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On Real Functions of Bounded Variation and an Application to Geometric Function Theory

O funkcjach rzeczywistych o wariacji ograniczonej i o ich zastosowaniu do geometrycznej teorii funkcji

О вещественных функциях ограниченной вариации и их применение в геометрической теории функции

1. Introduction. The main objective of this paper is to establish a theorem on the approximation of certain functions of bounded variation: Let $m, n \in \mathbf{N}$ and $f: [0,1] \to \mathbf{R}$ continuous such that

$$\begin{aligned} &\text{i)}\quad f(0)=0,\ f(1)=m-n, \\ &\text{ii)}\quad V_0^1(f) \leq m+n, \end{aligned} \tag{1}$$

where V_0^1 denotes the variation on $[0,1]$. For $c \in [0,1]$ define the step functions $g(\cdot, c): \mathbf{R} \to \mathbf{R}$ with

$$g(x,c) = \lim_{\epsilon \to +0} \tfrac{1}{2}([x-c+1+\epsilon] + [x-c+1-\epsilon]).$$

Theorem 1. *There exist numbers $c_j, d_j \in (0,1]$, $\mu \in [-\tfrac{1}{2}, \tfrac{1}{2}]$, such that for $x \in [0,1]$*

$$\left| f(x) - \sum_{j=1}^{m-1} g(x,c_j) + \sum_{j=1}^{n-1} g(x,d_j) - \mu \right| \leq \tfrac{1}{2}. \tag{2}$$

We have not been able to find a really elementary proof for this apparently simple result. The crucial part in our development is played by St. Banach's quite deep theorem on the indicatrix of a continuous function of bounded variation. However, Theorem 1 with the (best possible) constant ½ in (2) replaced by 1 is almost trivial.

We believe that Theorem 1 admits applications to various fields and we wish to point out the following corollary in function theory. A function F normalized by $F(0) = 0$, $F'(0) = 1$, analytic in the unit disc $\Delta = \{z : |z| < 1\}$ is called starlike of order a ($F \in S^*(a)$) if and only if

$$\operatorname{Re} \frac{zF'(z)}{F(z)} > a, \quad z \in \Delta.$$

For $m, n \in \mathbf{N}$ let $Q(m, n)$ be the class of functions

$$G = F_1 / F_2, \; F_1 \in S^*\left(1 - \frac{m}{2}\right), \; F_2 \in S^*\left(1 - \frac{n}{2}\right).$$

It is known that for any $G \in Q(m, n)$ the function

$$f(x) = \lim_{r \to 1} \arg G(r e^{2\pi i x})$$

has properties closely related to the assumptions of Theorem 1, for details see below. Using that theorem we obtain:

Theorem 2. *For $G \in Q(m, n)$ there exist $x_j, y_j \in \partial\Delta$ and $\mu \in \mathbf{R}$ such that*

$$\operatorname{Re} \left[e^{i\mu} \frac{\prod_{j=1}^{m-1} (1 + x_j z)}{\prod_{j=1}^{n-1} (1 + y_j z)} G(z) \right] > 0, \quad z \in \Delta. \tag{3}$$

This theorem generalizes a number of previous results and contains a considerable amount of new information. In fact, assume $F \in S^*(1 - m/2)$, $H \in S^*(½)$. Then according to our theorem we can find $x_j \in \partial\Delta$, $\mu \in \mathbf{R}$, such that

$$\operatorname{Re} \left[e^{i\mu} \prod_{j=1}^{m-1} (1 + x_j z) \frac{F(z)}{H(z)} \right] > 0, \quad z \in \Delta. \tag{4}$$

For $m = 1$ this corresponds to [6, Theorem 2.25]. For $n = 2$ and $H(z) = z / (1 + yz)$, $y \in \partial\Delta$, we obtain

$$\operatorname{Re} \left[e^{i\mu} (1 + yz)(1 + xz) F(z) \right] > 0, \quad z \in \Delta. \tag{5}$$

This formula played a major role in the first proof of the Pólya-Schoenberg conjecture [5]. The extension of (5) to $m > 2$ was given in [4]. Note that (4) is much stronger than (5).

Finally consider the class V_k of functions G normalized as above with $G' \neq 0$ in Δ and of boundary rotation at most $2k\pi$. It is known that $G \in V_k$ if and only if $G' \in Q(k + 1, k - 1)$. A corollary to (3) in this particular case is the following result:

Theorem 3. *Let $G \in V_k$, $k \in \mathbf{N}$, $k \geq 2$. Then there exist numbers $x_1, x_2 \in \partial\Delta$, $\mu \in \mathbf{R}$, such that*

$$|\arg(e^{j\mu}(1 + x_1 z)(1 + x_2 z) G'(z))| \leq (k-1)\frac{\pi}{2}. \tag{6}$$

It is natural to conjecture that this holds for $k \in \mathbf{R}$, $k \geq 2$, as well.

Theorem 3 is of particular interest when $k = 2$. It has already been known to Paatero [2] who introduced domains of bounded boundary variation without reference to analytic functions that a domain of boundary rotation at most 4π is schlicht. After introduction of the concept of close-to-convex domains (functions) it was easy to prove (compare [6, Corollary 2.27]) that any such domain is in fact close-to-convex (i.e. its complement can be covered by non-intersecting half lines). As a consequence of Theorem 3 and a recent result of Royster and Ziegler [3] we now have an even stronger conclusion. [1]

Theorem 4. *Let Ω be a domain of boundary rotation at most 4π (in the sense of Paatero). Then Ω is convex in at least one direction.*

It is known that any domain of boundary rotation 2π is convex (in every direction). It is likely that there is continuous passage connecting these two extreme cases for domains of boundary rotation at most $2k\pi$, $1 < k < 2$.

2. Proof of Theorem 1. Without loss of generality we may assume that f is nowhere constant, i.e. there is no intervall $(a, b) \subset [0, 1]$ such that f restricted to (a, b) is constant. Let τ be the set of numbers in $(0, 1)$ where f has a local extremum. For $y \in \mathbf{R}$ let

$$\nu(y) = \{x \in (0, 1) : f(x) = y\},$$

and for $y \in [0, 1]$

$$\nu_0(y) = \bigcup_{k \in \mathbf{Z}} \nu(y + k),$$

$$\lambda(y) = \nu_0(y) \setminus \tau.$$

We shall use # to indicate the cardinality of a set.

Lemma. i) *If $\tau = \emptyset$ or $f(\tau) \subset \mathbf{Z}$ we have $\#\lambda(0) \leq n + m - 2$.* ii) *If $f(\tau) \not\subset \mathbf{Z}$ then there exists $y_0 \in (0, 1)$ with $\#\lambda(y_0) \leq n + m - 1$.*

Proof. i) If $\tau = \emptyset$ then f is monotonic and thus

$$\#\lambda(0) = \#\nu_0(0) = |m - n| - 1 \leq n + m - 2.$$

If $\tau \neq \emptyset$ and $f(\tau) \subset \mathbf{Z}$ we have $V_a^b(f) = 1$ for any two subsequent elements a, b of $\nu_0(0)$ and therefore $\#\nu_0(0) \leq n + m - 1$. However, $\nu_0(0)$ contains at least one element of τ and the conclusion follows.

[1] A weaker form of this result is due to Renyi, A. Publ. Math. Debrecen, 1, (1949) 18–23.

ii) $\# \nu(y)$ is Banach's indicatrix which is measurable and satisfies (compare [1, p. 254])

$$\int_{-\infty}^{\infty} \# \nu(y)\,dy \leqslant V_0^1(f) \leqslant n+m,$$

hence

$$\int_0^1 \# \nu_0(y)\,dy \leqslant n+m. \tag{7}$$

Let us assume

$$\# \nu_0(y) \geqslant n+m, \quad y \in (0,1), \tag{8}$$

since otherwise we are done. If there exists $y_1 \in (0,1)$ for which strict inequality holds in (8) we may choose $n+m+1$ elements

$$a_1, \dots, a_r,\ b_1, \dots, b_s,\ c_1, \dots, c_t$$

from $\nu_0(y_1)$ where $r, s, t \geqslant 0$, $r+s+t = m+n+1$. Here a_j correspond to maxima, b_j to minima of f while $c_j \notin \tau$. Assume $r \leqslant s$. Since f is nowhere constant there exists $\varepsilon > 0$ such that for any $y \in [y_1, y_1+\varepsilon]$ the equation

$$y = f(x) - [f(x)], \quad x \in (0,1),$$

has at least $m+n+1$ solutions (each of the s minima b_j splits into at least two solutions which compensates the loss of the r solutions corresponding to the maxima a_j). Thus $\# \nu_0(y) \geqslant n+m+1$ for $y \in [y_1, y_1+\varepsilon]$ and with (7) we obtain

$$\int_{[0,1]\setminus[y_1, y_1+\varepsilon]} \# \nu_0(y)\,dy \leqslant n+m-(n+m+1)\varepsilon < (n+m)(1-\varepsilon). \tag{9}$$

Similary, if $r > s$ we find $\varepsilon > 0$ such that

$$\int_{[0,1]\setminus[y_1-\varepsilon, y_1]} \# \nu_0(y)\,dy < (n+m)(1-\varepsilon). \tag{10}$$

(9) or (10) show that $\# \nu_0(y) < n+m$ on a set positive measure and thus $\# \nu_0(y_0) < n+m$ for at least one $y_0 \in (0,1)$ which contradicts (8). Hence

$$\# \nu_0(y) = n+m, \; y \in (0,1). \tag{11}$$

From the assumption we have $x_0 \in \tau$, $f(x_0) \notin \mathbb{Z}$, which implies

$$y_0 = f(x_0) - [f(x_0)] \in (0,1).$$

Since $x_0 \in \nu_0(y_0)$ we get from (11): $\# \lambda(y_0) \leqslant n+m-1$.

Proof of Theorem 1. According to the Lemma we find $y_0 \in [0, 1)$ with $\# \lambda(y_0) < < n + m$. Since this set is finite it is clear that f is of increasing or decreasing type in every $c \in \lambda(y_0)$. (A function f is said to be of increasing type at c if there is an $\epsilon > 0$ such that $f(x) < f(c)$ for $x \in (c - \epsilon, c)$ and $f(x) > f(c)$ for $x \in (c, c + \epsilon)$); decreasing type is defined accordingly). Let $c_1, \ldots, c_r$ be the elements of $\lambda(y_0)$ where f is of increasing type and $d_1, \ldots, d_s$ the elements of $\lambda(y_0)$ where f is of decreasing type. Then by the Lemma we may assume

$$s + r \leqslant \begin{cases} n + m - 1, & y_0 \in (0, 1), \\ n + m - 2, & y_0 = 0. \end{cases} \tag{12}$$

Now let

$$h(x) = f(x) - \sum_{j=1}^{r} g(x, c_j) + \sum_{j=1}^{s} g(x, d_j), \quad x \in [0, 1]. \tag{13}$$

Consider the sets $I_k = [y_0 + k, y_0 + k + 1]$, $k \in \mathbf{Z}$, and two subsequent elements a, b of $\lambda(y_0)$. Since $\lambda(y_0) \cap (a, b) = \emptyset$ the range of f restricted to (a, b) is contained is a certain I_k and the same holds for h since in (a, b) f and h differ by an integral constant. The same argument works in the intervals $[0, a)$, $(b, 1]$ if a, b denote the smallest and the largest element of $\lambda(y_0)$, respectively. Now let $c \in \lambda(y_0)$ and assume that f is of increasing type at c. Then there exists $e > 0$ such that h (which has a jump of length -1 at c) maps $(c - \epsilon, c + e)$ into one of the sets I_k. The same conclusion holds when f is of decreasing type in $c \in \lambda(y_0)$. These considerations show that there must be one single set I_k which contains the range of $h(x)$, $x \in [0, 1]$. Since h is continuous at $x = 0, x = 1$ with $h(0) = = f(0) = 0$ we see that this set must be $[y_0 - 1, y_0]$ if $y_0 \neq 0$ or one of $[-1, 0]$, $[0, 1]$ if $y_0 = 0$. We need to distinguish three possible cases: $h(1) = 0, \pm 1$.

i) If $h(1) = 0$ we obtain from (13) at $x = 1$: $r - s = m - n$. We set

$$\tilde{h}(x) = h(x) \quad r_1 = r, \ s_1 = s. \tag{14}$$

ii) If $h(1) = 1$ such that the range of h lies in $[0, 1]$ we must have $y_0 = 0, r - s = = m - n - 1$. We set

$$\tilde{h}(x) = h(x) - g(x, 1), \ r_1 = r + 1, \ s_1 = s, \ c_{r_1} = 1 \tag{15}$$

iii) If $h(1) = -1$ such that the range of h lies in $[-1, 0]$ we must have $y_0 = 0$, $r - s = m - n + 1$. We set

$$\tilde{h}(x) = h(x) + g(x, 1), \ r_1 = r, s_1 = s + 1, \ d_{s_1} = 1. \tag{16}$$

Note that according to (12) we have in any of the three cases

$$\begin{aligned} r_1 - s_1 &= m - n, \\ r_1 + s_1 &\le m + n - 1. \end{aligned} \tag{17}$$

Also, the range of $\widetilde{h}$ lies in the same strip as the range of h and we obtain $\mu \in [-½, ½]$ such that

$$|\widetilde{h}(x) - \mu| \le ½, \ x \in [0, 1]. \tag{18}$$

From (17) we obtain $r_1 \le m-1$, $s_1 \le n-1$. If $r_1 = m-1$ (and thus $s_1 = n-1$) (18) is already the assertion (2). However, if $r_1 < m-1$ we choose an arbitrary $c \in (0, 1)$ and put

$$c = c_{r_1+1} = \ldots = c_{m-1} = d_{s_1+1} = \ldots = d_{n-1}.$$

Since $r_1 - s_1 = m - n$ we get for $x \in [0, 1]$

$$\widetilde{h}(x) = \widetilde{h}(x) - \sum_{j=r_1+1}^{m-1} g(x, c_j) + \sum_{j=s_1+1}^{n-1} g(x, c_j)$$

si that (2) follows from (18) also in this case.

3. Proofs of Theorems 2–4.

Proof of Theorem 2. Let $G = F_1/F_2$ where $F_1 \in S^*(1-(m/2))$, $F_2 \in S^*(1-(n/2))$. For $0 < r < 1$ let $G_r(z) = G(rz) = (F_1(rz)/r)/(F_2(rz)/r)$. Then $F_1(rz)/r$ and $F_2(rz)/r$ are starlike of the same respective orders and continuous in $|z| \le 1$. Assume Theorem 2 has been established for $G_r, 0 < r < 1$. Then an obvious limiting procedure gives the result for G. Thus it suffices to prove Theorem 2 for $G = F_1/F_2 \in Q(m, n)$ with F_1, F_2 continuous in $|z| \le 1$.

Let $F \in S^*(1-(m/2))$ be continuous in $|z| \le 1$. Then there exists $\widetilde{F} \in S^*(0)$ continuous in $|z| \le 1$ such that $F = z(\widetilde{F}/z)^{m/2}$. The function

$$V(x) = \frac{1}{\pi} \arg(\widetilde{F}(e^{2\pi i x}))$$

is continuous, monotonic increasing with $V(1) - V(0) = 2$. This proves the existence of two such functions V_1, V_2 such that

$$\frac{1}{\pi} \arg G(e^{2\pi i x}) = (n-m)x + \frac{m}{2} V_1(x) - \frac{n}{2} V_2(x). \tag{19}$$

Now let

$$f(x) = \frac{m}{2}(V_1(x) - V_1(0)) - \frac{n}{2}(V_2(x) - V_2(0)), \ x \in [0, 1] \tag{20}$$

f fulfills the assumptions of Theorem 1 and we find

$$p(x) = \sum_{j=1}^{m-1} g(x, c_j) - \sum_{j=1}^{n-1} g(x, d_j)$$

such that for a certain $\mu \in \mathbf{R}$

$$|f(x) - p(x) - \mu| \leq \tfrac{1}{2} \tag{21}$$

holds for $x \in [0, 1]$. For $c \in (0, 1]$ one easily deduces

$$\lim_{r \to 1} \frac{1}{\pi} \arg (1 - re^{2\pi i(x-c)}) = x - g(x, c) + \frac{1}{2} - c$$

and thus

$$p(x) = \lim_{r \to 1} \frac{1}{\pi} \arg \frac{\prod_{j=1}^{n-1} (1 + y_j z)}{\prod_{j=1}^{m-1} (1 + x_j z)} \mid (m-n) x \mid \phi \tag{22}$$

for $x \in [0, 1]$ and a certain constant ϕ. Here we used $y_j = \exp(i\pi(1 - 2c_j))$, $x_j = \exp(i\pi(1 - 2d_j))$, $z = r \cdot \exp(2\pi i x)$. A combination of (19)–(22) proves

$$\left| \lim_{r \to 1} \arg \left[e^{i\tilde{\mu}} \frac{\prod_{j=1}^{m-1} (1 + x_j z)}{\prod_{j=1}^{n-1} (1 + y_j z)} G(z) \right] \right| \leq \frac{\pi}{2}, \ x \in [0, 1].$$

where z is as above. That this relation extends to $z \in \Delta$ follows from a standard argument involving Poisson's integral formula and Lebesgue's dominated convergence theorem. Theorem 2 is proved.

Proof of Theorem 3. Since $G \in V_k$ if and only if $G' \in Q(k+1, k-1)$ we obtain from Theorem 2

$$|\arg (e^{i\mu} (1 + x_1 z)(1 + x_2 z) P(z) G'(z))| < \frac{\pi}{2}. \tag{23}$$

where

$$P(z) = \prod_{j=1}^{k-2} \frac{1 + u_j z}{1 + v_j z}, \ u_j, v_j \in \partial\Delta.$$

This implies $|\arg [e^{i\phi} P(z)]| < (k-2)\pi/2$ for a certain $\phi \in \mathbf{R}$ and $z \in \Delta$. The conclusion follows from (23).

Proof of Theorem 4. We may assume that there exists $G \in V_2$ with $G(\Delta) = \Omega$ since this can be achieved by translating and stretching Ω. These operations affect neither the assumption nor the conclusion of the theorem. Thorem 3 gives

$$\mathrm{Re}\,[e^{i\phi}(1 + x_1 z)(1 + x_2 z)\, G'(z)] > 0, z \in \Delta ,$$

for certain $\phi \in \mathbf{R}, x_1, x_2 \in \partial\Delta$. By an obvious extension of a recent result of Royster and Ziegler [3] we see that Ω is convex in at least one direction.

REFERENCES

[1] Natanson, I. P., *Theorie der Funktionen einer reellen Veränderlichen*, Zürich, Frankfurt, Thun 1975.

[2] Paatero, V., *Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind*, and: *Über Gebiete von beschränkter Randdrehung*, Ann. Acad. Sci. Fenn. A 33 (1931) and 37 (1933).

[3] Royster, W. C., Ziegler, M., *Univalent functions convex in one direction*, Publ. Math. Debrecen 23 (1976), 339–345.

[4] Ruschweyh, St., *On starlike functions*, Ann, Univ. Mariae Curie-Skłodowska, 28 (1974), 65–70.

[5] Ruschweyh, St., Sheil-Small, T., *Hadamard products of schlicht functions and the Pólya–Schonenberg conjecture*, Comment. Math. Helv., 48 (1973), 119–135.

[6] Schober, G., *Univalent Functions – Selected Topics*, Lecture Notes in Mathematics 478. Heidelberg, Berlin, New York 1975.

STRESZCZENIE

Główny wynik pracy (Tw. 1) dotyczy aproksymacji funkcji o wahaniu ograniczonym. Stosuje się to następnie do wykazania kilku twierdzeń o funkcjach jednolistnych.

РЕЗЮМЕ

Главный результат работы (Теорема 1) касается апроксимации функции с ограниченной вариацией. Применяется это для доказательства нескольких теорем об однолистных функциях.