ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXVIII, 1 SECTIO A 1984

Instytut Matematyki
Politechnika Poznańska
Instytut Matematyki
Uniwersytet im. Adama Mickiewicza

J. ALBRYCHT, A. MARCINIAK

Asymptotic Expansion of Total Error for a Discrete Mechanic Method

Asymptotyczne rozwinięcie błędu całkowitego w metodzie mechaniki dyskretnej

Асимптотическое разложение абсолютной величины ошибки в методе дискретной механики

1. Introduction. The numerical solutions of the dynamical problems obtained by the Greenspan's method [3–5, 8] have the analogous properties as the solutions of continuous mechanics equations, i.e. they conserve a total energy of isolated system of the material points [5], the linear and angular momentum of the system [8]. Moreover, basing on them, it may be proved an uniform motion of the center of mass of this system [8]. The Greenspan's dynamical equations, which are the difference equations, are invariant with respect to translation, with respect to rotation and under uniform motion of the frame of references [6]. As application of Greenspan's technique, the equations that describe the motion of material points in a rotating frame are also obtained [1, 11]. The discrete equations of relativistic mechanics are known too [2, 7, 8].

All mentioned above equations are of the first order-accuracy. Therefore they have a comparatively small precision. In this connexion it comes into existence the problem of improving this precision. It is generally known that in order to increase the precision up to the order J it must exists, for a method under consideration, an asymptotic expansion of the total discretization error of the same order [13].

In this paper, basing oneself on the theory given by H. J. Stetter [14], the existence of mentioned expansion for the Greenspan's discrete mechanics is deduced. Therby, a possibility for an precision-increasing for this method is also proved.

2. Preliminaries. Let us take into account the inertial frame of references $x_1x_2x_3$ and let $\Delta t = t_{k+1} - t_k$ ($k = 0, 1, 2, \ldots$) denotes an arbitrary given time interval. In this frame let us consider the motion of an isolated system of n material points P_i ($i = 1\,(1)\,N$) with coordinates $x_i^k = [x_{1i}^k, x_{2i}^k, x_{3i}^k]$ at the moment t_k and with velocities $v_i^k = [v_{1i}^k, v_{2i}^k, v_{3i}^k]$ at the same moment. D. Greenspan [3–5, 8] defines for this system the following dicrete dynamical equations

$$\frac{x_{li}^{k+1}-x_{li}^{k}}{\Delta_t}=\frac{v_{li}^{k+1}+v_{li}^{k}}{2},$$

(2.1)

$$\frac{v_{li}^{k+1}-v_{li}^{k}}{\Delta_t}=\frac{\mathcal{F}_{li}^{k}}{m_i};\quad l=1,2,3;\ i=1\,(1)\,N;\ k=0,1,2,\dots;$$

where, in gravitational case,

$$\text{(2.2)}\quad \mathcal{F}_{li}^{k}=-Gm_i\sum_{\substack{j=1\\ j\neq i}}^{N} m_j\,\frac{x_{li}^{k+1}+x_{li}^{k}-x_{lj}^{k+1}-x_{lj}^{k}}{r_{ij}^{k+1}\,r_{ij}^{k}\,(r_{ij}^{k+1}+r_{ij}^{k})},$$

and where G denotes the gravitational constant, m_i – the mass of P_i $(i=1\,(1)\,N)$ and

$$\text{(2.3)}\quad (r_{ij}^{k})^2=\sum_{l=1}^{3}(x_{li}^{k}-x_{lj}^{k})^2.$$

Undiminishing the generality, we restrict our consideration to the time interval [0,1] and record the discrete problem (2.1)–(2.3) in Stetter's notations [14].

For (2.1)–(2.3) let us define an input (continuous) problem $\mathcal{B}=\{E,E^0,\Phi\}$, where $\Phi: E\to E^0$ is of the form

$$\text{(2.4)}\quad \Phi(y)=\begin{bmatrix} y_p(0)-z_p^0\\ y_{p+3N}(0)-z_{p+3N}^0\\ y_p'-y_{p+3N}\\ y_{p+3N}'-f_p(y)\end{bmatrix}\in E^0 \text{ for } y\in E,$$

$$p=1\,(1)\,3N;$$

where $y_p\equiv x_{li};\ y_{p+3N}\equiv v_{li};\ l=[(p-1)/N]+1;\ i=p-[(p-1)/N]N;$

$$E=\underbrace{C^{(1)}[0,1]\times\dots\times C^{(1)}[0,1]}_{6N\text{ factors}},$$

$$E^0=\underbrace{\mathbb{R}\times\dots\times\mathbb{R}}_{6N\text{ factors}}\times\underbrace{C^{(1)}[0,1]\times\dots\times C^{(1)}[0,1]}_{6N\text{ factors}},$$

$$\text{(2.5)}\quad f_p(y)\equiv f_{li}[x(t)]=-G\sum_{\substack{j=1\\ j\neq i}}^{N} m_j\,\frac{x_{li}-x_{lj}}{r_{ij}^3}.$$

Let us introduce in the spaces E and E^0 the following norms

$$\|y\|_E = \max_{t\in[0,1]} \sum_{p=1}^{6N} |y_p(t)|,$$

(2.6)

$$\|d\|_{E^0} = \left\|\begin{bmatrix} d^0 \\ d \end{bmatrix}\right\|_{E^0} = \sum_{p=1}^{6N} |d_p^0| + \max_{t\in[0,1]} \sum_{p=1}^{6N} |d_p(t)|.$$

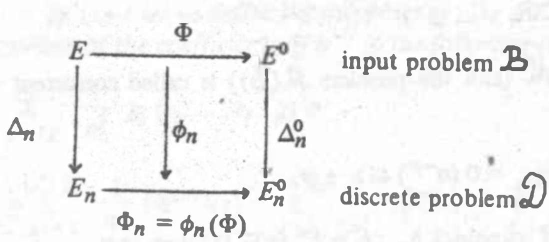
respectively. In accordance with Stetter's notations, the formulas of the discrete mechanics may be written in the form of a discrete problem $\mathcal{D} = \{E_n, E_n^0, \Phi_n\}$, where $\Phi_n : E_n \to E_n^0$ and, in our case, $E_n = (\mathbb{G}_n \to \mathbb{R}^{6N})$, $E_n^0 = (\mathbb{G}_n \to \mathbb{R}^{6N})$, $\mathbb{G}_n := \{(\nu/n) \in \mathbb{R} : \nu = 0\,(1)\,n\}$, and where (compare (2.1)–(2.2))

$$(2.7)\quad \Phi_n\left(\frac{\nu}{n}\right) = \begin{bmatrix} \eta_p(0) - z_p^0 \\ \eta_{p+3N}(0) - z_{p+3N}^0 \\ \dfrac{\eta_p\left(\frac{\nu}{n}\right) - \eta_p\left(\frac{\nu-1}{n}\right)}{\frac{1}{n}} - \dfrac{\eta_{p+3N}\left(\frac{\nu}{n}\right) + \eta_{p+3N}\left(\frac{\nu-1}{n}\right)}{2} \\ \dfrac{\eta_{p+3N}\left(\frac{\nu}{n}\right) - \eta_{p+3N}\left(\frac{\nu-1}{n}\right)}{\frac{1}{n}} - F_p\left[\eta\left(\frac{\nu}{n}\right), \eta\left(\frac{\nu-1}{n}\right)\right] \end{bmatrix};$$

$p = 1\,(1)\,3N$; $\nu = 1\,(1)\,n$;

$$(2.8)\quad F_p \equiv \frac{\mathcal{F}_{li}}{m_i} = -G \sum_{\substack{j=1 \\ j\neq i}}^{N} m_j \frac{x_{li}\left(\frac{\nu}{n}\right) + x_{li}\left(\frac{\nu-1}{n}\right) - x_{lj}\left(\frac{\nu}{n}\right) - x_{lj}\left(\frac{\nu-1}{n}\right)}{r_{ij}\left(\frac{\nu}{n}\right) r_{ij}\left(\frac{\nu-1}{n}\right)\left[r_{ij}\left(\frac{\nu}{n}\right) + r_{ij}\left(\frac{\nu-1}{n}\right)\right]}.$$

Apart from the discrete problem $\mathcal{D}$, let us define the discretization method $\mathcal{M} = \{E_n, E_n^0, \Delta_n, \Delta_n^0, \phi_n\}$ ($\mathcal{D} = \mathcal{M}(\mathcal{B})$), where the mappings Δ_n, Δ_n^0 and ϕ_n are defined as in the below diagram



Let us assume that

$$(2.9)\qquad (\Delta_n y)\left(\frac{\nu}{n}\right)=\begin{bmatrix} y_p\left(\frac{\nu}{n}\right)\\ y_{p+3N}\left(\frac{\nu}{n}\right)\end{bmatrix} \text{ for } y\in E,\qquad (\Delta_n^0 d)\left(\frac{\nu}{n}\right)=\begin{bmatrix} z_p^0\\ z_{p+3N}^0\\ z_p\left(\frac{\nu-1}{n}\right)\\ z_{p+3N}\left(\frac{\nu-1}{n}\right)\end{bmatrix} \text{ for } \begin{bmatrix} z^0\\ z(t)\end{bmatrix}\in E,$$

where $z = z(t)$ denotes the exact solution of the input problem $\mathcal{B}$. On the analogy of (2.6), we introduce the norms in E_n and E_n^0 respectively as follows

$$(2.10)\qquad \|\eta\|_{E_n} = \max_{\nu=0\,(1)\,n} \sum_{p=1}^{6N} \left|\eta_p\left(\frac{\nu}{n}\right)\right|,\qquad \|\delta\|_{E_n^0} = \sum_{p=1}^{6N} |\delta_p(0)| + \max_{\nu=1\,(1)\,n} \sum_{p=1}^{6N} \left|\eta_p\left(\frac{\nu}{n}\right)\right|.$$

In the further considerations we will use the definitions and the theorem quoted below [14].

Definition 1. The method $\mathcal{M}$ and the problem $\mathcal{M}(\mathcal{B})$ is called stable on $\mathcal{B}$ if the discrete problem $\mathcal{M}(\mathcal{B})$ is stable on $\{\Delta_n z\}$, where $z\in E$ denotes the exact solution of $\mathcal{B}$.

Definition 2. The discrete problem $\mathcal{D}$ is called stable on the sequence $\eta = \{\eta_n\}$, $\eta_n \in E_n$, if there exist the constants S and $\bar{r} > 0$ independent of n such that

$$\|\eta_n^{(1)} - \eta_n^{(2)}\|_{E_n} \leqslant S\, \|\Phi_n\, \eta_n^{(1)} - \Phi_n\, \eta_n^{(2)}\|_{E_n^0}$$

holds for each $\eta_n^{(i)}$ $(i = 1, 2)$ satisfying the condition

$$\|\Phi_n\, \eta_n^{(i)} - \Phi_n\, \eta_n\|_{E_n^0} < \bar{r}\,.$$

Definition 3. The method $\mathcal{M}$ (and the problem $\mathcal{M}(\mathcal{B})$) is called consistent with $\mathcal{B}$ with order p, if for all $y\in E$

$$\|\phi_n(\Phi)\,\Delta_n\, y - \Delta_n^0\, \Phi\, y\|_{E_n} = 0\,(n^{-p}) \text{ as } n\to\infty.$$

Definition 4. The sequence of mappings $\Lambda_n : E \to E^0$ $(n \in \mathbb{N})$ such that

$$\phi_n(\Phi)\,\Delta_n\, y \equiv \Phi_n\, \Delta_n\, y = \Delta_n^0\,(\Phi + \Lambda_n)\, y$$

for all y from a domain of Φ, is called a local error-mapping of the method $\mathcal{M}$ on $\mathcal{B}$.

Definition 5. We say that the local error-mapping $\{\Lambda_n\}$ of the discretization method $\mathcal{M}$ for $\mathcal{B}$ has an asymptotic expansion up to the order J, if there exist nonempty subset $D_J \subset E$ and the mappings $\lambda_j : D_J \to E^0$ $(j = 1\ (1)\ J)$ independent of n such that

$$\| \Delta_n^0 (\Lambda_n y - \sum_{j=1}^{J} \frac{1}{n^j} \lambda_j y) \| = 0\,(n^{-(J+1)}) \text{ as } n \to \infty$$

for all $y \in D_J$.

Definition 6. The mapping $\{\Lambda_n\}$ is called (J, p)-smooth, if there exists

$$\lambda_j^{(m)} y \, (\sum_{k=p}^{J} \frac{1}{n^k} e_k)^m \quad \text{for } m = 0\,(1) \left[\frac{J-1}{p} \right]$$

and

$$\sum_{j=1}^{J} \frac{1}{n^j} \left[\lambda_j y + \sum_{m=1}^{\left[\frac{J-j}{p}\right]} \frac{1}{m!} \lambda_j^{(m)} y \, (\sum_{k=p}^{J} \frac{1}{n^k} e_k)^m \right] =$$

$$= \Lambda_n (y + \sum_{k=p}^{J} \frac{1}{n^k} e_k) + 0\,(n^{-(J+1)})$$

holds for arbitrary $y, e_k \in D_{J_n}$ $(k = p\,(1)\,J)$.

Theorem 1 [14]. *Let $\mathcal{M} = \{E_n, E_n^0, \Delta_n, \Delta_n^0, \phi_n\}$ denotes the discretization method applied to $\mathcal{B} = \{E, E^0, \Phi\}$ with the exact solution z. Let us assume that for $\mathcal{M}$ and $\mathcal{B}$ hold:*

(i) *$\mathcal{M}$ is stable on $\mathcal{B}$;*

(ii) *$\mathcal{M}$ is consistent with $\mathcal{B}$ with order p and there exists the local error-mapping of $\mathcal{M}$ on $\mathcal{B}$, which has the asymptotic expansion up to the order J and $z \in D_J$;*

(iii) *the mapping Φ has Fréchet's derivatives up to the order* $[J/p]$, *that satisfy Lipschitz's condition in some domain of the form $B_R = \{y \in E : \| y - z \|_E < R, R > 0\}$ and the asymptotic expansion assumed in* (ii) *is* (J, p)*-smooth;*

(iv) *there exists $\Phi'(z)^{-1}$.*

For $j = 2p\ (1)\ J$ let us define the mappings $g_j : D_{J,p} \times D_{J,p+1} \times \ldots \times D_{J,j-p} \to E^0$ by comparison of the coefficients of n^{-j} in the following relation

$$\sum_{j=2p}^{J} \frac{1}{n^j} g_j (e_p, \ldots, e_{j-p}) =$$

$$= \sum_{m=2}^{[J/2]} \frac{1}{m!} [\Phi^{(m)}(z) + \sum_{j=1}^{J-mp} \frac{1}{n^j} \lambda_j^{(m)}(z)] (\sum_{k=p}^{J} \frac{1}{n^k} e_k)^m + 0\,(n^{-(J+1)}),$$

and for $j = p\ (1)\ 2p - 1$ let us put $g_j \equiv 0$.

If the elements $e_j \in E$ $(j = p\ (1)\ J)$ are defined as follows

$$\Phi'(z)\, e_j = [\, -\lambda_j z + \sum_{k=1}^{j-p} \lambda'_k(z)\, e_{j-k} + g_j\,(e_p, \ldots, e_{j-p})] \,,$$

then the total error $\{\epsilon_n\}$ *of the discretization method* $\mathcal{M}$ *for the input problem* $\mathcal{B}$ *has an unique asymptotic expansion up to the order J, i.e.*

$$\epsilon_n = \Delta_n \sum_{j=p}^{J} \frac{1}{n^j}\, e_j + 0\,(n^{-(J+1)})\,.$$

Using this Theorem, we will show that for the discrete mechanics method (2.7) applied to the input problem (2.4) a total error of discretization has an asymptotic expansion.

3. Stability. Let

$$(3.1)\quad 0 < r = \min_{\substack{t \in [0,\,1] \\ i,\,j=1\,(1)\,N}} r_{ij}(t)\,, \quad R = \max_{\substack{t \in [0,\,1] \\ i,\,j=1\,(1)\,N}} r_{ij}(t)\,, \quad M = \max_{i=1\,(1)\,N} m_i\,,$$

where $r_{ij}(t)$ is defined by (2.3) and m_i denotes the mass of P_i $(i = 1\,(1)\,N)$. Existence of such constants exclude the collisions among the bodies. Moreover, they assure that the mutual distances between them remain finite at any arbitrary moment.

Let the function $f_p\,[y(t)]$ given by (2.5) be approximated by the function $F_p\,[y(t + h), y(t)]$ with the following properties

$$F_p\,[y(t+h),\, y(t)] \to f_p\,[y(t)] \text{ as } h \to 0\,,$$
$$F_p\,[y(t),\, y(t)] \;=\; f_p\,[y(t)] \quad \text{(compare (2.5) and (2.8))},$$

Lemma 1 [9]. *If there exist constants* (3.1), *then for derivatives of the function* $F_p\,[\bar{y}(t), y(t)]$, *where* $\bar{y}(t) = y(t + h)$, *and* $p, q = 1\ (1)\ 3N$ *the following estimations*

$$\left.\begin{aligned} &\left|\frac{\partial F_p}{\partial \bar{y}_q}\right| \\ &\left|\frac{\partial F_p}{\partial \bar{y}_q}\right| \end{aligned}\right\} \leqslant \frac{MNG}{2r^3}\,(6Rr^2 + 1)$$

hold.

Lemma 2 [9]. *If there exist constants* (3.1), *then there exists* $\widetilde{M} \geqslant 1/3N$ *such that*

$$|\, F_p(y, \bar{y}) - F_p(x, \bar{x})\,| \leqslant \widetilde{M} \sum_{q=1}^{3N} (\,|\, y_q - x_q\,| + |\,\bar{y}_q - \bar{x}_q\,|\,);\ p = 1\ (1)\ 3N.$$

From the above lemmas it may be deduced the following theorem:

Theorem 2 [9]. *If there exist constants* (3.1), *then the discrete mechanics method*

(2.7)–(2.8) is stable on the input problem (2.4)–(2.5), and the constants $\bar{r}$ and S from the Definition 2 have the following values

$$\bar{r} = \infty,\ S = e^{3N\tilde{M}} \left\{ \frac{3N([\tilde{M}] - \tilde{M}) + 1}{3N[\tilde{M}] + 1} \right\}^{-3N[\tilde{M}]-1}.$$

4. Consistency and local error-mapping. In [9] it was shown that the method (2.7)–(2.8) is consistent with the problem (2.4)–(2.5). Now we deduce that it is consistent with order 1.

Theorem 3. *If there exist constants* (3.1), *then for the discrete mechanics method* (2.7)–(2.8)

$$\| \phi_n(\Phi)\Delta_n y - \Delta_n^0 \Phi y \|_{E_n^0} = 0(\frac{1}{n}) \text{ as } n \to \infty.$$

Proof. Let us consider the difference $\phi_n(\Phi)\Delta_n y(\nu/n) - \Delta_n^0 \Phi y(\nu/n) = (*)$. From (2.4), (2.7) and (2.9) we have

$$(4.1) \quad (*) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{y_p(\frac{\nu}{n}) - y_p(\frac{\nu-1}{n})}{\frac{1}{n}} - \dfrac{y_{p+3N}(\frac{\nu}{n}) + y_{p+3N}(\frac{\nu-1}{n})}{2} - \\ - y_p'(\frac{\nu-1}{n}) + y_{p+3N}(\frac{\nu-1}{n}) \\ \dfrac{y_{p+3N}(\frac{\nu}{n}) - y_{p+3N}(\frac{\nu-1}{n})}{\frac{1}{n}} - F_p[y(\frac{\nu}{n}),\ y(\frac{\nu-1}{n})] - \\ - y_{p+3N}'(\frac{\nu-1}{n}) + f_p[y(\frac{\nu-1}{n})] \end{bmatrix}$$

Since

$$y_p(\frac{\nu}{n}) = y_p(\frac{\nu-1}{n}) + \frac{1}{n} y_p'(\frac{\nu-1}{n}) + \frac{1}{n^2} y_p''(\xi), \text{ where } \xi \in (\frac{\nu-1}{n}, \frac{\nu}{n});$$

$$y_{p+3N}\left(\frac{\nu}{n}\right)=y_{p+3N}\left(\frac{\nu-1}{n}\right)+\begin{cases}\frac{1}{n}y'_{p+3N}\left(\frac{\nu-1}{n}\right)+\frac{1}{n^2}y''_{p+3N}(\xi_1), \\ \qquad\qquad \text{where } \xi_1\in\left(\frac{\nu-1}{n},\ \frac{\nu}{n}\right) \\ \text{or} \\ \frac{1}{n}y'_{p+3N}(\xi_2),\quad \text{where } \xi_2\in\left(\frac{\nu-1}{n},\ \frac{\nu}{n}\right);\end{cases}$$

$$f_p\left[y\left(\frac{\nu-1}{n}\right)\right]=F_p\left[y\left(\frac{\nu-1}{n}\right),y\left(\frac{\nu-1}{n}\right)\right],$$

then

$$(*)=\begin{bmatrix}0\\0\\\frac{1}{n}\left[y''_p(\xi)-y'_{p+3N}(\xi_2)\right]\\\frac{1}{n}\left[y''_{p+3N}(\xi_1)-\left\{F_p\left[y\left(\frac{\nu}{n}\right),y\left(\frac{\nu-1}{n}\right)\right]-F_p\left[y\left(\frac{\nu-1}{n}\right),y\left(\frac{\nu-1}{n}\right)\right]\right\}\right]\end{bmatrix},$$

If there exist constants (3.1), then the functions y''_p, y'_{p+3N} and y''_{p+3N} are bounded. Moreover,

$$F_p\left[y\left(\frac{\nu}{n}\right),y\left(\frac{\nu-1}{n}\right)\right]-F_p\left[y\left(\frac{\nu-1}{n}\right),y\left(\frac{\nu-1}{n}\right)\right]=$$
$$=\sum_{q=1}^{3N}\frac{\partial F_p(\tilde{y})}{\partial y_q}\left[y_q\left(\frac{\nu}{n}\right)-y_q\left(\frac{\nu-1}{n}\right)\right]=\frac{1}{n}\sum_{q=1}^{3N}\frac{\partial F_p(\tilde{y})}{\partial y_q}y'_q(\xi_3),$$

where $\tilde{y}\in\left(y\left(\frac{\nu-1}{n}\right),y\left(\frac{\nu}{n}\right)\right)$, $\xi_3\in\left(\frac{\nu-1}{n},\ \frac{\nu}{n}\right)$. From the Lemma 1 it follows that the function $\partial F_p/\partial y_q$ is bounded. Thus

$$F_p\left[y\left(\frac{\nu}{n}\right),y\left(\frac{\nu-1}{n}\right)\right]-F_p\left[y\left(\frac{\nu-1}{n}\right),y\left(\frac{\nu-1}{n}\right)\right]=0\left(\frac{1}{n}\right).$$

Therefore we have

$$(*)=\begin{bmatrix}0\\0\\0(1/n)\\0(1/n)\end{bmatrix},$$

Taking into account (2.10), i.e. the definition of norm in the space E_n^0, the Theorem 3 follows readily.

Now we show that there exists a mapping of the local error and that this mapping has an asymptotic expansion up to the order J.

Theorem 4. *For the method* (2.7)–(2.8) *applied to the problem* (2.4)–(2.5) *there exists a mapping of the local error having an asymptotic expansion up to the order J and* $z \in D_J = \underbrace{C^{(J+1)}[0,1] \times \ldots \times C^{(J+1)}[0,1]}_{6N \text{ factors}}$.

Proof. From the Definition 4 we have

$$(4.2)\quad (\Phi_n \Delta_n - \Delta_n^0 \Phi) y = \Delta_n^0 \Lambda_n y .$$

Since $\Phi_n = \phi_n(\Phi)$, when we may rewrite (4.2) in the following form

$$\phi_n(\Phi)\Delta_n y - \Delta_n^0 \Phi y = \Delta_n^0 \Lambda_n y .$$

Hence and from (4.1) the mapping $\{\Lambda_n\}$ may be defined as follows

$$(4.3)\quad \Lambda_n : y \longrightarrow \begin{bmatrix} 0 \\ 0 \\ \dfrac{y_p(t+\frac{1}{n}) - y_p(t)}{\frac{1}{n}} - \dfrac{y_{p+3N}(t+\frac{1}{n}) - y_{p+3N}(t)}{2} - y_p'(t) \\ \dfrac{y_{p+3N}(t+\frac{1}{n}) - y_{p+3N}(t)}{\frac{1}{n}} - \Big\{F_p[y(t+\frac{1}{n}), y(t)] - \\ - F_p[y(t), y(t)]\Big\} - y'_{p+3N}(t) \end{bmatrix} .$$

Therefore, there exists the mapping of local error. Now we show that $\{\Lambda_n\}$ has an asymptotic expansion up to the order J. We have

$$(4.4)\quad \frac{y_p(\frac{\nu}{n}) - y_p(\frac{\nu-1}{n})}{\frac{1}{n}} = y_p'(\frac{\nu-1}{n}) + \begin{cases} \displaystyle\sum_{j=1}^{J} \frac{1}{n^j} \frac{y_p^{(j+1)}(\frac{\nu-1}{n})}{(j+1)!} + 0(n^{-(J+1)}) \\ \text{or} \\ \displaystyle\sum_{j=1}^{J-1} \frac{1}{n^j} \frac{y_p^{(j+1)}(\frac{\nu-1}{n})}{(j+1)!} + 0(n^{-J}) , \end{cases}$$

if $y_p\ (t)\ \in\ C^{(J+1)}\ [0,1]$, but it follows from (2.4) and (2.5). By the same reason $y_{p+3N}\ (t) \in C^{(J+1)}\ [0,1]$, and so

$$\frac{y_{p+3N}(\frac{\nu}{n})-y_{p+3N}(\frac{\nu-1}{n})}{\frac{1}{n}} = y'_{p+3N}(\frac{\nu-1}{n}) + \sum_{j=1}^{J} \frac{1}{n^j} \frac{y_{p+3N}^{(j+1)}(\frac{\nu-1}{n})}{(j+1)!} + 0(n^{-(J+1)}),$$

(4.5)
$$\frac{y_{p+3N}(\frac{\nu}{n})-y_{p+3N}(\frac{\nu-1}{n})}{2} = \frac{1}{2n}\, y'_{p+3N}(\frac{\nu-1}{n}) + \sum_{j=1}^{J} \frac{1}{2n^{j+1}} \frac{y_{p+3N}^{(j+1)}(\frac{\nu-1}{n})}{(j+1)!} + 0\,(n^{-J}).$$

If we assume the existence of constants (3.1), then the derivatives of arbitrary order for F_p are restricted. Therefore

$$F_p[y(\frac{\nu}{n}), y(\frac{\nu-1}{n})] - F_p[y(\frac{\nu-1}{n}), y(\frac{\nu-1}{n})] = \sum_{j=1}^{J} \frac{d^j F_p}{j!} + 0(n^{-(J+1)}),$$

where

$$\frac{d^j F_p}{j!} = \frac{1}{j!} \sum_{i_1,\dots,i_j=1}^{3N} \frac{\partial^j f_p[y(\frac{\nu}{n})]}{\partial y_{i_1}\dots\partial y_{i_j}} \prod_{r=1}^{j} [y_{i_r}(\frac{\nu}{n}) - y_{i_r}(\frac{\nu-1}{n})].$$

Hence and from (4.4) we get

$$F_p[y(\frac{\nu}{n}), y(\frac{\nu-1}{n})] - F_p[y(\frac{\nu-1}{n}), y(\frac{\nu-1}{n})] = \sum_{j=1}^{J} \frac{1}{n^j} \Bigg\{ \sum_{i_1,\dots,i_j=1}^{3N} \frac{\partial^j f_p[y(\frac{\nu-1}{n})]}{\partial y_{i_1}\dots\partial y_{i_j}} \cdot$$

(4.6)
$$\cdot \prod_{r=1}^{j} \left[\sum_{k=1}^{J} \frac{1}{n^{k-1}} \frac{y_{i_r}^{(k)}(\frac{\nu-1}{n})}{k!} \right] \Bigg\} + 0(n^{-(J+1)}) =$$

$$= \sum_{j=1}^{J} \frac{1}{j!\,n^j} \sum_{k=1}^{j} \sum_{i_1,\dots,i_k=1}^{3N} \Bigg\{ \frac{\partial^k f_p[y(\frac{\nu-1}{n})]}{\partial y_{i_1}\dots\partial y_{i_k}} \cdot \sum_{\substack{l_1+\dots+l_k=j \\ l_i>0}} y_{i_1}^{(l_1)}(\frac{\nu-1}{n})\dots y_{i_k}^{(l_k)} \cdot$$

$$\cdot (\frac{\nu-1}{n}) \Bigg\} + 0\,(n^{-(J+1)}).$$

From the relations (4.3)–(4.6) we obtain

$$(4.7)\quad \Delta_n^0 \Lambda_n y(\frac{\nu}{n}) = \begin{bmatrix} 0 \\ 0 \\ \sum_{j=1}^{J} \frac{1}{n^j}\left[\frac{y_p^{(j+1)}(\frac{\nu-1}{n})}{(j+1)!} - \frac{y_{p+3N}^{(j)}(\frac{\nu-1}{n})}{2j!}\right] + 0(n^{-(J+1)}) \\ \sum_{j=1}^{J} \frac{1}{n^j}\left(\frac{y_{p+3N}^{(j+1)}(\frac{\nu-1}{n})}{(j+1)!} - \frac{1}{j!}\sum_{k=1}^{j}\sum_{i_1,\dots,i_k=1}^{3N}\left\{\frac{\partial^k f_p[y(\frac{\nu-1}{n})]}{\partial y_{i_1}\dots\partial y_{i_k}}\cdot\right.\right. \\ \left.\left.\cdot\sum_{\substack{l_1+\dots+l_k=j\\ l_i>0}} y_{i_1}^{(l_1)}(\frac{\nu-1}{n})\dots y_{i_k}^{(l_k)}(\frac{\nu-1}{n})\right\}\right) + 0(n^{-(J+1)}) \end{bmatrix}$$

The mappings λ_j (Def. 5) may be defined as follows

$$(4.8)\quad \lambda_j : y \to \begin{bmatrix} 0 \\ 0 \\ \frac{y_p^{(j+1)}(t)}{(j+1)!} - \frac{y_{p+3N}^{(j)}(t)}{2j!} \\ \frac{y_{p+3N}^{(j+1)}(t)}{(j+1)!} - \frac{1}{j!}\sum_{k=1}^{j}\sum_{i_1,\dots,i_k=1}^{3N}\left\{\frac{\partial^k f_p[y(t)]}{\partial y_{i_1}\dots\partial y_{i_k}}\cdot\right. \\ \left.\cdot\sum_{\substack{l_1+\dots+l_k=j\\ l_i>0}} y_{i_1}^{(l_1)}(t)\dots y_{i_k}^{(l_k)}(t)\right\} \end{bmatrix}$$

Finally let us take into considerations the norm $\|\cdot\|_{E_n^0}$ of the element $\Delta_n^0[\Lambda_n y - \sum_{j=1}^{J}\frac{1}{n^j}\lambda_j y]$.

From (2.10), (4.3),(4.7) and (4.8) we have

$$\|\Delta_n^0[\Lambda_n y - \sum_{j=1}^{J}\frac{1}{n^j}\lambda_j y]\|_{E_n^0} = \max_{\nu=1(1)n}\left\{\sum_{p=1}^{3N}\left|\sum_{j=1}^{J}\frac{1}{n^j}\left[\frac{y_p^{(j+1)}(\frac{\nu-1}{n})}{(j+1)!} - \right.\right.\right.$$

$$\left. - \frac{y_{p+3N}^{(j)}(\frac{\nu-1}{n})}{2j!} - \frac{y_p^{(j+1)}(\frac{\nu-1}{n})}{(j+1)!} + \frac{y_{p+3N}^{(j)}(\frac{\nu-1}{n})}{2j!}\right] + 0(n^{-(J+1)})\right| +$$

$$+ \sum_{p=1}^{3N}\left|\sum_{j=1}^{J}\frac{1}{n^j}\left[\frac{y_{p+3N}^{(j+1)}(\frac{\nu-1}{n})}{(j+1)!} - \frac{1}{j!}\sum_{k=1}^{j}(\cdot) - \frac{y_{p+3N}^{(j+1)}(\frac{\nu-1}{n})}{j!} + \frac{1}{j!}\sum_{k=1}^{j}(\cdot)\right] + \right.$$

$$\left.\left. + 0(n^{-(J+1)})\right|\right\} = \max_{\nu=1(1)n} 0(n^{-(J+1)}) = 0(n^{-(J+1)}),$$

where

$$(\cdot) = \sum_{i_1, \ldots, i_k = 1}^{3N} \frac{\partial^k f_p[y(\frac{\nu-1}{n})}{\partial y_{i_1} \ldots \partial y_{i_k}} \sum_{\substack{l_1 + \ldots + l_k = 1 \\ l_i > 0}} y_{i_1}^{(l_1)} (\frac{\nu-1}{n}) \cdot \ldots \cdot y_{i_k}^{(l_k)} (\frac{\nu-1}{n}) .$$

Since y is an arbitrary element of D_J, then the above conclusion is true for $y = z$ too (z denotes the exact solution of $\mathcal{B}$). Thus the Theorem 4 is proved.

5. (J, p)-smoothness of local error-mapping. Let us deduce first that the mapping Φ given by (2.4) has the Fréchet's derivatives.

Theorem 5. *The mapping Φ defined by* (2.4) *has the Fréchet's derivatives of arbitrary order. If there exist constants* (3.1), *then for this derivatives the Lipschitz's condition holds in the domain $B_R = \{y \in E: \| y - z \|_E < R, R > 0\}$, where z denotes the exact solution of $\mathcal{B}$.*

Proof. From definition of Fréchet's derivative we have

$$(4.9) \quad \underset{h \in E}{\Phi'(y)\, h} = \lim_{\alpha \to \infty} \frac{\Phi(y + \alpha h) - \Phi(y)}{\alpha} = \begin{bmatrix} h_p(0) \\ h_{p+3N}(0) \\ h'_p(t) - h_{p+3N}(t) \\ h'_{p+3N}(t) - \lim\limits_{\alpha \to \infty} \dfrac{f_p(y + \alpha h) - f_p(y)}{\alpha} \end{bmatrix} =$$

$$= \begin{bmatrix} h_p(0) \\ h_{p+3N}(0) \\ h'_p(t) - h_{p+3N}(t) \\ h'_{p+3N}(t) - \sum\limits_{q=1}^{3N} \dfrac{\partial f_p[y(t)]}{\partial y_q} h_q(t) \end{bmatrix},$$

$$(4.10) \quad \Phi^{(m)}(y)h^m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ - \sum\limits_{i_1, \ldots, i_m = 1}^{3N} \dfrac{\partial^m f_p[y(t)]}{\partial y_{i_1} \ldots \partial y_{i_m}} h_{i_1}(t) \ldots h_{i_m}(t) \end{bmatrix}.$$

This means that there exist Fréchet's derivatives of an arbitrary order for the mapping Φ. Now we deduce the Lipschitz's condition $\| \Phi'(y) - \Phi'(\bar{y}) \| \leqslant L \| y - \bar{y} \|$. From (4.9) we have

$$\| \Phi'(y) - \Phi'(\bar{y}) \| = \sup_{\| h \| \leq 1} \| \Phi'(y) h - \Phi'(\bar{y}) h \|_{E^0} =$$

$$= \sup_{\| h \| \leq 1} \max_{t \in [0,1]} \sum_{p=1}^{3N} \left| \sum_{q=1}^{3N} \left\{ \frac{\partial f_p [y(t)]}{\partial y_q} - \frac{\partial f_p [\bar{y}(t)]}{\partial y_q} \right\} h_q(t) \right| \leq$$

$$\leq \sup_{\| h \| \leq 1} \max_{t \in [0,1]} \sum_{p=1}^{3N} \sum_{q=1}^{3N} \left| \frac{\partial f_p [y(t)]}{\partial y_q} - \frac{\partial f_p [\bar{y}(t)]}{\partial y_q} \right| | h_q(t) | .$$

But

$$\frac{\partial f_p [y(t)]}{\partial y_q} = \frac{\partial f_p [\bar{y}(t)]}{\partial y_q} + \sum_{s=1}^{3N} \frac{\partial^2 f_p(\xi)}{\partial y_q \, \partial y_s} [y_s(t) - \bar{y}_s(t)] ,$$

where $\xi \in (y(t), \bar{y}(t))$. If there exist constants (3.1), then the derivatives $\dfrac{\partial^2 f_p}{\partial y_q \, \partial y_s}$ are bounded for example by some A. Hence

$$\| \Phi'(y) - \Phi'(\bar{y}) \| \leq \sup_{\| h \| \leq 1} \max_{t \in [0,1]} \sum_{p=1}^{3N} \sum_{q=1}^{3N} \sum_{s=1}^{3N} A \, | y_s(t) - \bar{y}_s(t) | \cdot | h_q(t) | =$$

$$= \sup_{\| h \| \leq 1} \max_{t \in [0,1]} 3NA \sum_{s=1}^{3N} | y_s(t) - \bar{y}_s(t) | \sum_{q=1}^{3N} | h_q(t) | \leq$$

$$\leq L \| y - \bar{y} \| \sup_{\| h \| \leq 1} \| h \| \leq L \| y - \bar{y} \| .$$

For the derivatives of higher order ($m \geq 2$) we have

$$\| \Phi^{(m)}(y) h^m - \Phi^{(m)}(\bar{y}) h^m \| =$$

$$= \max_{t \in [0,1]} \sum_{p=1}^{3N} \left| \sum_{i_1, \ldots, i_m = 1}^{3N} \left\{ \frac{\partial^m f_p [y(t)]}{\partial y_{i_1} \ldots \partial y_{i_m}} - \frac{\partial^m f_p [\bar{y}(t)]}{\partial y_{i_1} \ldots \partial y_{i_m}} \right\} \cdot h_{i_1}(t) \ldots h_{i_m}(t) \right|$$

and the proof is similar.

Theorem 6. *If there exist constants* (3.1), *then the mapping of the local error, i.e.* $\{\Lambda_n\}$ *defined by* (4.3), *is* (J, 1)*-smooth i.e. the mappings* (4.8) *have Fréchet's derivatives and for arbitrary* $y, e_k \in D_J = \underbrace{C^{(J+1)} [0,1] \times \ldots \times C^{(J+1)} [0,1]}_{6N \text{ factors}}$, $(k = 1 (1) J)$ *holds*

$$\sum_{j=1}^{J}\frac{1}{n^j}\,[\lambda_j y+\sum_{m=1}^{J-j}\frac{1}{m!}\,\lambda_j^{(m)}\,y\left(\sum_{k=1}^{J}\frac{1}{n^k}\,e_k\right)^m]=$$

(4.11)
$$=\Lambda_n\,(y+\sum_{k=1}^{J}\frac{1}{n^k}\,e_k)+0\,(n^{-(J+1)})\,.$$

Proof. From (4.8) we get

$$\lambda_j'(y)\,h=\lim_{\alpha\to 0}\frac{\lambda_j(y+\alpha h)-\lambda_j(y)}{\alpha}=\begin{bmatrix}0\\0\\ \dfrac{h_p^{(j+1)}(t)}{(j+1)!}-\dfrac{h_{p+3N}^{(j)}(t)}{2\,j!}\\ \\ \dfrac{h_{p+3N}^{(j+1)}(t)}{(j+1)!}-\lim\limits_{\alpha\to 0}\,(\cdot)\end{bmatrix},$$

where

$$\lim_{\alpha\to 0}\,(\cdot)=\frac{1}{j!}\sum_{k=1}^{j}\sum_{i_1,\ldots,i_k=1}^{3N}\lim_{\alpha\to 0}\left\{\frac{\partial^k f_p\,[y(t)+\alpha h(t)]}{\partial y_{i_1}\ldots\partial y_{i_k}}\,\cdot\right.$$

$$\cdot\sum_{\substack{l_1+\ldots+l_k=j\\ l_i>0}}\left[y_{i_1}^{(l_1)}(t)+\alpha h_{i_1}^{(l_1)}(t)\right]\ldots\left[y_{i_k}^{(l_k)}(t)+\alpha h_{i_k}^{(l_k)}(t)\right]-$$

$$\left.-\frac{\partial^k f_p\,[y(t)]}{\partial y_{i_1}\ldots\partial y_{i_k}}\sum_{\substack{l_1+\ldots+l_k=j\\ l_i>0}}y_{i_1}^{(l_1)}(t)\ldots y_{i_k}^{(l_k)}(t)\right\}.$$

Let us consider $\lim\limits_{\alpha\to 0}\{\cdot\}$. Since the derivatives $y_{i_s}^{(l_s)}(t)$ are bounded, then

$$\lim_{\alpha\to 0}\{\cdot\}=\lim_{\alpha\to 0}\frac{1}{\alpha}\left[\frac{\partial^k f_p\,(y+\alpha h)}{\partial y_{i_1}\ldots\partial y_{i_k}}-\frac{\partial^k f_p\,(y)}{\partial y_{i_1}\ldots\partial y_{i_k}}\right]\cdot$$

$$\cdot\sum_{\substack{l_1+\ldots+l_k=j\\ l_i>0}}y_{i_1}^{(l_1)}(t)\ldots y_{i_k}^{(l_k)}(t)+\lim_{\alpha\to 0}\frac{\partial^k f_p\,(y+\alpha h)}{\partial y_{i_1}\ldots\partial y_{i_k}}\,\cdot$$

$$\cdot\sum_{\substack{l_1+\ldots+l_k=j\\ l_i>0}}\sum_{s=1}^{k}y_{i_1}^{(l_1)}(t)\ldots y_{i_{s-1}}^{(l_{s-1})}(t)\,y_{i_{s+1}}^{(l_{s+1})}(t)\ldots y_{i_k}^{(l_k)}(t)\,h_{i_s}^{(l_s)}{}_t=(*).$$

But

$$\frac{\partial^k f_p(y+\alpha h)}{\partial y_{i_1}\dots\partial y_{i_k}} = \frac{\partial^k f_p(y)}{\partial y_{i_1}\dots\partial y_{i_k}} + \alpha \sum_{q=1}^{3N} \frac{\partial^{k+1} f_p(y)}{\partial y_{i_1}\dots\partial y_{i_k}\partial y_{i_q}} + \alpha^2 R\,,$$

where R is bounded, because $\partial^s f_p / \partial y_{i_1} \dots \partial y_{i_s}$ are bounded (it follows from the existence of constants (3.1)). Hence

$$(*) = \sum_{i_{k+1}=1}^{3N} \frac{\partial^{k+1} f_p(y)}{\partial y_{i_1}\dots\partial y_{i_{k+1}}} h_{i_{k+1}}(t) \sum_{\substack{l_1+\dots+l_k=j\\ l_i>0}} y_{i_1}^{(l_1)}(t)\dots y_{i_k}^{(l_k)}(t) +$$

$$+ \frac{\partial^k f_p(y)}{\partial y_{i_1}\dots\partial y_{i_k}} \sum_{\substack{l_1+\dots+l_k=j\\ l_i>0}} \sum_{s=1}^{k} h_{i_s}^{(l_s)} \prod_{\substack{q=1\\ q\neq s}}^{k} y_{i_q}^{(l_q)}\,.$$

Thus we have

$$(4.12)\quad \lambda_j'(y)\,h = \begin{bmatrix} 0 \\ 0 \\ \dfrac{h_p^{(j+1)}(t)}{(j+1)!} - \dfrac{h_{p+3N}^{(j)}(t)}{2\,j!} \\ \dfrac{h_{p+3N}^{(j+1)}(t)}{(j+1)!} - \dfrac{1}{j!} \displaystyle\sum_{k=1}^{j} \sum_{l=0}^{1} \sum_{i_1,\dots,i_{k+l}=1}^{3N} \cdot \\ \cdot \dfrac{\partial^{k+l} f_p[y(t)]}{\partial y_{i_1}\dots\partial y_{i_{k+l}}} h_{i_{k+1}}(t)\dots h_{i_{k+l}}(t) \cdot \\ \cdot \displaystyle\sum_{\substack{l_1+\dots+l_k=j\\ l_i>0}} \sum_{\substack{s_1,\dots,s_{1-l}=1\\ s_i\neq s_j \text{ for } i\neq j}}^{k} h_{i_{s_1}}^{(l_{s_1})}(t) \cdot \\ \cdot\dots\cdot h_{i_{s_{1-l}}}^{(l_{s_{1-l}})}(t) \displaystyle\prod_{\substack{q=1\\ q\neq s_i}}^{k} y_{i_q}^{(l_q)}(t) \end{bmatrix}.$$

In the analogous way it may be deduced that

$$(4.13)\quad \lambda_J^{(m)}(y)\,h^m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{1}{j!}\displaystyle\sum_{k=1}^{j}\sum_{l=0}^{m}\binom{m}{l}\sum_{i_1,\ldots,i_{k+l}=1}^{3N} \cdot \\ \cdot\ \dfrac{\partial^{k+l} f_p[y(t)]}{\partial y_{i_1}\ldots\partial y_{i_{k+l}}}\, h_{i_{k+1}}(t)\ldots h_{i_{k+l}}(t)\,\cdot \\ \cdot\ \displaystyle\sum_{\substack{l_1+\ldots+l_k=j\\ l_i>0}}\ \sum_{\substack{s_1,\ldots,s_{m-l}\\ s_i\neq s_j \text{ for } i\neq j}}^{k} h_{i_{s_1}}^{(l_{s_1})}(t)\,\cdot \\ \cdot\ldots\cdot h_{i_{s_{m-l}}}^{(l_{s_{m-l}})}(t)\displaystyle\prod_{\substack{q=1\\ q\neq s_i}}^{k} y_{i_q}^{(l_q)}(t) \end{bmatrix}.$$

This means that the mappings (4.8) have Fréchet's derivatives of an arbitrary order. Now we prove the formula (4.11). Let $y, e_k \in D_J$ and let us note that (compare (4.7))

$$(4.14)\quad \Lambda_n\, y = \begin{bmatrix} 0 \\ 0 \\ \displaystyle\sum_{j=1}^{J}\frac{1}{n^j}\left[\frac{y_p^{(j+1)}(t)}{(j+1)!}-\frac{y_{p+3N}^{(j)}(t)}{2j!}\right]+0\,(n^{-(J+1)}) \\ \displaystyle\sum_{j=1}^{J}\frac{1}{n^j}\left(\frac{y_{p+3N}^{(j+1)}(t)}{(j+1)!}-\frac{1}{j!}\sum_{k=1}^{j}\sum_{i_1,\ldots,i_k=1}^{3N}\left\{\frac{\partial^k f_p[y(t)]}{\partial y_{i_1}\ldots\partial y_{i_k}}\,\cdot\right.\right. \\ \left.\left.\cdot \displaystyle\sum_{\substack{l_1+\ldots+l_k=j\\ l_i>0}}\prod_{q=1}^{k} y_{i_q}^{(l_q)}\right\}\right)+0\,(n^{-(J+1)}) \end{bmatrix}$$

From (4.8), (4.12)–(4.14) we see that the equality (4.11) for the first and for the second component is self-evident. Hereunder the right-hand side of equality (4.11) for the i-th component ($i = 3, 4$) will be denoted by P_i and the left-hand side – by L_i. For the third component we have $\lambda_J^{(m)}(y)\,h^m = 0$ for $m \geqslant 2$, and so ($y_i \equiv y_i(t), e_i \equiv e_i(t)$)

$$P_3 \equiv [\![\Lambda_n(y+\sum_{k=1}^{J}\frac{1}{n^k}e_k)]\!]_3 = \sum_{j=1}^{J}\frac{1}{n^j}\left\{\frac{y_p^{(j+1)}}{(j+1)!}-\frac{y_{p+3N}^{(j)}}{2j!}+\right.$$

$$\left.+\sum_{k=1}^{J}\frac{1}{n^k}\left[\frac{e_{k,p}^{(j+1)}}{(j+1)!}-\frac{e_{k,p+3N}^{(j)}}{2j!}\right]\right\}+0\,(n^{-(J+1)}) =$$

$$= [\![\sum_{j=1}^{J} \frac{1}{n^j} [\lambda_j y + \lambda_j' y (\sum_{k=1}^{J} \frac{1}{n^k} e_k)] + O(n^{-(J+1)})]\!]_3 \equiv L_3 \,.$$

Finally, for the fourth component of (4.11) we get (for simplicity we denote $\sum_{k=1}^{J} \frac{1}{n^k} e_k$ by $\bar{e}$)

$$(4.15)\quad P_4 \equiv [\![\Lambda_n(y+\bar{e})]\!]_4 = \sum_{j=1}^{J} \frac{1}{n^j} \Bigg(\frac{y_{p+3N}^{(j+1)}}{(j+1)!} + \frac{\bar{e}_{p+3N}^{(j+1)}}{(j+1)!} - \frac{1}{j!} \sum_{s=1}^{j} \sum_{i_1,\dots,i_s=1}^{3N} \cdot$$

$$\cdot \left\{ \frac{\partial^s f_p(y+\bar{e})}{\partial y_{i_1} \dots \partial y_{i_s}} \cdot \sum_{\substack{l_1+\dots+l_s=j \\ l_i>0}} \prod_{q=1}^{s} (y_{i_q}^{(l_q)} + \bar{e}_{i_q}^{(l_q)}) \right\} \Bigg) + O(n^{-(J+1)}) =$$

$$= \sum_{j=1}^{J} \frac{1}{n^j} \Bigg(\frac{y_{p+3N}^{(j+1)}}{(j+1)!} + \frac{e_{p+3N}^{(j+1)}}{(j+1)!} - \frac{1}{j!} \sum_{s=1}^{j} \sum_{i_1,\dots,i_s=1}^{3N} \Bigg\{ \Bigg[\frac{\partial^s f_p(y)}{\partial y_{i_1} \dots \partial y_{i_s}} +$$

$$+ \sum_{l=1}^{J-1} \frac{1}{l!} \sum_{i_{s+1},\dots,i_{s+l}=1}^{3N} \frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \dots \partial y_{i_{s+l}}} \cdot \bar{e}_{i_{s+1}} \dots \bar{e}_{i_{s+l}} \Bigg] \cdot \sum_{\substack{l_1+\dots+l_s=j \\ l_i>0}} \cdot$$

$$\cdot \Bigg[\prod_{q=1}^{s} y_{i_q}^{(l_q)} + \sum_{t=1}^{J-1} \frac{1}{t!} \sum_{\substack{r_1,\dots,r_t=1 \\ r_i \neq r_j \text{ for } i\neq j}}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})} \dots \bar{e}_{i_{r_t}}^{(l_{r_t})} \cdot \sum_{\substack{q=1 \\ q\neq i}}^{s} y_{i_q}^{(l_q)} \Bigg] \Bigg\} \Bigg) +$$

$$+ O(n^{-(J+1)}) = \sum_{j=1}^{J} \frac{1}{n^j} \Bigg(\frac{y_{p+3N}^{(j+1)}}{(j+1)!} - \frac{1}{j!} \sum_{s=1}^{j} \sum_{i_1,\dots,i_s=1}^{3N} \Bigg[\frac{\partial^s f_p(y)}{\partial y_{i_1} \dots \partial y_{i_s}} \cdot$$

$$\cdot \sum_{\substack{l_1+\dots+l_s=j \\ l_i>0}} \prod_{q=1}^{s} y_{i_q}^{(l_q)} \Bigg] + \frac{\bar{e}_{p+3N}^{(j+1)}}{(j+1)!} - \frac{1}{j!} \sum_{s=1}^{j} \Bigg\{ \sum_{i_1,\dots,i_s=1}^{3N} \Bigg[\frac{\partial^s f_p(y)}{\partial y_{i_1} \dots \partial y_{i_s}} \cdot$$

$$\cdot \sum_{\substack{l_1+\dots+l_s=j \\ l_i>0}} \sum_{r=1}^{s} \bar{e}_{i_r}^{(l_r)} \prod_{\substack{q=1 \\ q\neq r}}^{s} y_{i_q}^{(l_q)} \Bigg] + \sum_{i_1,\dots,i_{s+1}=1}^{3N} \Bigg[\frac{\partial^{s+1} f_p(y)}{\partial y_{i_1} \dots \partial y_{i_{s+1}}} \bar{e}_{i_{s+1}} \cdot$$

$$\cdot \sum_{\substack{l_1+\dots+l_s=j \\ l_i>0}} \prod_{q=1}^{s} y_{i_q}^{(l_q)} \Bigg] \Bigg\} - \frac{1}{j!} \sum_{s=1}^{j} \Bigg\{ \sum_{i_1,\dots,i_s=1}^{3N} \cdot \Bigg[\frac{\partial^s f_p(y)}{\partial y_{i_1} \dots \partial y_{i_s}} \cdot$$

$$\cdot \sum_{\substack{l_1+\dots+l_s=j \\ l_i>0}} \sum_{t=2}^{J-j} \frac{1}{t!} \sum_{\substack{r_1,\dots,r_t=1 \\ r_i\neq r_j}}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})} \dots \bar{e}_{i_{r_t}}^{(l_{r_t})} \cdot \prod_{\substack{q=1 \\ q\neq r_i}}^{s} y_{i_q}^{(l_q)} \Bigg] +$$

$$+ \sum_{l=2}^{J-j} \frac{1}{l!} \sum_{i_1,\dots,i_{s+l}=1}^{3N} \Bigg[\frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \dots \partial y_{i_{s+l}}} \cdot \bar{e}_{i_{s+1}} \dots \bar{e}_{i_{s+l}} \sum_{\substack{l_1+\dots+l_s=j \\ l_i>0}} \cdot$$

$$\cdot \prod_{q=1}^{s} y_{i_q}^{(l_q)} \Bigg] + \sum_{l=1}^{J-j} \frac{1}{l!} \sum_{i_1,\dots,i_{s+l}=1}^{3N} \Bigg[\frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \dots \partial y_{i_{s+l}}} \cdot \bar{e}_{i_{s+1}} \dots \bar{e}_{i_{s+l}} \cdot$$

$$\cdot \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \sum_{t=1}^{J-j} \frac{1}{t!} \cdot \sum_{\substack{r_1,\ldots,r_t=1 \\ r_i \neq r_j}}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})} \ldots \bar{e}_{i_{r_t}}^{(l_{r_t})} \prod_{\substack{q=1 \\ q \neq r_i}}^{s} y_{i_q}^{(l_q)} \Big]\Big\}\Big) +$$

$$+ 0(n^{-(J+1)}) = [\![\sum_{j=1}^{J} \frac{1}{n^j} [\lambda_j y + \lambda_j'(y) e] + 0(n^{-(J+1)})]\!]_4 - \frac{1}{j!} \sum_{s=1}^{j} \cdot$$

$$\cdot \Big(\sum_{m=2}^{J-j} \frac{1}{m!} \Big\{ \sum_{i_1,\ldots,i_s=1}^{3N} \Big[\frac{\partial^s f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_s}} \cdot \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \sum_{\substack{r_1,\ldots,r_m=1 \\ r_i \neq r_j}}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})} \ldots \bar{e}_{i_{r_m}}^{(l_{r_m})} \cdot$$

$$\cdot \prod_{\substack{q=1 \\ q \neq r_i}}^{s} y_{i_q}^{(l_q)} \Big] + \sum_{i_1,\ldots,i_{s+m}=1}^{3N} \Big[\frac{\partial^{s+m} f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_{s+m}}} \bar{e}_{i_{s+1}} \ldots \bar{e}_{i_{s+m}} \cdot \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \cdot$$

$$\cdot \prod_{q=1}^{s} y_{i_q}^{(l_q)} \Big]\Big\} + \sum_{l=1}^{J-j} \frac{1}{l!} \sum_{i_1,\ldots,i_{s+l}=1}^{3N} \Big[\frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_{s+l}}} \cdot \bar{e}_{i_{s+1}} \ldots \bar{e}_{i_{s+l}} \cdot$$

$$\cdot \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \sum_{t=1}^{J-j} \frac{1}{t!} \cdot \sum_{\substack{r_1,\ldots,r_t=1 \\ r_i \neq r_j}}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})} \ldots \bar{e}_{i_{r_t}}^{(l_{r_t})} \prod_{\substack{q=1 \\ q \neq r_i}}^{s} y_{i_q}^{(l_q)} \Big]\Big) + 0(n^{-(J+1)}).$$

Now we show that

$$(4.16) \quad \sum_{l=1}^{J-j} \frac{1}{l!} \sum_{i_1,\ldots,i_{s+l}=1}^{3N} \Big[\frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_{s+l}}} \bar{e}_{i_{s+1}} \ldots \bar{e}_{i_{s+l}} \cdot \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \cdot$$

$$\cdot \sum_{t=1}^{J-j} \frac{1}{t!} \sum_{\substack{r_1,\ldots,r_t=1 \\ r_i \neq r_j}} \bar{e}_{i_{r_1}}^{(l_{r_1})} \ldots \bar{e}_{i_{r_t}}^{(l_{r_t})} \prod_{\substack{q=1 \\ q \neq r_i}}^{s} y_{i_q}^{(l_q)} \Big] = \sum_{m=2}^{J-j} \frac{1}{m!} \sum_{l=1}^{m-1} \binom{m}{l} \cdot$$

$$\cdot \sum_{i_1,\ldots,i_{s+l}=1}^{3N} \Big[\frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_{s+l}}} \bar{e}_{i_{s+1}} \ldots \bar{e}_{i_{s+l}} \cdot \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \sum_{\substack{r_1,\ldots,r_{m-l}=1 \\ r_i \neq r_j}}^{s} \cdot$$

$$\cdot \bar{e}_{i_{r_1}}^{(l_{r_1})} \ldots \bar{e}_{i_{r_{m-l}}}^{(l_{r_{m-l}})} \prod_{\substack{q=1 \\ q \neq r_i}}^{s} y_{i_q}^{(l_q)} \Big] + 0(n^{-(J-j+1)}).$$

Let us denote the left-hand side of (4.16) by L and the right-hand side by P. We have

$$L = \sum_{l=1}^{J-j} \frac{1}{l!} \sum_{i_1,\ldots,i_{s+l}=1}^{3N} \left\{ \frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_{s+l}}} \bar{e}_{i_{s+1}} \ldots \bar{e}_{i_{s+l}} \cdot \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \cdot \right.$$

$$\cdot \left[\sum_{r_1=1}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})} \prod_{\substack{q=1 \\ q\neq r_1}}^{s} y_{i_q}^{(l_q)} + \frac{1}{2!} \sum_{\substack{r_1,r_2=1 \\ r_1\neq r_2}} \bar{e}_{i_{r_1}}^{(l_{r_1})} \bar{e}_{i_{r_2}}^{(l_{r_2})} \prod_{\substack{q=1 \\ q\neq r_i}}^{s} y_{i_q}^{(l_q)} + \ldots + \frac{1}{(J-j)!} \cdot \right.$$

$$\left.\left. \cdot \sum_{\substack{r_1,\ldots,r_{J-j}=1 \\ r_i\neq r_j}}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})} \ldots \bar{e}_{i_{r_{J-j}}}^{(l_{r_{J-j}})} \cdot \prod_{\substack{q=1 \\ q\neq r_i}}^{s} y_{i_q}^{(l_q)} \right]\right\} = \sum_{l=1}^{J-j-1} \frac{1}{l!} \sum_{i_1,\ldots,i_{s+l}=1}^{3N} \cdot$$

$$\cdot \left[\frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_{s+l}}} \bar{e}_{i_{s+1}} \ldots \bar{e}_{i_{s+l}} \cdot \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \sum_{r_1=1}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})} \prod_{\substack{q=1 \\ q\neq r_1}}^{s} y_{i_q}^{(l_q)} \right] +$$

$$+ 0\,(n^{-(J-j+1)}) + \sum_{l=1}^{J-j-2} \frac{1}{l!} \frac{1}{2!} \sum_{i_1,\ldots,i_{s+l}=1}^{3N} \left[\frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_{s+l}}} \bar{e}_{i_{s+1}} \ldots \bar{e}_{i_{s+l}} \cdot \right.$$

$$\left. \cdot \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \sum_{\substack{r_1,r_2=1 \\ r_1\neq r_2}}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})} \bar{e}_{i_{r_2}}^{(l_{r_2})} \prod_{\substack{q=1 \\ q\neq r_i}}^{s} y_{i_q}^{(l_q)} \right] + 0\,(n^{-(J-j+1)}) + \ldots +$$

$$+ \frac{1}{(J-j-1)!} \sum_{i_1,\ldots,i_{s+1}=1}^{3N} \left[\frac{\partial^{s+1} f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_{s+1}}} \bar{e}_{i_{s+1}} \cdot \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \cdot \right.$$

$$\left. \cdot \sum_{\substack{r_1,\ldots,r_{J-j-1}=1 \\ r_i\neq r_j}}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})} \ldots \bar{e}_{i_{r_{J-j-1}}}^{(l_{r_{J-j}})} \prod_{\substack{q=1 \\ q\neq r_i}}^{s} y_{i_q}^{(l_q)} \right] + 0\,(n^{-(J-j+1)}),$$

i.e..

$$L = \sum_{m=2}^{J-j} \sum_{l=1}^{J-j-(m-1)} \frac{1}{l!\,(m-1)!} \sum_{i_1,\ldots,i_{s+l}=1}^{3N} \frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_{s+l}}} \cdot$$

$$\cdot \bar{e}_{i_{s+1}} \ldots \bar{e}_{i_{s+l}} \sum_{\substack{l_1+\ldots+l_s=j \\ l_i>0}} \sum_{\substack{r_1,\ldots,r_{m-1} \\ r_i\neq r_j}}^{s} \cdot \bar{e}_{i_{r_1}}^{(l_{r_1})} \ldots \bar{e}_{i_{r_{m-1}}}^{(l_{r_{m-1}})} \prod_{\substack{q=1 \\ q\neq r_i}}^{s} y_{i_q}^{(l_q)} +$$

$$+ 0\,(n^{-(J-j+1)}).$$

From (4.16) follows that $L = P$ if and only if

$$(4.17)\quad \sum_{m=2}^{J-j}\ \sum_{l=1}^{J-j-(m-1)} A_{ml} = \sum_{m=2}^{J-j}\ \sum_{l=1}^{m-1} B_{ml}\,.$$

where

$$A_{ml} = \frac{1}{l!\,(m-1)!} \sum_{i_1,\ldots,i_{s+l}=1}^{3N} \frac{\partial^{s+l} f_p(y)}{\partial y_{i_1}\ldots\partial y_{i_{s+l}}}\, \bar{e}_{i_{s+1}}\ldots\bar{e}_{i_{s+l}}\ \cdot$$

$$\cdot \sum_{\substack{l_1+\ldots+l_s=j\\ l_i>0}}\ \sum_{\substack{r_1,\ldots,r_{m-1}=1\\ r_i\neq r_j}}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})}\ldots\bar{e}_{i_{r_{m-1}}}^{(l_{m_{r-1}})} \prod_{\substack{q=1\\ q\neq r_i}}^{s} y_{i_q}^{(l_q)}\,,$$

$$B_{ml} = \frac{1}{l!\,(m-l)!} \sum_{i_1,\ldots,i_{s+l}=1}^{3N} \frac{\partial^{s+l} f_p(y)}{\partial y_{i_1}\ldots\partial y_{i_{s+l}}}\, \bar{e}_{i_{s+1}}\ldots\bar{e}_{i_{s+l}}\ \cdot$$

$$\cdot \sum_{\substack{l_1+\ldots+l_s=j\\ l_i>0}}\ \sum_{\substack{r_1+\ldots+r_{m-l}=1\\ r_i\neq r_j}}^{s} \bar{e}_{i_{r_1}}^{(l_{r_1})}\ldots\bar{e}_{i_{r_{m-l}}}^{(l_{r_{m-l}})} \prod_{\substack{q=1\\ q\neq r_i}}^{s} y_{i_q}^{(l_q)}\,.$$

Let us note that

$$A_{m1} = B_{m1} \quad \text{for } m = 2\,(1)\,J-j,$$

$$A_{m2} = B_{m+1,2} \quad \text{for } m = 2\,(1)\,J-j-1,$$

$$A_{m3} = B_{m+2,3} \quad \text{for } m = 2\,(1)\,J-j-2,$$

. . . .

$$A_{m,J-j-m+1} = B_{J-j,J-j-m+1} \quad \text{for } m = 2,$$

i.e.

$$A_{pq} = B_{p+q-1,q} \quad \text{for } q = 1\,(1)\,J-j-1,\ \ p = 2\,(1)\,J-j-q+1.$$

This equality precises the relationship between all components of the left and right sides of (4.17). So we have

$$\sum_{m=2}^{J-j}\ \sum_{l=1}^{J-j-(m-1)} A_{ml} = \sum_{q=1}^{J-j-1}\ \sum_{p=2}^{J-j-q+1} A_{pq} =$$

$$= \sum_{q=1}^{J-j-1}\ \sum_{p=2}^{J-j-q+1} B_{p+q-1,q} = \sum_{m=2}^{J-j}\ \sum_{l=1}^{m-1} B_{ml}\,.$$

Thus the equality (4.16) is truthful. Since (4.15) and (4.16) yield

$$P_4 = \Big[\!\Big[\sum_{j=1}^{J} \frac{1}{n^j} \left[\lambda_j y + \lambda_j'(y) e\right] + 0\left(n^{-(J+1)}\right) \Big]\!\Big]_4 + \sum_{j=1}^{J} \frac{1}{n^j} \Bigg(\sum_{m=2}^{J-j} \frac{1}{m!} \Big\{ - \frac{1}{j!} \cdot$$

$$\cdot \sum_{s=1}^{j} \sum_{l=0}^{m} \sum_{i_1, \ldots, i_{s+l}=1}^{3N} \left[\frac{\partial^{s+l} f_p(y)}{\partial y_{i_1} \ldots \partial y_{i_{s+l}}} \right] \overline{e}_{i_{s+1}} \ldots \overline{e}_{i_{s+l}} \cdot \sum_{\substack{l_1 + \ldots + l_s = j \\ l_i > 0}} \cdot$$

$$\cdot \sum_{\substack{r_1, \ldots, r_{m-l}=1 \\ r_i \neq r_j}}^{s} \overline{e}_{i_{r_1}}^{(l_{r_1})} \ldots \overline{e}_{i_{r_{m-l}}}^{(l_{r_{m-l}})} \cdot \prod_{\substack{q=1 \\ q \neq r_i}}^{s} y_{i_q}^{(l_q)} \Big\} + 0\left(n^{-(J-j+1)}\right) \Bigg),$$

then from (4.13) follows that

$$P_4 = \Big[\!\Big[\sum_{j=1}^{J} \frac{1}{n^j} \Big[\lambda_j y + \lambda_j'(y) e + \sum_{m=2}^{J-j} \frac{1}{m!} \lambda_j^{(m)}(y) e^m\Big] + 0\left(n^{-(J+1)}\right) \Big]\!\Big]_4 = L_4 .$$

Thereby the equality (4.11) is proved. This brings the proof of Theorem 6 to an end.

6. Expansion of total error. The Theorems 2–6 ensure the fulfilment of conditions (i)–(iii) of the Theorem 1. From (4.9) it follows that there exist $\Phi'(z)^{-1}$, where z denotes the exact solution of (2.4)–(2.5). So the condition (iv) of the Theorem 1 is satisfied too. Thereby we have

Theorem 7. *If there exist constants* (3.1), *then the method of discrete mechanics* (2.7)–(2.8) *has an unique asymptotic expansion of the total discretization error up to the order J, i.e.*

$$\epsilon_n = \Delta_n \sum_{j=1}^{J} \frac{1}{n^j} e_j = 0\left(n^{-(J+1)}\right),$$

where $e_j \in \underbrace{C^{(1)}[0,1] \times \ldots \times C^{(1)}[0,1]}_{6N \text{ factors}}$.

From the above theorem follows [13] that if $\{\mathbb{G}_k\}$ is the sequence of nets on the time interval $[0, 1]$, where $\mathbb{G}_k = 1/\beta_k n$, $\{\beta_k\}$ is the exactly increasing sequence of natural numbers and $\eta_{(k)p}$ $(p = 1\,(1)\,6N)$ denotes the solution obtained by the method (2.7)–(2.8) on the net $\mathbb{G}_k$, then the solution

$$\eta_p\left(\frac{\nu}{n}\right) = \sum_{k=1}^{J} a_k \eta_{(k)p}\left(\frac{\nu}{n}\right), \quad \nu = 0\,(1)\,n,$$

where the coefficients a_k are calculated from equations

$$\sum_{k=1}^{J} a_k = 1, \quad \sum_{k=1}^{J} a_k / (\beta_k n)^j = 0, \quad j = 1\,(1)\,J-1,$$

has a precision of the order J.

REFERENCES

[1] Albrycht, J., Marciniak, A., *Orbit calculations near by the equilibrium points by a discrete mechanics method*, Celestial. Mech. 24 (1981), 391–405.

[2] Albrycht, J., Marciniak, A., *Discrete dynamical equations in Minkowski space*, Internat. J. Theoret. Phys. 20 (1981), 821–830.

[3] Greenspan, D., *A new explicit discrete mechanics with applications*, J. Franklin Inst. 294 (1972), 231–240.

[4] Greenspan, D., *New forms of discrete mechanics*, Kybernetes 1 (1972), 87–101.

[5] Greenspan, D., *An algebraic, energy conserving formulation of classical molecular and Newtonian n-body interaction*, Bull. Amer. Math. Society 2 (1972), 423–427.

[6] Greenspan, D., *Symmetry in discrete mechanics*, Found. Phys. 2 (1973), 247–253.

[7] Greenspan, D., *The arithmetic basis of special relativity*, Internat. J. Theoret. Phys. 15 (1976), 557–574.

[8] Greenspan, D., *Arithmetic Applied Mathematics*, Pergamon Press, Oxford 1980.

[9] Marciniak, A., *Application of discrete mechanics to the numerical solution of n-body problem and verification of this theory for the Solar system* (in Polish), Doctor thesis, A. Mickiewicz University, Poznań 1981.

[10] Marciniak, A., *Discrete mechanics and its application to the solution of n-body problem*, Acta Appl. Math. (to apear).

[11] Marciniak, A., *Discrete Hill's equations*, Comp. Meth. Appl. Mech. Engin. 37 (1983), 15–24.

[12] Marciniak, A., *Discrete mechanics of arbitrary order*, Comp. Meth. Appl. Mech. Engin. (to appear).

[13] Marcuk, G. J., Sajdurov, V. V., *An Increase of Precision for Solutions of Difference Schemes* (in Rusian), Izdatel'stvo "Nauka", Moscow 1979.

[14] Stetter, H. J., *Analysis of Discretization Methods for Ordinary Differential Equations*, Springer--Verlag, Berlin 1973.

STRESZCZENIE

W pracy tej autorzy badają istnienie asymptotycznego rozwinięcia dla błędu w metodzie Greenspana mechaniki dyskretnej. Pewne zastosowanie tych wyników zostało podane w pracy drugiego z autorów [12].

РЕЗЮМЕ

В этой работе авторы исследуют существование асимптотического разложения ошибки в методе Гринспана дискретной механики. Некоторые применения этих результатов приведены в работе второго автора [12].