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The Marx Conjecture for Starlike Functions. II

Hipoteza Marksa dla funkcji gwiaździstych. II

Гипотеза Маркса для звездообразных функций. II

1. Introduction. Let Δ denote the unit disc $\{z : |z| < 1\}$ and let S^* denote the class of starlike functions, that is, the class of all functions $f(z)$ which are analytic and univalent in Δ, normalized by $f(0) = 0$, $f'(0) = 1$, and which map Δ onto a region which is starshaped with respect to the origin. Let $k(z)$ denote the Koebe function

$$k(z) = z/(1-z)^2. \tag{1.1}$$

Given $z_0 \in \Delta$ and $r = |z_0|$, define the sets

$$\begin{cases} K_1(r) = \{w : w = k'(z), \quad |z| \leq r\}, \\ K(r) = \{w : w = \log k'(z), \ |z| \leq r\} \\ M(z_0) = \{w : w = \log f'(z_0), \ f \in S^*\}, \end{cases} \tag{1.2}$$

where the branch of the logarithm is fixed by setting $\log f'(0) = \log k'(0) = 0$.

We observe that if $f(z) \in S^*$, then so is $f_\alpha(z) = e^{-i\alpha} f(e^{i\alpha} z)$, α real, and $\log f'_\alpha(r) = \log f'(re^{i\alpha})$. Therefore $M(z_0) = M(|z_0|)$ and hence it suffices to let $z_0 = |z_0|$ in studying $M(z_0)$.

In 1932, A. Marx [5] showed that if $|z_0| \leq \sin \pi/8 = 0.382..$, then $f'(z_0) \in K_1(|z_0|)$ and conjectured that this would be true for any $z_0 \in \Delta$. This could be written as the

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conjecture that $f'(z) \prec k'(z)$ for any $f \in S^*$ (if one allows this use of subordination even th ough $k'(z)$ is not univalent in Δ).

Marx's result was based on the fact that every normalized convex univalent function $F(z)$ satisfied the condition Re $\{F(z)/z\} > ½$. Hence, the function $2F(z)/z - 1$ is a normalized function with positive real part and has a Herglotz representation. Thus, there exists a measure $\mu(x)$ of total mass 1 on the circle $|x| = 1$ such that

$$F(z) = z \int_{|x|=1} \frac{d\mu(x)}{1-xz} .$$

(Marx used the approximation by finite sums.) Since $F(z) \in S^*$ if and only if $f(z) = zF'(z)$ for some convex function $F(z)$, it follows that for any $f \in S^*$ there is a measure $\mu(x)$ of total mass 1 such that

$$f'(z) = \int_{|x|=1} \frac{1+xz}{(1-xz)^3}\, d\mu(x) .$$

Thus, $f'(r)$ always lies in the closed convex hull of the set $K_1(r)$. Marx obtained his bound from this observation.

Robinson [6] studied the relationship between the subordination of two functions and the subordination of transforms of these functions. In particular, he considered conditions under which $f \prec g$ implies $f' \prec g'$. He was able to prove that if $zf'(z)/f(z) \prec (1+z)/(1-z)$ (as is true for $f \in S^*$), then $f'(z) \prec (1+z)/(1-z) = k'(z)$ for $|z| < ½(5 - 17^{1/2}) = 0.438...$ That is, he showed that the Marx conjecture holds for $r \leq 0.438...$

Somewhat later, Robinson [7] proved that if B and C are any two complex numbers, not both 0, then any extremal function for the problem of maximizing Re $\{B \log f'(z) + C \log f(z)/z\}$ in the class S^* was a function which maps Δ onto the exterior of at most two radial slits. He then proved that $\log k'(z)$ is univalent in Δ, that $K(r)$ is convex if $r \leq 0.6$, and that the extremal functions have at most one slit if $r < 0.62$. That is, he proved that

$$M(r) \subset K(r) \tag{1.3}$$

if $r < 0.6$.

This essentially replaced the original Marx conjecture with what we can call the Marx-Robinson Conjecture: that (1.3) holds for all $r < 1$.

Duren [1] improved Robinson's results to show that (1.3) holds for $r < 0.736...$ Since the method of proof used the convexity of $M(r)$, Duren calculated the actual radius of convexity of $M(r)$ which is $r = 0.886...$

Hummel [3] showed the existence of a counterexample to (1.3) when $r = 0.99$ and stated that computations suggested the existence of such counterexamples for $r \geq 0.94$.

It is clear that the truth of the Marx-Robinson conjecture for $|z| \leq r$ implies the truth of the original Marx conjecture for the same disc. The converse is true if the region $K(r)$ is contained in the strip $|\text{Im}\{w\}| < \pi$. Numerical computations (discussed in section 4 below) show that the boundary of $K(r)$ first touches the lines $|\mathrm{Im}\{w\}| = \pi$ when $r = 0.810465...$ However, a counterexample to the Marx-Robinson conjecture will define a counterexample to the Marx conjecture even if r is largerthan this value provided that $w_0 = \log f'(r) \notin K(r)$ and the line $\mathrm{Re}\{w\} = \mathrm{Re}\{w_0\}$ intersects $K(r)$ in a segment of length less that 2π.

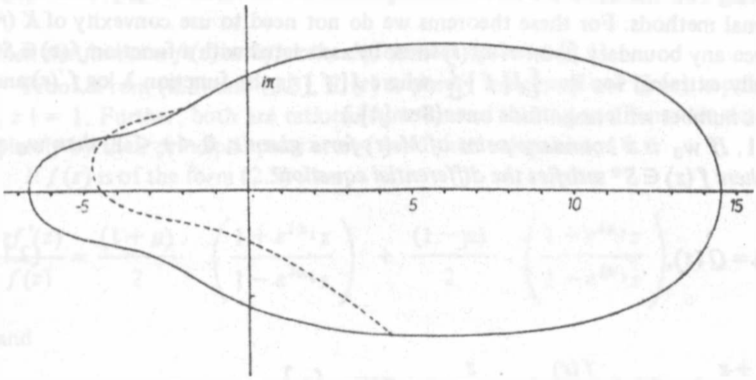


Figure 1. $K(r)$ for $r = 0.99$

Figure 1 shows the region $K(r)$ for $r = 0.99$. The dashed curve follows the values of $\log f'(r)$ for the functions in S^* of the form $f(z) = z(1 - e^{i\alpha_1}z)^{-1-\mu}(1 - e^{i\alpha_2}z)^{-1+\mu}$, $\alpha_1 = 2.2089323, \alpha_2 = 5.9854563$, as μ varies between -1 and $+1$. The value $i\pi$ is marked on the imaginary axis. It is clear that some of the μ define functions which are counterexamples to the Marx conjecture. Computations discussed in section 4 of this paper indicate that such counterexamples exist for $r > 0.93919...$ In every case, such counterexamples seem to produce only points which are properly contained in the convex hull of $K(r)$. We note that the radius of convexity of $K(r)$, 0.886.., does not appear to be the bound for the Marx-Robinson conjecture, as was observed by Robinson. See [1].

Based on the numberical results of this paper, it seems reasonable to conjecture that

1) $M(r) \subset K(r)$ for $r \leqslant 0.9391924...$ and for no larger r.

2) The original Marx conjecture holds in the same range.

3) $M(r)$ is properly contained in the convex hull of $K(r)$ for $r > 0.8863486...$

The set $K_1(r)$ is doubly connected for $r > 0.810465...$ We know, as Marx showed in his original paper, that if $f \in S^*$ then $f'(r)$ is in the convex hull of $K_1(r)$. However, all of the counterexamples discovered are such that $f'(r)$ in fact lies in the hole in the center of $K_1(r)$ (i.e. in the bounded component of $\mathbb{C} - K_1(r)$). This was pointed out to the writer in a personal comunication by R. Boutellier who suggests the addition conjecture

4) For any $f \in S^*$, $f'(r)$ lies in the region bounded by the outer boundary of $K_1(r)$.

2. Results based on variational methods. In this section, and in the next, many of our results will be based on functions of the form (2.5) given in Theorem 2 below. These are the 'two slit functions'. These functions depend on three real parameters, α_1, α_2, and μ. In addition, we use the real parameter $r = |z|$. We find it convenient to introduce the conventions

$$z_1 = re^{i\alpha_1}, \ z_2 = re^{i\alpha_2}.$$

The following two theorems are direct consequences of the results of [2], and follow from variational methods. For these theorems we do not need to use convexity of $K(r)$ (or $M(r)$) since any boundary point of $M(r)$ must be associated with a function $f(z) \in S^*$ which is locally extremal for Re $\{J[f]\}$ where $J[f]$ is the function $\lambda \log f'(r)$ and λ is a complex number with magnitude one. (See [4].)

Theorem 1. *If w_0 is a boundary point of $M(r)$ for a given r, $0 < r < 1$, then $w_0 = \log f'(r)$ where $f(z) \in S^*$ satisfies the differential equation*

$$\frac{zf'(z)}{f(z)} R(z) = Q(z), \tag{2.1}$$

where

$$R(z) = \lambda\left(\frac{r+z}{r-z}\right) - 2\lambda\left(\frac{f(r)}{f'(r)}\right)\frac{z}{(r-z)^2} + 2i\,\mathrm{Im}\,\{\lambda\} - \bar{\lambda}\left(\frac{rz+1}{rz-1}\right) + 2\bar{\lambda}\left(\overline{\frac{f(r)}{f'(r)}}\right)\frac{z}{(rz-1)^2}, \tag{2.2}$$

$$Q(z) = \lambda\left(1 + \frac{rf''(r)}{f'(r)}\right)\left(\frac{r+z}{r-z}\right) - \lambda\frac{2rz}{(r-z)^2} + 2\,\mathrm{Re}\,\{\lambda\} + \bar{\lambda}\left(1 + \frac{rf''(r)}{f'(r)}\right)\left(\frac{rz+1}{rz-1}\right) - \bar{\lambda}\frac{2rz}{(rz-1)^2}, \tag{2.3}$$

and λ is a complex parameter with $|\lambda| = 1$ such that

$$\mathrm{Im}\left\{\frac{\lambda f''(r)}{f(r)}\right\} \doteq 0. \tag{2.4}$$

Theorem 2. *Any $f(z) \in S^*$ satisfying the conditions of Theorem 1 is of the form*

$$f(z) = \frac{z}{(1 - e^{i\alpha_1}z)^{1+\mu}(1 - e^{i\alpha_2}z)^{1-\mu}}. \tag{2.5}$$

where α_1, α_2, and μ are real, and $-1 \leq \mu \leq 1$.

We now observe a consequence of (2.5).

Theorem 3. *Let $f(z)$ be an arbitrary function of the form (2.5). Let λ be such that*

(2.4) *holds. Let the functions* $R(z)$ *and* $Q(z)$ *be defined by formulas* (2.2) *and* (2.3). *If both*

$$R(e^{-i\alpha_1}) = 0, \tag{2.6}$$

$$R(e^{-i\alpha_2}) = 0, \tag{2.7}$$

then the function $f(z)$ *satisfies the differential equation* (2.1).

Proof. From (2.2) and (2.3), $R(z)$ is purely imaginary and $Q(z)$ is purely real when $|z| = 1$. Further, both are rational functions of order 4 and hence are determined completely by their principial parts at $z = r$ and their values at $z = 0$.

If $f(z)$ is of the form (2.5) then

$$\frac{zf'(z)}{f(z)} = \frac{(1+\mu)}{2}\left(\frac{1+e^{i\alpha_1}z}{1-e^{i\alpha_1}z}\right) + \frac{(1-\mu)}{2}\left(\frac{1+e^{i\alpha_2}z}{1-e^{i\alpha_2}z}\right) \tag{2.8}$$

and

$$\frac{f''(z)}{f'(z)} = \frac{\mu(e^{i\alpha_1}-e^{i\alpha_2}) - 2e^{i(\alpha_1+\alpha_2)}z}{1+\mu(e^{i\alpha_1}-e^{i\alpha_2})z - e^{i(\alpha_1+\alpha_2)}z^2} + \frac{(2+\mu)e^{i\alpha_1}}{1-e^{i\alpha_1}z} + \frac{(2-\mu)e^{i\alpha_2}}{(1-e^{i\alpha_2}z)}.$$

Setting $z = r$ and putting these into (2.3), simple computations show that $Q(0) = \lambda$ and

$$Q(z) = -\frac{2r^2\lambda}{(z-r)^2} - \frac{2r^2\lambda f''(r)/f'(r) + 4r\lambda}{(z-r)} + g(z)$$

where $g(z)$ is regular at $z = r$. (The hypothesis that $\lambda f''(r)/f'(r)$ is real is needed in this computation.)

This defines the right hand side of (2.1). If $L(z)$ is the left hand side of (2.1), we see that $L(z)$ appears to be a rational function of order 6. However, since by hypothesis $R(e^{-i\alpha_1}) = R(e^{-i\alpha_2}) = 0$, the poles of (2.8) are cancelled. That is, $L(z)$ is a rational function of order 4. Further, from (2.8), $zf'(z)/f(z)$ is purely imaginary on $|z| = 1$. Thus $L(z)$ is also determined completely by its value at 0 and its principal part at $z = r$. Again, straightforward calculations show that $L(0) = \lambda$ and $L(z)$ has the same principal part as $Q(z)$ at $z = r$. It follows that $L(z) = Q(z)$, i.e. $f(z)$ satisfies the differential equation.

At this point, we seem to have the problem under control. The functions (2.5) depend on three real parameters, (2.4) defines λ, and hence when (2.6) and (2.7) are satisfied, we expect to have a single free parameter left which will then define the boundary of $M(r)$. Unfortunately, we have the following two theorems.

Theorem 4. *For every real* α, *the Koebe function* $k_\alpha(z) = z/(1-e^{i\alpha}z)^2$ *satisfies* (2.1) *if* λ *is chosen so that* $\lambda k''_\alpha(r)/k'_\alpha(r)$ *is real.*

Proof. Set

$$R_1(z)=\lambda\left(\frac{r+z}{r-z}\right)-\frac{2\lambda f(r)}{f'(r)}\frac{z}{(r-z)^2}+\lambda=\frac{2\lambda r}{(r-z)}\left(1-\frac{f(r)}{rf'(r)}\frac{z}{(r-z)}\right) \quad (2.9)$$

Then we see from (2.2) that $R(z)=R_1(z)-\overline{R_1(1/\bar{z})}$ holds in general. However, if $|z|=1$ this implies

$$R(z)=2i\,\mathrm{Im}\left\{R_1(z)\right\},\ |z|=1. \quad (2.10)$$

Thus, if $|z|=1$ and $R_1(z)$ is real, then $R(z)=0$.

If we set $f(z)=k_\alpha(z)$, then a straightforward calculation shows that

$$R_1(e^{-i\alpha})=-\frac{2\lambda re^{i\alpha}(2+re^{i\alpha})}{(1-re^{i\alpha})(1+re^{i\alpha})}=-\frac{\lambda rf''(r)}{f'(r)}$$

which is real by hypothesis. Hence $R(e^{-i\alpha})=0$ and the conclusion of the theorem follows from Theorem 3. setting $\alpha_1=\alpha_2=\alpha$.

Theorem 5. *Let α_1, α_2, and μ be given, with $-1<\mu<1$. Define $f(z)$ by (2.5) and let λ be such that (2.4) is satisfied. If (2.6) holds, then so does (2.7).*

Proof. The hypothesis of the theorem is equivalent to $R_1(e^{-i\alpha_1})$ being real, where $R_1(z)$ is defined by (2.9), since (2.10) holds in this case. Similarly, it suffices to prove that $R_1(e^{-i\alpha_2})$ is real.

One verifies easily that if $f(z)$ is of the form (2.5) then

$$R_1(e^{-i\alpha_1})=-\frac{2\lambda z_1}{(1-z_1)}\frac{[2-z_2-z_1z_2+\mu(z_1-z_2)]}{[1-z_1z_2+\mu(z_1-z_2)]},$$

$$R_1(e^{-i\alpha_2})=-\frac{2\lambda z_2}{(1-z_2)}\frac{[2-z_1-z_1z_2+\mu(z_1-z_2)]}{[1-z_1z_2+\mu(z_1-z_2)]},$$

where we use the convention $z_\nu=re^{i\alpha_\nu}$ as mentioned at the beginning of this section. Set

$$\beta=2\lambda rf'(r)/f''(r)=2\lambda\left(\frac{\mu(z_1-z_2)-2z_1z_2}{1-z_1z_2+\mu(z_1-z_2)}+\frac{(2+\mu)z_1}{(1-z_1)}+\frac{(2-\mu)z_2}{(1-z_2)}\right) \quad (2.11)$$

and

$$G_1=\frac{\beta}{R_1(e^{-i\alpha_1})}+2,\ G_2=\frac{\beta}{R_1(e^{-i\alpha_2})}+2. \quad (2.12)$$

Then a straightforward but somewhat tedious computation shows that

$$\begin{cases} G_1 = \dfrac{(1-\mu)(z_1-z_2)\,[2-2z_1z_2+\mu(z_1-z_2)]}{z_1(1-z_2)\,[2-z_2-z_1z_2+\mu(z_1-z_2)]}, \\ \\ G_2 = -\dfrac{(1+\mu)(z_1-z_2)\,[2-2z_1z_2+\mu(z_1-z_2)]}{z_2(1-z_1)\,[2-z_2-z_1z_2+\mu(z_1-z_2)]}, \end{cases} \tag{2.13}$$

and more easily that

$$\frac{1-\mu}{G_1} + \frac{1+\mu}{G_1} = 1 \tag{2.15}$$

If neither G_1 nor G_2 is zero or ∞, then this implies the conclusion of the theorem since by hypothesis, $R_1(e^{-i\alpha_1})$ is real, and so is β as defined by (2.11). Thus from (2.12), G_1 is real and (2.14) implies G_2 is real, which in turn implies $R_1(e^{-i\alpha_2})$ real from (2.12).

The theorem holds trivially if $\alpha_1 = \alpha_2$, so we may assume $z_1 \neq z_2$. If $G_1 = \infty$ then $2 - z_2 - z_1 z_2 + \mu(z_1 - z_2) = 0$ and clearly $G_2 \neq \infty$. Then from (2.14) G_2 is real and the conclusion follows as before. If $G_1 = 1-\mu$, then $G_2 = \infty$, but then $R_1(e^{-i\alpha_2}) = 0$ and the conclusion still follows.

We see that $G_1 = 0$ if and only if $\mu = 2(z_1 z_2 - 1)/(z_1 - z_2)$ and that $G_2 = 0$ if and only if $G_1 = 0$. However, the given conditions, $-1 < \mu < 1$, $|z_1| = |z_2| = r$, $z_1 \neq z_2$ make this impossible, as is seen by considering the linear fractional transformation $\omega = 2(z_1\zeta - 1)/(z_1 - \zeta)$. $G_1 = 0$ if and only if $\omega = \mu$ when $\zeta = z_2$. Here, $\zeta = 1, -1$, and $1/z_1$ map to $\omega = 2, -2$, and 0 respectively. Hence the line segment $[-2, 2]$ is an arc of a circle from -1 to $+1$ passing through $1/z_1$ (or the real axis less the interval $(-1, 1)$ if z_1 is real). In any case, this arc is exterior to the unit circle, and the image of $|\zeta| = r$ cannot cross the real axis between -1 and $+1$.

At this point we see that one of the needed conditions has evaporated, and, since [3] shows that not all Koebe functions are extremal, the set of solutions of (2.1) contains functions which are extraneous.

3. The envelope of the family of two slit functions. Since the differential equation does not contain (directly) sufficient information, we turn to a study of the family of two slit functions (2.5). If $f(z)$ is of the form (2.5), we compute $W = \log f'(r)$ and set

$$W(r, \alpha_1, \alpha_2, \mu) = \log[1 + \mu(z_1 - z_2) - z_1 z_2] - 2\log(1 - z_1)(1 - z_2) + \mu \log\left(\frac{1 - z_2}{1 - z_1}\right) \tag{3.1}$$

where as before $z_\nu = re^{i\alpha_\nu}$. The branch of $\log f'(z)$ is fixed by letting $\log f'(0) = 0$. We assume that the branches of the terms in (3.1) are chosen to give this value of $\log f'(r)$.

If we fix r, α_1, and α_2, then as μ varies from -1 to $+1$, the values of $W(r, \alpha_1, \alpha_2, \mu)$ trace out an arc, which must be contained in $M(r)$. From Theorems 1 and 2, every boundary point of $M(r)$ must be contained in one of these arcs. We thus turn to envelope theory and prove:

Theorem 6. *Let r with $0 < r < 1$ be given. Suppose w_0 is a boundary point of $M(r)$ which is not $\log k'_\alpha(r)$ for some Koebe function $k_\alpha(z) = z/(1 - e^{i\alpha}z)^2$. Then there exist α_1 and α_2 with $\alpha_1 \neq \alpha_2$, and μ with $-1 < \mu < 1$ such that the three complex numbers $\partial W / \partial\alpha_1$, $\partial W / \partial\alpha_2$, and $\partial W / \partial\mu$ at this r, α_1, α_2, and μ are linearly dependent over the reals. This is equivalent to the three quantities*

$$W_\nu = a_\nu\mu + b_\nu,\ \nu = 1, 2, 3, \tag{3.2}$$

being linearly dependent over the reals, where

$$\begin{aligned} a_1 &= iz_1(z_1 - z_2)/(1 - z_1), & b_1 &= iz_1(2 - z_2 - z_1z_2)/(1 - z_1), \\ a_2 &= iz_2(z_1 - z_2)/(1 - z_2), & b_2 &= iz_2(2 - z_1 - z_1z_2)/(1 - z_2), \\ a_3 &= (z_1 - z_2)\log(1 - z_1)(1 - z_2), & b_3 &= (z_1 - z_2) + (1 - z_1z_2)\log\left(\frac{1 - z_1}{1 - z_2}\right). \end{aligned} \tag{3.3}$$

Proof. We remark that it is easy to show that when the conditions of this theorem are satisfied, not only will the three quantities $\partial W / \partial\alpha_1$, $\partial W / \partial\alpha_2$, $\partial W / \partial\mu$ lie on the same straight line through the origin, but also $\partial W / \partial\alpha_1$ and $\partial W / \partial\alpha_2$ (or equivalently W_1 and W_2) will in fact lie on a single ray from the origin.

If w_0 is a boundary point of $M(r)$ bot not of $K(r)$, then there exist α_1, α_2, and μ as in the theorem. Since $-1 < \mu < +1$, each of these three can be varied freely in some neighborhood and $W(r, \alpha_1, \alpha_2, \mu)$ will cover a neighborhood of w_0 unless the rank of the Jacobian matrix $(\partial W / \partial\alpha_1, \partial W / \partial\alpha_2, \partial W / \partial\mu)$ is less than two, i.e. unless these three quantities are linearly dependent over the reals.

One easily varifies that the W_ν are real multiples of $[(1 - z_1z_2) + \mu(z_1 - z_2)]$ times the respective partial derivatives. Using a proof similar to that given to Theorem 5, it is easily shown that this common factor is non-zero.

The following theorem offers an interesting insight into Theorem 3.

Theorem 7. *Let $f(z)$ be of the form (2.5). Suppose λ satisfies (2.4) and $R(e^{-i\alpha_1}) = 0$. Then $\partial W / \partial\alpha_1$ and $\partial W / \partial\alpha_2$ are linearly dependent over the reals.*

Proof. From (2.13) and (3.3) it follows that $G_1W_1 + G_2W_2 = 0$ or equivalently that

$$G_1 \frac{\partial W}{\partial\alpha_1} + G_2 \frac{\partial W}{\partial\alpha_2} = 0.$$

The hypotheses of the theorem imply that G_1 and G_2 are real. Neither is zero, as was shown in the proof of Theorem 5. So if neither G_1 nor G_2 is ∞, the theorem follows. However, if G_1, say, is ∞, then from (2.13) and (3.3) it is clear that $\partial W / \partial\alpha_1 = 0$ and the theorem still holds.

From these results, we see that the entire content of the variational method is contained in Theorem 2. Thus it is necessary to base the study of the problem on a study of the family of two slit functions.

4. Methods of computation. Numerical methods were used to investigate this problem. Where possible, standard, well tested subroutines were used. Thus, for example, the problem of finding the value of $r = 0.810465...$ at which the boundary of $K(r)$ first touches the line $\operatorname{Im}\{w\} = \pi$ was solved purely numerically as follows.

Set $\phi(z) = \log k'(z) = \log(1+z) - 3\log(1-z)$. Then $z\phi'(z) = 2z(2+z)/(1-z^2)$. Let $z = re^{i\alpha}$. For a fixed r, a standard zero-finding routine was used to solve for the α for which $\operatorname{Re}\{z\phi'(z)\} = 0$. This locates the 'top' point on $K(r)$ and allows one to compute $\phi(z)$ as a function of r. Another copy of the same zero finding routine was used to solve for the r at which this $\operatorname{Im}\{\phi(z)\} = \pi$. This was easy to program and required a negligible amount of computer time. All computations were done in double precision (about 18 decimal places accuracy). This allowed all results to be obtained with more than eight digit accuracy without any dificulty with roundoff errors.

The major computational work was based on Theorem 6. Two functions, $F(r, \alpha_1, \alpha_2)$ and $G(r, \alpha_1, \alpha_2)$ where defined as follows. Given any r, α_1, and α_2, set $z_1 = re^{i\alpha_1}$ and $z_2 = re^{i\alpha_2}$, and define the complex numbers a_ν, b_ν, $\nu = 1, 2, 3$ by (3.3). When the $W_\nu = a_\nu\mu + b_\nu$ are linearly dependent over the reals, we must have

$$P_\nu = \operatorname{Im}\{W_\nu \overline{W}_3\} = A_\nu\mu^2 + B_\nu\mu + C_\nu = 0,$$

for $\nu = 1, 2$, where the quantities A_ν, B_ν, C_ν are defined by

$$\begin{cases} A_\nu = \operatorname{Im}\{a_\nu \overline{a}_3\}, \\ B_\nu = \operatorname{Im}\{b_\nu \overline{a}_3 + \overline{b}_3 a_\nu\}, \\ C_\nu = \operatorname{Im}\{b_\nu \overline{a}_3\} \end{cases} \tag{4.1}$$

for $\nu = 1, 2$.

Treat P_1 and P_2 as polynomials in μ and apply the Euclidean algorithm to eliminate μ. Thus when $P_1 = P_2 = 0$ we must have $D_\nu\mu + E_\nu = 0$ for $\nu = 1, 2$ where

$$\begin{cases} D_1 = A_2B_1 - A_1B_2 \\ E_1 = A_2C_1 - A_1C_2 \\ D_2 = A_1C_2 - A_2C_1 = -E_1 \\ E_2 = B_1C_2 - B_2C_1. \end{cases} \tag{4.2}$$

Then, these two linear expressions being zero simultaneously implies $D_1E_2 + E_1^2 = 0$.

If $D_1\mu + E_1 = 0$ and $D_1 \neq 0$ then $\mu = -E_1/D_1$, so $|\mu| < 1$ if and only if $|D_1| >$

$> | E_1 |$. However in any case, if $D_1 \mu + E_1 = 0$ and $| \mu | \leq 1$, then $| D_1 | - | E_1 | \geq 0$. Thus if we set

$$\begin{cases} F(r, \alpha_1, \alpha_2) = D_1 E_2 + E_1^2 \\ G(r, \alpha_1, \alpha_2) = | D_1 | - | E_1 |, \end{cases} \tag{4.3}$$

we have proved

Theorem 8. *Let r with $0 < r < 1$ be given. Suppose w_0 is a boundary point of $M(r)$ which is not a boundary point of $K(r)$. Then there exists a function of the form* (2.5) *such that $w_0 = W(r, \alpha_1, \alpha_2, \mu)$ and*

$$\begin{cases} F(r, \alpha_1, \alpha_2) = 0, \\ G(r, \alpha_1, \alpha_2) \geq 0. \end{cases} \tag{4.4}$$

We observe that condition (4.4) is necessary but not sufficient for $W(r, \alpha_1, \alpha_2, \mu)$ to be a boundary point. In particular, whenever $W_3 = 0, F = 0$ even though W_1 and W_2 may not be linearly dependent.

Given r, α_1, and α_2 we set $z_1 = re^{i\alpha_1}$ and $z_2 = re^{i\alpha_2}$. Then using (3.3), (4.1), (4.2), and (4.3) we can readily compute $F(r, \alpha_1, \alpha_2)$ and $G(r, \alpha_1, \alpha_2)$. The behavior of these functions is indicated in Figures 2 and 3 which show the curves along which $F = 0$ and $G = 0$ for $r = 0.99$ and $r = 0.935$, respectively. These are shown in the triangular region $0 \leq \alpha_1 \leq 2\pi, 0 \leq \alpha_2 \leq \alpha_1, 0 \leq \alpha_2 \leq 2\pi - \alpha_1$ since an inspection of the definitions shows that $F(r, -\alpha_1, -\alpha_2) = F(r, \alpha_1, \alpha_2), G(r, -\alpha_1, -\alpha_2) = G(r, \alpha_1, \alpha_2), F(r, \alpha_2, \alpha_1) = F(r, \alpha_1, \alpha_2)$, and $G(r, \alpha_2, \alpha_1) = G(r, \alpha_1, \alpha_2)$. Of course, both F and G are periodic in both α_1 and α_2 with period 2π.

The curves of Figures 2 and 3 were prepared by computing points along these curves. Starting at an approximate zero, a numerical approximation to the gradient was computed and a zero of the function was searched for along this gradient. The next starting point was found by moving a short distance orthogonal to the gradient. F has a zero of order 8 in

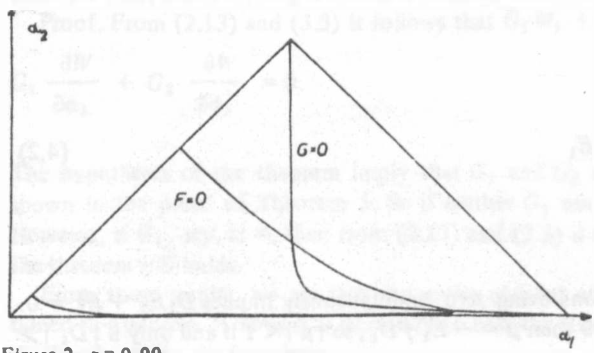


Figure 2. $r = 0.99$

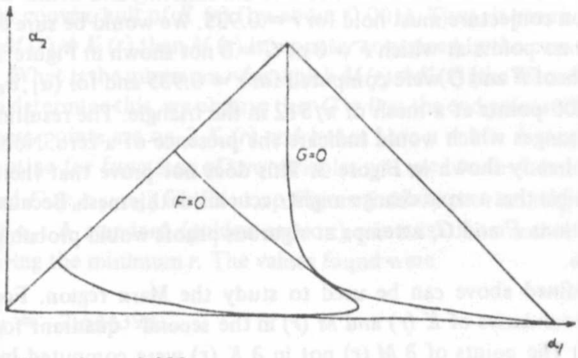


Figure 3. $r = 0.935$

$(\alpha_1 - \alpha_2)$, so near the line $\alpha_1 = \alpha_2$ the gradient of F was approximated by $(1, 1)$ or $(-1, -1)$ rather than being computed. All computations of F and G were done in double precision and appear to be accurate to about 14 decimal places. The zeros of F and G were located with an accuracy of 10^{-4} or better, which is less than the width of the plotted curve.

In both of these figures, the curve of $G = 0$ extends from the point (π, π) to the point $(2\pi, 0)$. G is greater than zero to the right of this curve. The curve along which $F = 0$ joins two points on the line $\alpha_1 = \alpha_2$ and is tangent to the line $\alpha_2 = 0$ at $(\pi, 0)$. The portion of this curve extending to the left (smaller values of α_1) from $(\pi, 0)$ is the arc on which $W_3 = 0$ and hence represents the spurious zeros of F mentioned above. However, we see that $G < 0$ along all points of this portion of the curve so none are candidates for extreme points on $\partial M(r) - \partial K(r)$.

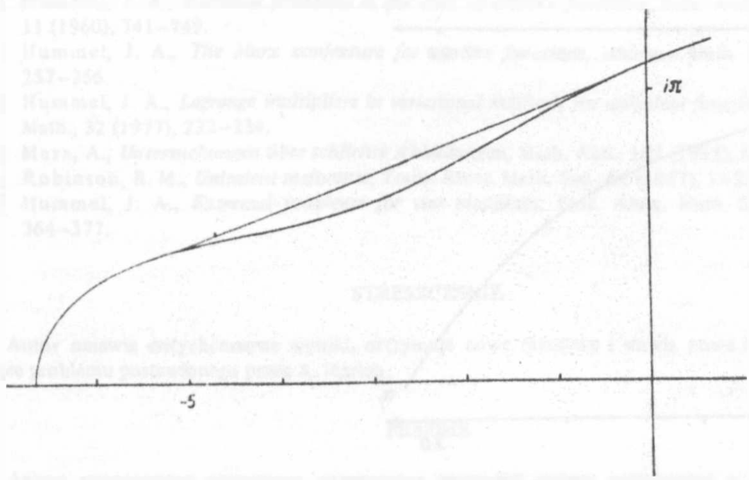
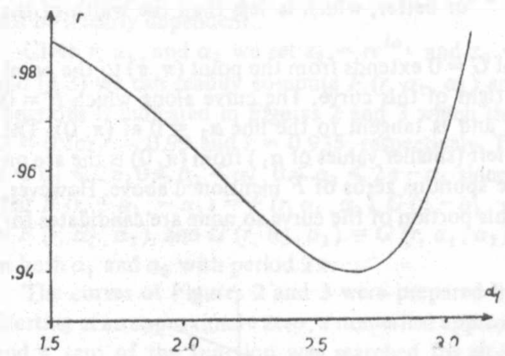


Figure 4. $M(r) - K(r)$ for $r = 0.99$

In Figure 3, we see that the curves of $F = 0$ and $G = 0$ are disjoint, and hence we suspect that the Marx-Robinson conjecture must hold for $r = 0.935$. We would be sure of this if we knew that there are no points at which $F = 0$ or $G = 0$ not shown in Figure 3.

To investigate this, the values of F and G were computed for $r = 0.935$ and for (α_1, α_2) at the set of more than 261,000 points at a mesh of $\pi/512$ in the triangle. The resulting data were inspected for sign changes which would indicate the presence of a zero. None were found other than those already shown in Figure 3. This does not prove that there are no others. It is always possible that a rapid change might occur inside this mesh. Because of the complexity of the functions F and G, attemps at rigorous proofs would probably best start fresh from Theorem 6.

The functions F and G defined above can be used to study the Marx region. For example, Figure 4 shows the boundaries of $K(r)$ and $M(r)$ in the second quadrant for $r = 0.99$. (Compare Figure 1.) The points of $\partial M(r)$ not in $\partial K(r)$ were computed by fixing an α_1 and searching for an α_2 at which $F(r, \alpha_1, \alpha_2) = 0$. Then if $G > 0$ at this point the value of μ $(= -E_1 / D_1)$ was determined and $w_0 = W(r, \alpha_1, \alpha_2, \mu)$ was calculated. This was done for enough α_1 to give enough points to produce Figure 4. The symmetric points would of course also occur in the third quadrant.



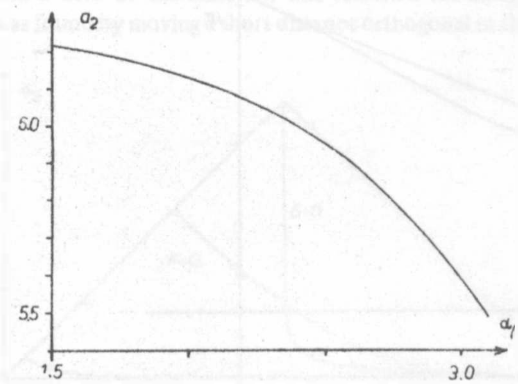


Figure 5. Values of r, α_1, α_2, for which $F = G = 0$

Observe that the arc of $\partial M(r) - \partial K(r)$ has a slight curvature and lies properly inside the convex hull of $K(r)$ (by about 0.001). Thus, it seems reasonable to conjecture that if $M(r) \neq K(r)$ then $M(r)$ is properly contained in the convex hull of $K(r)$.

What is the minimum r for which $M(r) \not\subset K(r)$ (the Marx-Robinson radius)? To attempt to determine this, we observe that $G = 0$ at the end points of the arc of $\partial M(r) - \partial K(r)$ since these points are on $\partial K(r)$ and hence have $\mu = \pm 1$. A simple secant method zero finding routine for functions of two variables was used to find simultaneous zeros of $F(r, \alpha_1, \alpha_2)$ and $G(r, \alpha_1, \alpha_2)$ for fixed α_1. Figure 5 shows the resulting values of r and α_2 as functions of α_1. A standard (golden section) minimization routine was used to solve for the α_1 giving the minimum r. The values found were

$$\alpha_1 = 2.644398...$$

$$\alpha_2 = 5.8675868...$$

$$r = 0.9391922419...$$

This value of r was computed to 14 places and the digits shown are certainly accurate. The values of α_1 and α_2 are of course only accurate to half as many places.

If the functions F and G have no other zeros than those along the curves indicated in Figure 2 and 3, then these computations would constitute a proof of the conjecture that the above r is the actual Marx-Robinson radius.

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STRESZCZENIE

Autor omawia dotychczasowe wyniki, otrzymuje nowe rezultaty i stawia nowe hipotezy dotyczące problemu postawionego przez A. Marksa.

РЕЗЮМЕ

Автор оговаривает известные результаты, получает новые результаты и формулирует новые гипотезы, связанные с известной проблемой А. Маркса.