslant \epsilon_1(X) r(\{x_n\}, C)$.

In special case $\epsilon_1(X) = 0$ i.e. if X is Δ-uniformly convex, then any bounded sequence x_n has in any closed convex set C, compact asymptotic center. In this case if $T: C \to C$ is nonexpansive we obtain a more or less constructive method for searching for fixed points of T. Taking $x_0 \in C$ we obtain convex compact T-invariant set $A(\{T^n x_0\}, C)$ and we can restrict our search only to this set. If $0 < \epsilon_1(X) < 1$ such convex compact set can be also constructed with the use of Theorem 4 but it requires countably many steps. Put $C_0 = C$. Choose $\{y_n^0\}$ satisfying $\lim_{n\to\infty} y_n^0 - Ty_n^0 = 0$ and put $C_1 = A(\{y_n^0\}, C_0)$, then choose $\{y_n^1\}$ in the same way in C_1 and $r(\{y_n^1\}, C_1) \leqslant \alpha(C_1)$, and put $C_2 = A(\{y_n^1\}, C_1)$. Proceeding this way we obtain the sequence $C_0 \supset C_1 \supset C_2 \supset \ldots$ of closed convex T-invariant sets with $\lim_{n\to\infty} \alpha(C_n) = 0$. $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is convex, compact and T-invariant.

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STRESZCZENIE

Praca zawiera definicję i omówienie własności nowej funkcji „mierzącej" wypukłość kul w przestrzeniach Banacha nazwanej modułem niezwartej wypukłości.

РЕЗЮМЕ

Статья содержит определение и изложение некоторых свойств нововведунной функции, названной модулем некомпактной выпуклости, измеряющей выпуклость шаров в банаховых пространствах.