ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA

LUBLIN–POLONIA

VOL. XXXVI/XXXVII, 11 SECTIO A 1982/1983

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A Property of Convex Mappings

Własność odwzorowań wypukłych

Свойство выпуклых отображений

Let S represent the class of functions $f(z)$ regular and univalent in the open unit disc Δ, $\Delta = \{z \in C: |z| < 1\}$, with the usual normalization

$$f(0) = 0 \text{ and } f'(0) = 1\,; \tag{1}$$

and, for a in Δ, let

$$f(z;a) = \int_0^z \frac{f'\left(\dfrac{\zeta + a}{1 + \bar{a}\zeta}\right)}{f'(a)}\, d\zeta\,. \tag{2}$$

For any admissible value of a, $f(z; a)$ is locally univalent throughout the disc Δ and $f(z; 0) = f(z)$. It is reasonable to ask about conditions on a and $f(z)$ under which $f(z;a)$ is in S and the purpose of this note is to do so.

If $f(z)$ is normalized as above and if Re $\{f'(z)\} > 0$, z in Δ, then it is well-known that $f(z)$ is close-to-convex and hence in S, [1]. It follows that $f(z; a)$ is also close-to-convex for each a in Δ. On the other hand, if we let $k(z) = z / (1 - z)^2$, the Koebe function, and let

$$k(z;a) = \int_0^z \frac{k'\left(\dfrac{\zeta + a}{1 + \bar{a}\zeta}\right)}{k'(a)}\, d\zeta = z + A_2(a)\, z^2 + \ldots$$

then

$$A_2(a) = \frac{k''(a)}{2\,k'(a)} \cdot (1-|a|^2) = \frac{(2+a)(1-|a|^2)}{1-a^2}\,. \tag{3}$$

for any a in Δ. If we choose $0 < a < 1$, then

$$A_2(a) = 2 + a > 2\,;$$

this means that $k(z;a)$ is not one-to-one [2] and hence not in S. Consequently every neighborhood of the origin contains points a such that for some function $f(z)$ in S the function $f(z;a)$ is not in S.

These examples show that both extremes of behavior are possible for various subclasses of S. The next result shows that the operator (2) does preserve univalence when $f[\Delta]$ is a convex domain.

Theorem. *If $f(z)$ is convex and in S, then $f(z;a)$ is close-to-convex and in S provided $|a| < \sqrt{2}/2$. This conclusion is best possible.*

By subjecting the integral (2) to a change of variable replacing $[(\zeta + a)/(1 + \bar{a}\zeta)]$ by a new variable, say τ, and by suppressing constants which play no role in univalence or in the definitions of convexity and close-to-convexity, we see that the univalence of $f(z,a)$ is equivalent to the univalence of

$$F(z;a) = \int_0^z \frac{f'(\tau)}{(1-a\tau)^2}\,d\tau\,. \tag{4}$$

Now, if $|a| < 2^{-1/2}$, then

$$\left|\arg\frac{F'(z,a)}{f'(z)}\right| \leqslant 2\left|\arg(1-az)\right| \leqslant 2\arcsin|a| < \frac{\pi}{2} \tag{5}$$

and $F(z;a)$ is close-to-convex with respect to $f(z)$.

To show that the constant $2^{-1/2}$ is best possible we construct an example such that $F(z;a)$ fails to be univalent in Δ for some a, $|a| = 2^{-1/2}$. If, in (4), we choose $f(z) = z/(1-bz)$, then $f(z)$ is a convex mapping of the disk and

$$F(z;a,b) = \int_0^z \frac{d\tau}{(1-a\tau)^2(1-b\tau)^2}\,. \tag{6}$$

If we choose $a = b$, in (6), then $F(z;a,b)$ fails to be univalent only when $|a| > \sqrt{3}/2$; however, if we let a and b be positive and different, then

$$F(z;a,b) = (b-a)^{-2}\left\{\frac{a^2 z}{1-az} + \frac{b^2 z}{1-bz} + \frac{2ab}{b-a}\log\left(\frac{1-bz}{1-az}\right)\right\} \tag{7}$$

and we can show $F(z;\ a,b)$ is not univalent in the disc for appropriate choices of a and b.

$F(z;\ a,b)$ has real coefficients and maps the interval $[-1,1]$ onto the real axis; if $F(z;a,b)$ is univalent no point in the upper half of Δ maps onto or below the real axis. If we let $z = e^{i\theta}$, then

$$\operatorname{Im} F(e^{i\theta};\ a,b) = \frac{a^2 \sin\theta}{1+a^2-2a\cos\theta} + \frac{b^2\sin\theta}{1+b^2-2b\cos\theta} +$$
$$+\frac{2ab}{b-a}\left\{\tan^{-1}\left(\frac{a\sin\theta}{1-a\cos\theta}\right) - \tan^{-1}\left(\frac{b\sin\theta}{1-b\cos\theta}\right)\right\}$$

Letting $a_0 = \sqrt{2}/2$, $b_0 = (\sqrt{2}/3) + (\sqrt{3}/6)$ and $\theta = \pi - (1/10^2)$, we find, with the aid of an Apple II (programmed, with our thanks, by W. E. Baxter), that $\operatorname{Im} F(e^{i\theta};\ a_0, b_0) < 0$. This shows that $F(e^{i\theta};\ a_0, b_0)$ is not univalent in Δ. This concludes proof of the theorem.

At this point one might ask if the conclusion about close-to-convexity in the theorem can be replaced by convexity. If we choose $f(z) = z\ /\ (1-\epsilon z)$, $|\epsilon| = 1$, then the coefficient of z^2 for $f(z;a)$ in (2) is $A_2(a) = [\epsilon\,(1-|a|^2)]\ /\ (1-\epsilon a)$; now, for any a in Δ, $a = \rho e^{i\theta}$, let $\epsilon = e^{-i\theta}$, then we find that $A_2(a) = 1+\rho = 1+|a|$. However, for a convex function, $|\ A_2(a)\ |$ cannot exceed 1, consequently $f(z;a)$ is not convex; and we conclude that for each a in Δ there is a convex function $f(z)$ such that $f(z;a)$ is not convex.

Had we been able to show that $f(z;a)$ was convex for each convex $f(z)$ and a in some neighborhood of the origin, we would have shown that the transformation (2) generates a variation for the class of convex functions. It may be possible to show this for other subclasses of S or to replace the linear transformation in (2) by some other mapping of Δ into Δ.

REFERENCES

[1] Kaplan, W., *Close-to-convex schlicht functions*, Michigan Math. J., 1 (1952), 169–185.
[2] Nehari, Z., *Conformal Mapping*, New York 1952.

STRESZCZENIE

Dowodzi się, że jeżeli f jest funkcją holomorficzną i wypukłą w kole $|z| < 1$ zaś $|a| < 1/\sqrt{2}$ to całka $\int_0^z f'[(a+u)/(1+\bar{a}u)]\,du$ jest funkcją jednolistną. Stała $1/\sqrt{2}$ jest najlepszą z możliwych.

РЕЗЮМЕ

Доказывается, что если f функция голоморфна и выпукла в круге $|z| < 1$ и $|a| < 1/\sqrt{2}$ тогда интеграл $\int_0^z f'\,[(a+u)/(1+\bar{a}u)]\,du$ функция однолистна. Константа $1/\sqrt{2}$ самая лучшая из возможных.