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Starlike Continued Fractions and Zeros of Polynomials

Ułamki ciągłe gwiaździste i zera wielomianów

Звездообразные непрерывные дробы и нули полинотов

1. Introduction. This paper contains two distinct application of the value region technique commonly associated with continued fractions. The first application is the determination of the radius of starlikeness of order α, $0 \leqslant \alpha < 1$, of a class of analytic functions in the unit disk D, each member of which has a certain S-fraction expansion. The second is the derivation of zero-free regions for certain classes of polynomials. The proofs herein for these applications are dependent on a pair of interrelated lemmas.

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers in the disk $|w| \leqslant 1/4$, where $a_{m+j} = 0$ $(j = 1, 2, \ldots)$ when $a_m = 0$, then the S-fraction

$$f(z) = \frac{z}{1} + \frac{a_1 z}{1} + \frac{a_2 z}{1} + \cdots + \frac{a_n z}{1} + \cdots \tag{1}$$

converges in D to an analytic function. Let W be the class of analytic functions f in D that have a representation (1), where $|a_n| \leqslant 1/4$ $(n = 1, 2, \ldots)$. Some years ago, Thale [6] proved that if $f \in W$, then f is univalent for $|z| \leqslant 12\sqrt{2} - 16$ and Perron [4] proved this result is sharp in the class W. F. V. Atkinson [1] showed by a formidable computational method that the radius of univalence for the class W is indeed the radius of starlikeness of the class. We present here a simpler method to verify this fact along with some extensions.

Theorem 1. *If $f \in W$ then for $|z| \leqslant 4r(1-r)$, $0 \leqslant r \leqslant 1 - 1/\sqrt{3}$, the expression $w = zf'/f$ is in the intersection of the regions*

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$$\left| w - \frac{r(1-r)}{1-2r} \right| < \frac{(1-r)^2}{1-2r}, \quad \operatorname{Re} w > \frac{1-2r-r^2}{(1+r)(1-2r)}. \tag{2}$$

In particular, each $f \in W$ is univalent and starlike of order α, $0 \leq \alpha < 1$, for $|z| \leq 4 r_\alpha (1 - r_\alpha)$, where

$$r_\alpha = \frac{2(1-\alpha)}{2-\alpha+\sqrt{9\alpha^2-16\alpha+8}} \tag{3}$$

The function $f_0(z) = 2z / [3 - \sqrt{1+z}]$ is in W and is starlike of exact order α in $|z| < 4 r_\alpha (1 - r_\alpha)$ and is not univalent in any disk $|z| < \rho$ when $\rho > 4 r_0 (1 - r_0) = 12\sqrt{2} - 16$.

The method for the proof of Theorem 1 was suggested by the argument used to verify the following generalization of the work of Saff and Varga [5] (see also [2]).

Theorem 2. *Let the sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ be defined by $p_{-1}(z) \equiv 0$, $p_0(z) \equiv 1$, and*

$$p_n(z) = (\beta_n + z)\, p_{n-1}(z) - e_n z\, p_{n-2}(z) \tag{4}$$

where $e_{n+1} > 0$ and $s_n = \min\limits_{1 \leq j \leq n} \operatorname{Re} \beta_j > 0$ $(n = 1, 2, \ldots)$. If for a fixed positive integer n we select an $\alpha \in (0, s_n]$ such that

$$c = \max_{2 \leq j \leq n} \frac{\tfrac{1}{2} e_j}{\operatorname{Re} \beta_j - \alpha}$$

is finite, then the polynomials $\{p_j\}_{j=1}^{n}$ have no zeros in the region containing the origin given by

$$c\,|z| - (1-c) \operatorname{Re} z \leq \alpha, \quad z \neq -\alpha. \tag{6}$$

The region (6) is hyperbolic when $c < \tfrac{1}{2}$ (the right-half), parabolic when $c = \tfrac{1}{2}$ (see [2], [5]), and elliptic when $c > \tfrac{1}{2}$. In each case, $z = -\alpha$ is a vertex of this conic.

2. Preliminaries. Our first lemma can be shown to be a special case of a general value region result of Lane [3]. Since the proof is short and requires less effort than establishing the relation to previous work, it is included.

Lemma 1. *Let c and d be real numbers, $d > 0$. Let E be the disc $\{\zeta \in \mathbb{C} : |\zeta - c| < d\}$. For $\alpha > 0$, we have $\operatorname{Re} w > 1 - \alpha$, where $w = 1 + \zeta z$, for all $\zeta \in E$ if and only if*

$$d\,|z| - c \operatorname{Re} z \leq \alpha. \tag{7}$$

Furthermore, $\operatorname{Re} w \geq 1 - \alpha$ for all $\zeta \in \overline{E}$, the closure of E, if and only if (7) holds.

Proof. Let $\zeta = c + dte^{i\phi}$ and $z = \rho e^{i\theta}$, where $0 \leq t < 1$, $\rho > 0$, and ϕ, θ are in the interval $(-\pi, \pi]$. Then

$$\operatorname{Re} w = 1 + \operatorname{Re}(\zeta z) = 1 + c\rho\cos\theta + dt\rho\cos(\theta+\phi) \geq 1 + c\rho\cos\theta - dt\rho >$$

$$> 1 + c\rho\cos\theta - d\rho = 1 + c\operatorname{Re} z - d|z|.$$

Therefore (7) implies $\operatorname{Re} w > 1 - \alpha$. For the converse, let $\phi = \pi - \theta$ and we have $\operatorname{Re} w = 1 + c\operatorname{Re} z - dt|z| > 1 - \alpha$ for all $t \in [0, 1)$. Let $t \to 1$ to obtain (7). The second part of the lemma is an obvious consequence of what has been proved.

This lemma is adequate to prove Theorem 2. Our second lemma uses this result and is the essence of our proof of Theorem 1.

Lemma 2. *Let r be a real number $0<r<1-1/\sqrt{3}$. For each ζ in $|\zeta - r^2/(1-r^2)| \leq r/(1-r^2)$ and for each z in*

$$\left|z - \frac{r(1-r)}{1-2r}\right| \leq \frac{(1-r)^2}{1-2r}, \tag{8}$$

we have $w = 1 + \zeta z$ is in the intersection of the regions of (2)

Proof. By Lemma 1, we have $\operatorname{Re} w \geq 1 - r(1-r)/[(1+r)(1-2r)]$ if and only if z is in or on the ellipse (with eccentricity r)

$$|z| - r\operatorname{Re} z = (1-r)^2/(1-2r).$$

The circle of curvature at the right end point of the majors axis for this ellipse is the boundary of the disk (8). Therefore, the points of (8) are in or on the ellipse and the last inequality of (2) has been established.

Since $w = 1 + \zeta z$ and ζ is in a given disk, we have

$$|w - 1 - r^2 z/(1-r^2)| \leq r|z|/(1-r^2).$$

The point w of (9) are in the disk (8) if the distance between the centers of these disks plus the radius of the smaller does not exceed the radius of the larger. Thus, the condition

$$\left|1 + \frac{r^2 z}{1-r^2} - \frac{r(1-r)}{1-2r}\right| + \frac{r|z|}{1-r^2} \leq \frac{(1-r)^2}{1-2r} \tag{10}$$

implies the disc (9) is contained in the disk (8). This inequality is satisfied if it is valid for $z = c + de^{i\phi}$, where $\phi \in (-\pi, \pi]$ and c, d are respectively the center, radius of the disk (8). For this choice of z, we rewrite (10) as

$$H(\phi) \equiv |r(1-r)e^{i\phi} - A| + |r + (1-r)e^{i\phi}| \leq (1-r)^2(1+r)/r,$$

where for brevity we have set $A = 1 - (1-r)(1-2r^2)/r$. For a fixed r, the derivative $H'(\phi)$ is zero when $\sin\phi = 0$ or when

$$\frac{A}{(A^2 + r^2(1-r)^2 - 2Ar(1-r)\cos\phi)^{1/2}} = \frac{1}{(r^2 + (1-r)^2 + 2r(1-r)\cos\phi)^{1/2}}.$$

This condition implies $A > 0$ and

$$\cos\phi = \frac{2A^2 + r(1-r)}{2A(1+A)}.$$

The latter cannot occur for $\phi \neq 0$ if $2A^2 + r(1-r) \geq 2A(1+A)$, that is, if $2 - 4r - 3r^2 + 3r^3 = (1+r)(2 - 6r^2 + 3r^3) \geq 0$. This inequality implies $r \leq 1 - 1/\sqrt{3}$. Since $H(\pi) < H(0) = (1-r)^2(1+r)/r$, we conclude that (10) holds for all z in the disc (8) and, hence, the disk (9) is a subset of the disk (8). This completes the proof of the lemma.

There is a minimal subset V of the region defined by (2) such that $w = 1 + \zeta z \in V$ for all $z \in V$ and ζ in the disk of Lemma 2. This value region is closed, symmetric relative to the real axis, and the interval $(1 - 2r - r^2)/(1 - r - 2r^2) \leq x \leq (1-r)/(1-2r)$ is in V. If we could explicitly identify V, we would locate the region of values of $w = zf'/f$ when $|z| \leq p < 1$.

3. Proof of Theorem 1. For a given function (1) in W and a fixed positive integer n, define $f_0(z) = z$ and

$$f_{p+1}(z) = \frac{z}{1 + a_{n-p} f_p(z)} \quad (p = 0, 1, 2, \ldots, n-1),$$

where a_j is the coefficient of the $j + 1$st partial numerator of (1). Each f_p is in W. By a formal computation,

$$\frac{zf'_{p+1}}{f_{p+1}} = 1 - \frac{a_{n-p} f_p}{1 + a_{n-p} f_p} \frac{zf'_p}{f_p} \quad (z \in D). \tag{11}$$

It is known that for $f \in W$ we have $|f(z)| \leq 4r$ when $|z| \leq 4r(1-r)$, $0 \leq r < ½$ [7, p. 105]. For this choice of z,

$$\left|\zeta - \frac{r^2}{1-r^2}\right| \leq \frac{r}{1-r^2},$$

where $\zeta = -a_{n-p} f_p / (1 + a_{n-p} f_p)$, $|a_{n-p}| \leq ¼$. By (11) $\sigma_{p+1} = 1 + \zeta\sigma_p$, where $\sigma_p = zf'_p / f_p$ $(p = 0, 1, 2, \ldots, n)$. Since $\sigma_0 = 1$ is in the disk (8), it follows by Lemma 2 and induction that the nth approximant f_n of f is in (2). Since $f_n \to f$ uniformly on compact subsets of D [7, p. 42], we conclude $w = zf'/f$ is in the intersection of the regions (2) For a given $\alpha \in [0, 1)$, the condition $\operatorname{Re} w \geq \alpha$ that f be starlike of order α in $|z| < 4r(1-r)$ requires $1 - \alpha - (2-\alpha)r + (2\alpha - 1)r^2 \geq 0$. The positive zero of this quadratic is r_α given in (3). The function

$$f_0(z) = \frac{z}{1} - \frac{\frac{1}{4}z}{1} + \frac{\frac{1}{4}z}{1} + \cdots + \frac{\frac{1}{4}z}{1} + \cdots = \frac{2z}{3-\sqrt{1-z}}$$

is clearly in W and for this function

$$\left.\frac{z f_0'(z)}{f_0(z)}\right|_{z=-4r(1-r)} = \frac{1-2r-r^2}{1-r-2r^2}.$$

This proves the order of starlikeness is exact for each choice of $\alpha \in [0, 1)$. Since $f_0'(z) = 0$ when $r = \sqrt{2}-1$, we conclude that the radius of starlikeness ($\alpha = 0$) is also the radius of univalence for the class W. This completes the proof.

4. Proof of Theorem 2 and Applications. The sequence (4) is the sequence of denominators of the continued fraction

$$z - \frac{\epsilon_1 z}{\beta_1 + z -} \frac{\epsilon_2 z}{\beta_2 + z -} \cdots \frac{\epsilon_n z}{- \beta_n + z -} \cdots.$$

If $\{q_j\}_{j=-1}^{\infty}$ is the sequence of numerators of this continued fraction, then the approximants are

$$w_n = \frac{q_n}{p_n} = s_1 \cdot s_2 \cdot \ldots \cdot s_n(z) \quad (n = 1, 2, 3, \ldots)$$

where

$$s_j(v) = z\left(1 - \frac{\epsilon_j}{\beta_j + v}\right) \quad (j = 1, 2, \ldots)$$

By the determinant formula [7, p. 16]

$$\begin{vmatrix} q_{n-1} & q_n \\ p_{n-1} & p_n \end{vmatrix} = \epsilon_1 \epsilon_2 \ldots \epsilon_n z^n \neq 0 \text{ for } z \neq 0$$

and we conclude q_n, p_n have no common zero. Indeed, $p_n(0) = \beta_1\beta_2 \ldots \beta_n \neq 0$ by (4) and the hypothesis.

If $\operatorname{Re} \nu > -\alpha$, then $\zeta = 1 - e_j / (\beta_j + \nu)$ is in the disk

$$\left| \zeta - 1 + \frac{e_j}{2(\operatorname{Re}\beta_j - \alpha)} \right| < \frac{e_j}{2(\operatorname{Re}\beta_j - \alpha)} \quad (j = 2, 3, \ldots, n),$$

where $\alpha \in (0, s_n]$ has been selected such that (5) is finite. This disk is contained in the disk $|\zeta - 1 + c| < c$, where c is defined by (5), provided the distance between the centers of these disks plus the radius of the smaller does not exceed the radius of the larger. That is, the condition

$$\left| c - \frac{e_j}{2(\operatorname{Re}\beta_j - \alpha)} \right| + \frac{e_j}{2(\operatorname{Re}\beta_j - \alpha)} \leqslant c$$

is sufficient for the proposed inclusion. By (5) the last condition is obviously the case for $j = 2, 3, \ldots, n$. Thus, $\operatorname{Re} s_j(v) > -\alpha$ for $\operatorname{Re} v > -\alpha$ ($j = 2, 3, \ldots, n$) by Lemma 1 whenever z is in the region $c\,|z| - (1-c)\operatorname{Re} z \leqslant \alpha$.

Let z be in the region (6). Since $z \neq -\alpha$, we have $\operatorname{Re} z > -\alpha$. Thus, $\operatorname{Re} s_j(z) > -\alpha$ ($j = 2, 3, \ldots, n$) by what has already been proved. Using induction and Lemma 1, we conclude that $\operatorname{Re}\zeta(z) > -\alpha$, where $\zeta(z) = s_2 \circ s_3 \circ \ldots \circ s_j(z)$. It follows that $\operatorname{Re}(\beta_1 + \zeta(z)) > \operatorname{Re}\beta_1 - \alpha \geqslant 0$ and, in particular, $\beta_1 + \zeta(z) \neq 0$ for z in the region (6). This proves the jth approximant

$$w_j = z + \frac{e_1 z}{\beta_1 + \zeta(z)} \quad (j = 2, 3, \ldots, n)$$

is analytic for z in the region (6). In particular, the denominator p_j of this jth aproximant is not zero for such z. Since $\operatorname{Re} p_1(z) = \operatorname{Re}\beta_1 + \operatorname{Re} z > \operatorname{Re}\beta_1 - \alpha \geqslant 0$, we have proved the theorem.

For applications of this theorem, let $f(z) = 1 + \sum_{j=1}^{\infty} a_j z^j$, $a_j \neq 0$ ($j = 1, 2, \ldots$), be a formal power series. The partial sums $t_n(z) = 1 + \sum_{j=1}^{\infty} a_j z^j$ satisfy the identity of Euler

$$\frac{t_n(z)}{a_n} = \left(\frac{a_{n-1}}{a_n} + z\right)\frac{t_{n-1}(z)}{a_{n-1}} - \frac{a_{n-2}}{a_{n-1}}\, z\, \frac{t_{n-2}(z)}{a_{n-2}} \quad (n = 1, 2, \ldots),$$

where we set $t_{-1} = 0$, $a_{-1} = a_0 = t_0 = 1$. Thus, the polynomials $p_n = t_n / a_n$ are generated by a recurrence formula (4), where $\beta_n = a_{n-1}/a_n$ and $e_n = a_{n-2}/a_{n-1}$. If $a_j > 0$ ($j = 1, 2, \ldots, n-1$) and $s_n = \min\{1/a_1, a_1/a_2, \ldots, a_{n-2}/a_{n-1}, \operatorname{Re}(a_{n-1}/a_n)\} > 0$, the hypothesis of Theorem 2 is satisfied and there is a choice of $\alpha > 0$ such that (5) is finite.

The polynomials $\{t_j\}_{j=1}^{n}$ are therefore not zero in the region containing the origin given by (6), where c is defined for the particular α by (5).

In particular, if $f(z) = e^z$, then $a_n = 1/n!$, $\beta_n = n$, $e_n = n-1$, $s_n = \min(1, 2, \ldots, n) = 1$, and for $\alpha = 1$

$$c = \max_{2 \leqslant j \leqslant n} \frac{\tfrac{1}{2}(j-1)}{j-1} = \tfrac{1}{2}.$$

The region (6) is the parabolic region $|z| - \operatorname{Re} z \leq 2, z \neq -1$, of Saff and Varga [5]. Unfortunately we have not improved the known result in this case. However, if

$$t_n(z) = 1 + z + z^2/2! + \ldots + z^{n-1}/(n-1)! + a z^n/n!,$$

where a is complex such that $|a - ½| \leq ½$, then $t_n \neq 0$ for z in this parabolic region.

For a new application, consider the hypergeometric function

$$_1F_0(a;z) = 1 + az + \frac{a(a+1)}{2!} z^2 + \frac{a(a+1)(a+2)}{3!} z^3 + \ldots,$$

where $a > 1$. We have

$$\beta_n = \frac{a_{n-1}}{a_n} = \frac{n}{a+n-1} = e_{n+1} \; (n = 1, 2, \ldots)$$

and we can choose $\alpha = 1/a$. Then $c = ½(a+1)/(a-1)$ and the partial sums of $_1F_0$ are not zero in the elliptical region

$$|z| + \frac{3-a}{a+1} \operatorname{Re} z < \frac{2(a-1)}{a(a+1)}, \; z \neq -\frac{1}{a}.$$

Next, consider for $a > 0$

$$_1F_1(1,a;z) = 1 + \frac{1}{a} z + \frac{1}{a(a+1)} z^2 + \ldots .$$

We have $\beta_n = a + n - 1 \geq a > 0$, $\alpha = a$, and $c = a/2$ provided $a \geq 1$. The partial sums are not zero in $|z| + (1 - 2/a) \operatorname{Re} z \leq 2, z \neq -a$. This region is elliptic for $a > 1$. If $0 < a < 1$, then $c = (n + a - 2)/2(n-1)$ and the partial sums are not zero in the hyperbolic region (right-half).

$$|z| + \frac{a-n}{a+n-2} \operatorname{Re} z \leq \frac{2a(n-1)}{a+n-2}, \; z \neq -a.$$

This region contains the parabolic region $|z| - \operatorname{Re} z \leq 2a, z \neq -a$. There are other applications suggested by those in [5].

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STRESZCZENIE

Wykorzystuje się obszary zmienności pewnych funkcjonałów oraz technikę ułamków ciągłych do wyznaczania promieni gwiaździstości pewnych rodzin funkcji holomorficznych w kole jednostkowym.

РЕЗЮМЕ

Используются мажорантные области некоторых функционалов и техника непрерывных дробей для получения радиусов звездообразности некоторых классов функций голоморфных в единичном круге.