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On a Problem of A. A. Goldberg *

O pewnym problemie A. A. Goldberga

Об одной проблеме А. А. Гольдберга

1. Goldberg [2] considered the following problem. Let K_1 denote the class of functions f meromorphic in the unit disc for which the multiplicities with which the values 0, 1 and ∞ are taken are finite and distinct. Let $r(f)$ denote the maximum modulus for a point where one of these values is assumed and let A_1 denote the greatest lower bound of these quantities for $f \in K_1$. Let K_2, K_3, K_4 be the classes obtained by replacing meromorphic by regular, rational and polynomial respectively and A_2, A_3, A_4 the corresponding greatest lower bounds. Then Goldberg concluded that

$$0 < A_1 = A_3 < A_2 = A_4 .$$

He also obtained explicit numerical upper and lower bounds for A_2 and an explicit numerical upper bound for A_1. He did not however obtain such a lower bound for A_1. The object of this paper is to provide such a bound (which is better than Goldberg's lower bound for A_2). The same order of ideas also gives an upper bound for A_1 significantly better than Goldberg's, .

I want to express my thanks to James M. Anderson of University College, London, who brought this problem to my attention and supplied the reference to Goldberg's paper.

2. **Definition 1.** A function f meromorphic in $|z| < 1$ is said to satisfy condition C if the multiplicities with which f takes the values 0, 1 and ∞ in $|z| < 1$ are finite and distinct.

Theorem 1. *If the function f meromorphic in $|z| < 1$ satisfies condition* C *and does not take the value* 0, 1 *or* ∞ *in* $r < |z| < 1$, $0 < r < 1$, *then* $r \geqslant .00037008$.

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Let Δ denote the sphere punctured at $0, 1, \infty$. The mapping $w = f(z)$ carries a circumference $|z| = s$, $r < s < 1$, into a path in Δ. The covering surface of Δ determined by the cyclic subgroup of the fundamental group corresponding to this path is a doubly-connected Riemann domain $\mathfrak{R}$. $\mathfrak{R}$ is conformally equivalent to a domain obtained from the upper half-plane by identifying points congruent under the corresponding subgroup of the group of linear transformations with integral coefficients and determinant 1 generated by a hyperbolic transformation T. If T has fixed points ζ_1, ζ_2 is has the representation (with suitable choice of notation)

$$\frac{\omega - \zeta_1}{\omega - \zeta_2} = \lambda \frac{\zeta - \zeta_1}{\zeta - \zeta_2} \quad (\lambda > 1)$$

which is to be appropriately modified in case either of ζ_1, ζ_2 is the point at infinity. In any case $\mathfrak{R}$ has module $\pi (\log \lambda)^{-1}$. It is well known [3, 4] that the module of $r < |z| < 1$ is at most this size. Thus

$$\frac{1}{2\pi} \log r^{-1} \leqslant \pi (\log \lambda)^{-1}$$

or

$$r \geqslant \exp(-2\pi^2 (\log \lambda)^{-1}).$$

On the other hand if T is given by

$$\frac{a\zeta + b}{c\zeta + d}, \quad ad - bc = 1,$$

it is well known that

$$\lambda + \lambda^{-1} + 2 = (a + d)^2.$$

Since T represents a covering transformation of the universal covering surface of Δ it is well known that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (taken to have determinant 1) will be congruent modulo 2 to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; see for example [1, p. 270]. Thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has the form

$$\begin{pmatrix} 1 + 2m & 2k \\ 2\ell & 1 + 2n \end{pmatrix}$$

k, ℓ, m, n integral, with

$$1 + 2m + 2n + 4mn - 4k\ell = 1$$

so that

$$a + d = 2 - 4mn + 4k\ell .$$

If $a + d = \pm 2$, T is parabolic (or the identity), thus when T is hyperbolic $a + d$ is divisible by 6. Hence

$$\lambda \geq 17 + (288)^{1/2} .$$

Carrying out the numerical calculation we find

$$r > .00037008 .$$

Definition 2. Let K_1 denote the class of functions f meromorphic in the unit disc satisfying condition C. Let $r(f)$ denote the maximum modulus of a point in $|z| < 1$ where f takes one of the values $0, 1, \infty$. Let $A_1 = \text{g.}\ell\text{.b.}_{f \in K_1} r(f)$.

Corollary. $A_1 > .00037$.

3. **Lemma.** *There exists a meromorphic function in K_1 for which all of the points at which it takes the values $0, 1, \infty$ lie in a continuum whose complement with respect to the unit disc is a doubly-connected domain whose module is arbitrarily close to*

$$\pi (\log (17 + (288)^{1/2}))^{-1} .$$

There does exist a path in Δ for which the corresponding linear transformation actually has the value $a + d = 6$ and which has positive distinct indexes about 0 and 1, for example, a path consisting of one simple loop enclosing 0 and 1 followed by a simple loop enclosing just 1. The associated doubly-connected covering surface $\mathfrak{dR}$ of Δ is conformally equivalent to a ring $\mathfrak{R}$: $s < |z| < 1$ under a mapping from $\mathfrak{R}$ onto $\mathfrak{dR}$. The image of $|z| = s + \epsilon$ for sufficiently small $\epsilon > 0$ is an analytic curve Γ which lies on the boundary of the Riemann image $\mathfrak{dR}_\epsilon$ of $s + \epsilon < |z| < 1$. It is well known that there is a Riemann domain Ξ homeomorphic to a disc coverning the sphere, bounded by Γ and lying locally on the opposite side of Γ from $\mathfrak{dR}_\epsilon$. Ξ and $\mathfrak{dR}_\epsilon$ together make up a simply-connected hyperbolic Riemann surface $\mathfrak{S}$. We map $\mathfrak{S}$ conformally onto the unit disc by ψ. Then ψ^{-1} provides the desired function in K_1.

Theorem 2. $A_1 < .00149$.

If we choose ψ so that the origin lies in $\psi(\Xi)$ in the Lemma it is well known that the diameter of $\psi(\Xi)$ is less than $4(s + \epsilon)$. Since ϵ can be chosen arbitraily close to 0 the result follows.

Evidently more detailed geometric considerations would provide some improvement in this bound.

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STRESZCZENIE

Niech K_1 będzie klasą funkcji f meromorficznych, które przyjmują każdą z wartości 0, 1, ∞ skończoną i różną ilość razy. Niech

$$r(f) = \max\left\{|z| : z \in f^{-1}\left(\{0, 1, \infty\}\right)\right\}$$

oraz

$$A_1 = \inf\left\{r(f) : f \in K_1\right\}.$$

W pracy otrzymano oszacowanie A_1 od dołu i od góry, ulepszające oszacowanie otrzymane przez Goldberga.

РЕЗЮМЕ

Пусть K_1 будет классом функций f мероморфных в единичном круге, которые принимают каждое значение 0, 1, ∞ с конечной и разной кратностью. Пусть

$$r(f) = \max\left\{|z| : z \in f^{-1}\left(\{0, 1, \infty\}\right)\right\}.$$

$$A_1 = \inf\left\{r(f) : f \in K_1\right\}.$$

В этой работе получены оценки A_1 снизу и сверху улучшающие оценки полученные Гольдбергом.