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Distortion Estimates for Quasisymmetric Functions

Oszacowania zniekształcenia dla funkcji quasisymetrycznych

Оценки искажения для квазисимметрических функций

1. Introduction. An increasing homeomorphism f of the real line $\mathbb{R}$ onto itself is called K-quasisymmetric (K-qs), if

$$\frac{1}{K} \le \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \le K \tag{1}$$

holds for $x \in \mathbb{R}$, $t \neq 0$. A function is quasisymmetric (qs) if it is K-qs for some K. Qs functions are exactly the boundary values of those quasiconformal mappings of the upper half plane onto itself that fix the point at infinity [1].

We write

$$N(K) = \{f : f \text{ is } K\text{-qs}, f(0) = 0, f(1) = 1\},$$

$$N_0(K) = \{f : f \text{ is } K\text{-qs}, f(-1) = -1, f(1) = 1\},$$

introducing thus two normalizations for qs functions. We define

$$M(x, K) = \sup \{f(x) : f \in N(K)\},$$

$$m(x, K) = \inf \{f(x) : f \in N(K)\}.$$

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We define similarly $M_0(x, K)$ and $m_0(x, K)$ for the class $N_0(K)$ and note that sup (inf) can be replaced by max (min), since $N(K)$ and $N_0(K)$ are compact [1].

If f is qs, we write

$$q(f) = \inf \{K : f \text{ is } K\text{-qs}\}$$

and note that f is $q(f)$-qs.

Let S_1 and S_2 be adjacent line segments, and let $|S|$ be the length of S. For $c \geq 1$, $K \geq 1$ there is a number Q such that

$$\frac{1}{Q} \leq \frac{|f(S_1)|}{|f(S_2)|} \leq Q \tag{1.2}$$

whenever f is K-qs and $1/c \leq |S_1|/|S_2| \leq c$. We denote the infimum of such numbers Q by $q(c, K)$ and note that (1.2) remains valid for $Q = q(c, K)$.

Our first result shows the connection between these concepts.

Theorem 1. *We have*

$$q(c, K) + 1 = M(c+1, K) \tag{1.3}$$

and

$$q(\rho_1, \rho_2) = \sup \{q(f_2 \circ f_1) : f_i \text{ is } \rho_i\text{-qs}, i = 1, 2\}. \tag{1.4}$$

The functions $M(x, K)$ and $m(x, K)$ are related for $x > 1$, as our next result shows.

Theorem 2. *Assume that* $x > 1$ *and* $y > 1$. *Then* $1/x + 1/y = 1$ *if and only if*

$$1/M(x, K) + 1/m(y, K) = 1. \tag{1.5}$$

Moreover, $(x-1)(y-1) = 4$ *if and only if*

$$(M_0(x, K) - 1)(m_0(y, K) - 1) = 4. \tag{1.6}$$

It seems to be of interest to obtain good bounds for $M(x, K)$ and $m(x, K)$, particularly in view of Theorem 1. Kelingos proved that for $x \geq 1$ [3, Theorem 1], $M(x, K) \leq (2x)^a$ and $m(x, K) \geq (x/2)^b$ where $a = \log_2(K+1)$ and $b = \log_2(1 + 1/K)$. This together with Theorem 1 shows e.g. that $\log_2(q(c, K) + 1) \leq (1 + \log_2(c+1)) \log_2(K+1)$, $\log_2(q(f_2 \circ f_1) + 1) \leq (1 + \log_2(q(f_1) + 1)) \log_2(q(f_2) + 1)$. This is already better than the estimate for $q(f_2 \circ f_1)$ obtained by using quasiconformal extensions of f_1 and f_2, namely (cf. [3]) $\log q(f_2 \circ f_1) \leq (\text{const.})\, q(f_1)\, q(f_2)$.

We will sharpen the above bounds for $M(x, K)$ and $m(x, K)$, and among other things we will show (improving the above a and b) that the correct exponent is the exponent α ($\alpha \geq 1$ for M, $0 < \alpha \leq 1$ for m) such that $q(g_\alpha) = K$, where $g_\alpha(x) = |x|^\alpha \operatorname{sign} x$. The functions g_α were studied by Beurling and Ahlofors in [1]. Since the precise statement of our results needs some preparation, we postpone it until the appropriate sections.

Finally we prove a result needed in [2].

Theorem 3. *For every $K \geq 1$, the functions $M(x, K)$, $m(x, K)$ and $q(c, K)$ are continuous.*

2. Proof of Theorems 1 and 2. We recall [3] that if f is qs, if L_i $(i = 1, 2)$ are linear and if $g(x) = -f(-x)$, then $q(f) = q(g) = q(L_1 \circ f \circ L_2)$.

2.1. Proof of Theorem 1. Let S_1 and S_2 be adjacent line segments with $1/c \leq |S_1| / |S_2| \leq c$, $c \geq 1$. We can assume that the shorter of S_1 and S_2, say S_1, is $[0, 1]$, and that S_2 is $[1, \ell + 1]$, $\ell \leq c$, since this can be achieved by using linear transformations. It is easy to see that when we are looking for the maximum of $|f(S_2)| / |f(S_1)|$, we can assume that the longer of S_1 and S_2 is mapped onto the longer of $f(S_1)$ and $f(S_2)$.

Since now $|f(S_2)|/|f(S_1)| = f(\ell + 1) - 1$, $\ell \leq c$, we deduce $q(c, K) \leq M(c + 1, K) - 1$. On the other hand, if $S_1 = [0, 1]$, $S_2 = [1, c + 1]$ and if $f \in N(K)$ is such that $f(c + 1) = M(c + 1, K)$, then $q(c, K) \geq |f(S_2)| / |f(S_1)| = M(c + 1, K) - 1$. This gives (1.3).

To prove (1.4), we define

$$f_1(x) = x, \quad x \leq 1,$$

$$f_1(x) = 1 + \rho_1(x - 1), \quad x \geq 1.$$

Then $q(f_1) = \rho_1$. Let $f_2 \in N(\rho_2)$ be such that

$$f_2(\rho_1 + 1) = M(\rho_1 + 1, \rho_2). \tag{2.1}$$

Then with $f = f_2 \circ f_1$, we have by (1.3) and (2.1),

$$q(f) \geq \frac{f(2) - f(1)}{f(1) - f(0)} = f_2(\rho_1 + 1) - 1 = q(\rho_1, \rho_2).$$

Hence the right hand side of (1.4) is at least equal to $q(\rho_1, \rho_2)$.

On the other hand, if f_i is ρ_i-qs, $i = 1, 2$, and if S_1 and S_2 are adjacent segments of equal length, then

$$\frac{1}{\rho_1} \leq \frac{|f_1(S_1)|}{|f_1(S_2)|} \leq \rho_1,$$

so that by the definition of $q(c, K)$, we have

$$\frac{1}{q(\rho_1, \rho_2)} \leq \frac{|f_2(f_1(S_1))|}{|f_2(f_1(S_2))|} \leq q(\rho_1, \rho_2).$$

Hence $q(f_2 \circ f_1) \leq q(\rho_1, \rho_2)$. Theorem 1 is proved.

2.2. Proof of Theorem 2. Assume first that $1/x + 1/y = 1$, $x, y > 1$. If $f \in N(K)$, we define $L_1(t) = (1 - x)t + x$, $L_2(t) = (1 - z)t + z$, where $1/z + 1/f(x) = 1$. If

$g = L_2 \circ f \circ L_1$, then $g \in N(K)$ and $g(y) = z$. For $f(x) = M(x, K)$, we obtain $m(y, K) \leqslant \leqslant g(y) = z$, hence $1/m(y, K) + 1/M(x, K) \geqslant 1$. Next we interchange x and y and choose f so that $f(y) = m(y, K)$. Then $M(x, K) \geqslant g(x) = z$, where $1/z + 1/m(y, K) = 1$. Hence $1/m(y, K) + 1/M(x, K) \leqslant 1$. Thus $1/m(y, K) + 1/M(x, K) = 1$.

On the other hand, assume that (1.5) holds. Let z be such that $1/z + 1/y = 1$. Then by the first part of Theorem 2, we have $1/M(z, K) + 1/m(y, K) = 1$. Since (1.5) holds and since $M(x, K)$ is strictly increasing, we must have $z = x$.

Next we note that $f_0 \in N_0(K)$ if and only if $f \in N(K)$, where

$$f(x) = ½ \left\{ f_0(2x - 1) + 1 \right\}, \tag{2.2}$$

$$f_0(x) = 2f \left\{ ½ (x + 1) \right\} - 1. \tag{2.3}$$

Therefore

$$M(x, K) = ½ \left\{ M_0(2x - 1, K) + 1 \right\}, \tag{2.4}$$

$$M_0(x, K) = 2M(½ (x + 1), K) - 1, \tag{2.5}$$

and similar equations are true for $m(x, K)$ and $m_0(x, K)$. Now (1.6) follows from (1.5) by using these relations. Theorem 2 is proved.

Remark. The values of $M(x, K)$ for $x \geqslant 1$ naturally determine $M(x, K)$ and $m(x, K)$ for $x < 1$. Using linear transformations as in the proof of Theorem 2 we can deduce

$$M(x, K) = 1/m(1/x, K), \ 0 < x \leqslant 1, \tag{2.6}$$

$$m(x, K) = 1/M(1/x, K), \ 0 < x \leqslant 1, \tag{2.7}$$

and

$$M(x, K) = 1 - m(1 - x, K), \ x \leqslant 0, \tag{2.8}$$

$$m(x, K) = 1 - M(1 - x, K), \ x \leqslant 0. \tag{2.9}$$

Similar results are true for $M_0(x, K)$ and $m_0(x, K)$.

3. Estimates for $M_0(x, K)$ and $m_0(x, K)$. In the rest of the paper, the normalization $f(-1) = -1$, $f(1) = 1$ is more convenient, so that we will consider only $M_0(x, K)$ and $m_0(x, K)$. If not otherwise mentioned, K will be fixed but arbitrary, and we often write $M_0(x)$ and $m_0(x)$ instead of $M_0(x, K)$ and $m_0(x, K)$. By using (2.4), the reader can obtain the corresponding results for $M(x, K)$ and $m(x, K)$.

The correct orders of magnitude of $M_0(x)$ and $m_0(x)$ are given for suitable α by the functions

$$g_\alpha(x) = |x|^\alpha \operatorname{sign} x, \tag{3.1}$$

Beurling and Ahlfors [1, p. 132–134] studied these functions and proved the following result.

Lemma 1. *The map g_α is qs for $\alpha > 0$, and $q(g_\alpha) = K_\alpha$ is determined as follows. Let t_α be the solution of*

$$(t+1)^{1-\alpha} + (t-1)^{1-\alpha} = 2\,, \tag{3.2}$$

so that $1 < t_\alpha < 2$. Further set

$$\begin{aligned} q_\alpha &= [(t_\alpha+1)^\alpha - 1]\,[(t_\alpha-1)^\alpha + 1]^{-1} = [(t_\alpha+1)/(t_\alpha-1)]^{\alpha-1} = \\ &= 2\,(t_\alpha+1)^{\alpha-1} - 1. \end{aligned} \tag{3.3}$$

Then $K_\alpha = q_\alpha$ for $\alpha > 1$, $K_\alpha = 1/q_\alpha$ for $0 < \alpha < 1$, and $K_1 = 1$.

The quantity q_α is a continuous strictly increasing function of α. Thus if $K > 1$, the equation $q_\alpha = K$ has exactly one positive root which we denote by $\alpha_1(K)$. Similarly the equation $1/q_\alpha = K$ has exactly one positive root which we denote by $\alpha_2(K)$. Thus α_1 is strictly increasing while α_2 is strictly decreasing. Further, g_{α_1} and g_{α_2} are K-qs.

More precisely, Beurling and Ahlfors showed that for $x \in R$, $t > 0$,

$$\frac{1}{K_\alpha} \leqslant \frac{g_\alpha(x+t) - g_\alpha(x)}{g_\alpha(x) - g_\alpha(x-t)} \leqslant K_\alpha \tag{3.4}$$

with equality on the right hand side if $t = xt_\alpha$, $x > 0$, when $\alpha \geqslant 1$, and on the left hand side if $t = xt_\alpha$, $x > 0$, $\alpha < 1$.

The functions g_α belong to both $N(K)$ and $N_0(K)$. This gives immediately

Lemma 2. *If $x \geqslant 1$, we have*

$$M_0(x, K) \geqslant x^{\alpha_1(K)}$$

$$m_0(x, K) \leqslant x^{\alpha_2(K)}\,.$$

These are the correct bounds up to multiplicative constants depending on K. We proceed to prove

Theorem 4. *Suppose that we have the inequalities*

$$M_0(x) \leqslant c_1 x^{\alpha_1}\,,\quad 0 < x_1 \leqslant x \leqslant x_2\,, \tag{3.5}$$

and

$$m_0(x) \geqslant c_2 x^{\alpha_2},\quad 0 < x_3 \leqslant x \leqslant x_4\,. \tag{3.6}$$

If

$$x_2/x_1 \geqslant (t_1+1)/(t_1-1) = K^{1/(\alpha_1-1)},\quad t_1 = t_{\alpha_1}, \tag{3.7}$$

then (3.5) *holds for* $x \geqslant x_1$, *and if*

$$x_4/x_3 \geqslant (t_2+1)/(t_2-1) = K^{1/(1-\alpha_2)},\ t_2 = t_{\alpha_2}, \tag{3.8}$$

then (3.6) *holds for* $x \geqslant x_3$.

3.1. To prove Theorem 4, we need a lemma.

Lemma 3. *If* $a, b > 0$, *then*

$$M_0(a+2b) \leqslant KM_0(a) + (K+1)\,M_0(b), \tag{3.9}$$

$$m_0(a+2b) \geqslant (1/K)\,m_0(a) + (1+1/K)\,m_0(b). \tag{3.10}$$

We set $x = b$, $x - t = -a$, so that $x + t = a + 2b$. Then (1.1) shows that for $f \in N_0(K)$ we have $f(a+2b) \leqslant (K+1)f(b) - Kf(-a)$. Since $f \in N_0(K)$, so does $-f(-x)$. Thus $f(b) \leqslant M_0(b)$ and $-f(-a) \leqslant M_0(a)$, so that $f(a+2b) \leqslant (K+1)\,M_0(b) + KM_0(a)$. This gives (3.9). Similarly $f(a+2b) \geqslant (1+1/K)f(b) - f(-a)/K$, and this yields (3.10) and completes the proof of Lemma 3. We can now complete the proof of Theorem 4. Assume that (3.5) and (3.7) hold. We write $\rho = t_1 + 1$, $r = \rho/(t_1 - 1)$ and suppose that (3.5) holds for $x_1 \leqslant x \leqslant x_2\rho^n$, $n \geqslant 0$. This is true for $n = 0$, so that if we can prove (3.5) for $x_2\rho^n < x \leqslant x_2\rho^{n+1}$, it follows by induction that (3.5) holds for $x \geqslant x_1$. To do this, pick x, $x_2\rho^n < x \leqslant x_2\rho^{n+1}$, and apply (3.9) with $a + 2b = x$ and $a = x/r$. This gives $b = x/(t_1+1)$, and our hypotheses ensure that $x_1 \leqslant b = x/\rho$ and $x_1 \leqslant a < x/\rho$. Thus $M_0(a) \leqslant c_1 a^{\alpha_1}$, $M_0(b) \leqslant c_1 b^{\alpha_1}$.

We recall that equality holds on the right hand side of (3.4) if $\alpha = \alpha_1$, $K_\alpha = K$, and $x = b$, $t = xt_1 = bt_1$. This gives (for our x)

$$x^{\alpha_1} = (K+1)\,b^{\alpha_1} + Ka^{\alpha_1}. \tag{3.11}$$

On the other hand, Lemma 3 yields

$$M_0(x) = M_0(a+2b) \leqslant c_1(Ka^{\alpha_1} + (1+K)\,b^{\alpha_1}) = c_1 x^{\alpha_1},$$

by (3.11). Thus (3.5) holds for $x \geqslant x_1$. Similarly we deduce that (3.6) remains valid for $x \geqslant x_3$. This completes the proof of Theorem 4.

Before continuing we note that t_α is a strictly increasing function of α. To see this we write

$$x = (t+1)^{1-\alpha},\ y = (t-1)^{1-\alpha},\ \phi(t,\alpha) = x + y.$$

Then if $\alpha \neq 1$,

$$\frac{1}{1-\alpha}\,\frac{\partial\phi}{\partial t} = (t+1)^{-\alpha} + (t-1)^{-\alpha} > 0$$

and if $\phi(t,\alpha) = 2$,

$$\tfrac{1}{2}(1-\alpha)\frac{\partial\phi}{\partial\alpha} = -\tfrac{1}{2}(x\log x + y\log y) < -\frac{x+y}{2}\log\left(\frac{x+y}{2}\right) = 0,$$

since $x \log x$ is a convex function of x for $x > 0$. Thus

$$\frac{dt}{d\alpha} = -\frac{\partial\phi}{\partial\alpha} \Big/ \frac{\partial\phi}{\partial t} > 0 .$$

Also as was noted by Beurling and Ahlfors [1], t_α tends to $1, \sqrt{2}, 2$ as α tends to $0, 1, \infty$ respectively. Thus t_α increasses strictly from 1 to 2 as α increases from 0 to ∞.

4. Estimates for $\alpha_2(K)$ and $c_2(K)$. We define

$$c_1(K) = \sup_{x \geq 1} M_0(x, K)\, x^{-\alpha_1(K)} \tag{4.1}$$

and

$$c_2(K) = \inf_{x \geq 1} m_0(x, K)\, x^{-\alpha_2(K)}. \tag{4.2}$$

If $x_2 > x_1 > 0$, then the supremum (4.1) over $x_1 \leq x \leq x_2$ is certainly finite. It follows from Theorem 4 that $c_1(K) < \infty$. Similarly, $c_2(K) > 0$. Moreover, by Theorem 4 and Theorem 3, to be proved in section 6, $c_1(K)$ and $c_2(K)$ are attained. Clearly $c_1(K) \geq 1$ and $c_2(K) \leq 1$ (consider $x = 1$). Now we derive bounds in the opposite direction.

Theorem 5. *We have for $K > 1$,*

$$\alpha_2(K)\log 2 < \log\,[1 + 2/(K-1)]\,. \tag{4.3}$$

Further, we have

$$c_2(K)\, x^{\alpha_2(K)} \leq m_0(x, K) \leq x^{\alpha_2(K)},\quad x \geq 1\,, \tag{4.4}$$

where $c_2(K) \geq 1/9$ for all K, and for $K \geq 3$,

$$c_2(K) \geq K^{-6/K} = 1 + O(\log K)\, K^{-1} \tag{4.5}$$

as $K \to \infty$.

Remark. In fact it can be shown from (3.2) and (3.3) (see also [1, p. 134]) that

$$t_{\alpha_2} = 1 + \alpha_2(K)\log 2 + O(\alpha_2^2 \log \alpha_2),\ \alpha_2 \to 0\,, \tag{4.6}$$

and

$$\alpha_2(K) = 2\,(K\log 2)^{-1} + O(\log K)\, K^{-2},\ K \to \infty. \tag{4.7}$$

It follows from Lemma 1 that K, $\alpha = \alpha_2(K)$ and $t = t_\alpha$ are related by

$$1/K = 2(t+1)^{\alpha-1} - 1 \tag{4.8}$$

and

$$(t+1)^{1-\alpha} + (t-1)^{1-\alpha} = 2\,. \tag{4.9}$$

This gives

$$2K/(K+1) + \left\{[2K/(K+1)]^{1/(1-\alpha)} - 2\right\}^{1-\alpha} = 2\,, \tag{4.10}$$

or

$$[2K/(K+1)]^{1/(1-\alpha)} = 2 + [2/(K+1)]^{1/(1-\alpha)}, \tag{4.11}$$

which implies

$$[2K/(K+1)]^{1/(1-\alpha)} < (2^{1-\alpha} + 2/(K+1))^{1/(1-\alpha)},$$

i.e.

$$(K-1)/(K+1) < 2^{-\alpha}.$$

This yields (4.3)

Lemma 2 and (4.2) yield (4.4). To prove the estimate for $c_2(K)$, we first note that if $\alpha = ½$, then by (4.9), $t = 5/4$, and by (4.8), $K = 3$. For $1 \leq K \leq 3$, we thus have

$$\frac{t+1}{t-1} \leq 9,$$

since t increases with α_2 and so with $1/K$, as we noted at the end of section 3.

Thus, by Theorem 4,

$$c_2(K) = \inf_{1 \leq x \leq 9} m_0(x)\,x^{-\alpha} \geq 9^{-\alpha} \geq 1/9\,.$$

If $K > 3$ it follows from Theorem 4 that

$$c_2(K) = \inf_{1 \leq x \leq K^{1/(1-\alpha)}} m_0(x)\,x^{-\alpha} \geq K^{-\alpha/(1-\alpha)} \geq K^{-2\alpha},$$

since $0 < \alpha < ½$. Using (4.3) we deduce that

$$\log c_2(K) \geq -2(\log 2)^{-1}(\log K)\log[(K+1)/(K-1)]\,.$$

Also for $K \geq 3$,

$$K \log [(K+1)/(K-1)] = (1/K^{-1}) \log [(1+K^{-1})/(1-K^{-1})] \leq 3 \log 2 .$$

Therefore

$$c_2(K) \geq \exp \left\{ -6 (\log K)/K \right\} \geq \exp(-2 \log 3) = 1/9 ,$$

which gives (4.5) and shows that $1/9 \leq c_2(K) \leq 1$ for $K \geq 1$. This proves Theorem 5.

For later use we apply (3.10) with $a = b$ to obtain $m_0(3a) \geq (1 + 2/K)\, m_0(a)$. Since $m_0(1) = 1$, we get by induction

$$m_0(3^n) \geq (1 + 2/K)^n, \; n \geq 0 . \tag{4.12}$$

Taking $n = 1$ and applying Lemma 2 with $x = 3$ we deduce that

$$\alpha_2(K) \log 3 \geq \log (1 + 2/K) .$$

5. **Estimates for $\alpha_1(K)$ and $c_1(K)$.** First we prove a result analogous to Theorem 5.

Theorem 6. *We have*

$$0 < \alpha_1(K) - \log[(3/2)(K+1)]/\log 3 \leq \log [2K/(K+1)]/(3 \log 3) < \\ < \log 2/\log 27 < 0.211 . \tag{5.1}$$

Moreover, we have for $x \geq ½$,

$$x^{\alpha_1(K)} \leq M_0(x, K) \leq c_1(K)\, x^{\alpha_1(K)}, \tag{5.2}$$

where

$$c_1(K) \leq 2K + 1 . \tag{5.3}$$

Remark. As $K \to \infty$, we have

$$\alpha_1(K) \log 3 = \log K + a + O(1)/\log K \tag{5.4}$$

where $a = \log 3 - (2/3) \log 2 = 0.6365...$

It was proved in [1, p. 133] that $\sqrt{2} < t < 2$, $t = t_\alpha$, $\alpha = \alpha_1(K)$. Thus (3.7) is certainly satisfied if $x_2/x_1 \geq (\sqrt{2} + 1)/(\sqrt{2} - 1) = 3 + 2\sqrt{2} \approx 5.83$.

Suppose that $x_1 = ½$, $x_2 = 3$. We take $a = b$ in (3.9) to obtain $M_0(3a) \leq (2K + 1)\, M_0(a)$. Since $M_0(1) = 1$, induction gives

$$M_0(3^n) \leq (2K + 1)^n, \; n \geq 0. \tag{5.5}$$

Taking $n = 1$ and applying Lemma 2 with $x = 3$ we get

$$\alpha_1(K) \leqslant \log(2K+1)/\log 3 . \tag{5.6}$$

Thus for $\frac{1}{2} \leqslant x \leqslant 1$,

$$M_0(x)x^{-\alpha} \leqslant 2^{\alpha} \leqslant (2K+1)^{\log 2/\log 3},$$

while for $1 \leqslant x \leqslant 3$, $M_0(x)x^{-\alpha} < M_0(3) \leqslant 2K+1$. Thus (3.5) holds for $\frac{1}{2} \leqslant x \leqslant 3$ with $c_1 = 2K+1$ and so for $x \geqslant \frac{1}{2}$. This proves (5.2) subject to (5.3).

It remains to prove (5.1). By Lemma 1, K, t and α are related by $(t+1)^{1-\alpha} + (t-1)^{1-\alpha} = 2$ and $K = 2(t+1)^{\alpha-1} - 1$. Eliminating t we get

$$2/(K+1) + \left\{[2/(K+1)]^{1/(1-\alpha)} - 2\right\}^{1-\alpha} = 2$$

or

$$[2/(K+1)]^{1/(1-\alpha)} = 2 + [2K/(K+1)]^{1/(1-\alpha)} < 3, \tag{5.7}$$

since $\alpha > 1$. Thus $\alpha > 1 + \log[\frac{1}{2}(K+1)]/\log 3$, which is the left hand inequality in (5.1).

On the other hand (5.7) yields $[2/(K+1)]^{1/(1-\alpha)} > 3\,[2K/(K+1)]^{1/3(1-\alpha)}$, since $0 < x < 1$ implies $3x^{1/3} < 2 + x$. This gives the second inequality (5.1) and completes the proof of Theorem 6.

The equation (5.4) follows from a more detailed investigation of (5.7) when K is large.

We also note that $\log(t+1) = \log\left\{\frac{1}{2}(K+1)\right\}/(\alpha-1)$. Here the left hand side increases strictly from $\log(1+\sqrt{2})$ to $\log 3$ as K increases from 1 to ∞.

5.1. We sharpen (5.3) for large K.

Theorem 7. *We have*

$$\log 4 \leqslant \liminf_{K\to\infty} \frac{c_1(K)\log\log K}{K} \leqslant \limsup_{K\to\infty} \frac{c_1(K)\log\log K}{K} \leqslant \log 9. \tag{5.8}$$

We have seen that

$$\sup_{1\leqslant x\leqslant 3} M_0(x)x^{-\alpha} \leqslant c_1(K) \leqslant \sup_{\frac{1}{2}\leqslant x\leqslant 3} M_0(x)x^{-\alpha}, \tag{5.9}$$

where $\alpha = \alpha_1(K)$. We choose $\log x = (\log K)^{-p}$, where $p > 1$, and take $y > 1$ so that $(x-1)(y-1) = 4$. Then $y - 1 \sim 4(\log K)^p$ as $K \to \infty$. Now Theorem 5 yields $\log m_0(y) \leqslant \leqslant \alpha_2 \log y \leqslant (1+o(1))\, p\, \alpha_2 \log\log K < (1+o(1))\, p \left\{2/(K\log 2)\right\} \log\log K$. Thus by Theorem 2, $M_0(x) - 1 = 4/(m_0(y)-1) > (1+o(1))(\log 4)K/(p\log\log K)$.

On the other hand, $\log(x^{\alpha}) = \alpha\log x = O(\log K)^{1-p} = o(1)$, since $p > 1$. Thus for this particular x, $M_0(x)x^{-\alpha} > (1+o(1))(\log 4)K/(p\log\log K)$. Since p was arbitrary, we obtain the left hand inequality in (5.8).

Next if $0 < p < 1$, $0 \leqslant \log x \leqslant (\log K)^{-p}$, and if $(x-1)(y-1) = 4$, then $y \geqslant (4 + o(1))(\log K)^p$. Thus (4.12) gives $\log m_0(y) \geqslant (1+o(1))\log(1+2/K)\log y/\log 3 \geqslant$

$\geqslant (1 + o(1))\, 2\, p\, (\log \log K) / (K \log 3)$. Thus $M_0(x) - 1 = 4/(m_0(y) - 1) \leqslant \leqslant (1 + o(1))\, 2 K (\log 3) / (p \log \log K)$, so that

$$M_0(x)\, x^{-\alpha} \leqslant (1 + o(1))\, 2 K (\log 3) / (p \log \log K)\,. \tag{5.10}$$

On the other hand, if $(\log K)^{-p} \leqslant \log x \leqslant \log 3$, then by (5.5), $M_0(x) \leqslant 2K + 1$, and by (5.1) $\log x^{\alpha} = \alpha \log x \geqslant (1 + o(1)) \log K \log x / \log 3 > (\log K)^{1-p/2}$ and $x^{\alpha} > \log K$ if K is large. Therefore $M_0(x)\, x^{-\alpha} = O(K)/\log K$. Similarly if $\frac{1}{2} \leqslant x \leqslant 1$, $M_0(x)\, x^{-\alpha} \leqslant \leqslant 2^{\alpha} = O(K^{\log 2/\log 3})$, by (5.6). Hence (5.9) and (5.10) give

$$c_1(K) \leqslant (1 + o(1))\, 2 K (\log 3) / (p \log \log K).$$

Since p was arbitrary, $0 < p < 1$, we obtain the right hand inequality (5.8), and Theorem 7 is proved.

5.2. When x is near to one, $M_0(x)$ grows faster than x^{α_1}. The following bounds come from Theorems 2, 5 and 6 by a straightforward calculation.

Theorem 8. *If* $1 < x \leqslant 3$, *then*

$$\frac{2}{3} \leqslant \frac{4}{(x+3)^{\alpha_2} - (x-1)^{\alpha_2}} \leqslant \frac{M_0(x) - 1}{(x-1)^{\alpha_2}} \tag{5.11}$$

and

$$\frac{4}{c_1 6^{\alpha_1}} \leqslant \frac{4}{c_1 (x+3)^{\alpha_1} - (x-1)^{\alpha_1}} \leqslant \frac{m_0(x) - 1}{(x-1)^{\alpha_1}} \leqslant \leqslant \frac{4}{(x+3)^{\alpha_1} - (x-1)^{\alpha_1}} \leqslant 4^{1-\alpha_1}, \tag{5.12}$$

where $\alpha_i = \alpha_i(K)$, $c_i = c_i(K)$, $i = 1, 2$. *Moreover, if* $1 < x \leqslant 1 + 4.9^{-K}$ *then*

$$\frac{M_0(x) - 1}{(x-1)^{\alpha_2}} \leqslant \frac{4}{c_2 (x+3)^{\alpha_2} - (x-1)^{\alpha_2}} \leqslant \frac{4^{1-\alpha_2}}{(1 + 9^{-K})^{\alpha_2} / 9 - 9^{-K\alpha_2}}\,. \tag{5.13}$$

We prove (5.13). Suppose that $1 < x \leqslant 1 + 4.9^{-K}$, and define y by $(x-1)(y-1) = 4$. Then $y \geqslant 1 + 9^{K} > 9^{K}$. By (4.13), we have $K_{\alpha_2} \log 3 \geqslant K \log (1 + 2/K)$, and since $K \log (1 + 2/K)$ increases from $\log 3$ to 2 as K increases from 1 to ∞, we have $K\alpha_2 \geqslant 1$ and $y^{\alpha_2} > 9$. Thus by theorem 5, $c_2\, y^{\alpha_2} > 1$.

Applying Theorems 2 and 5, we obtain

$$0 < c_2\, y^{\alpha_2} - 1 \leqslant m_0(y) - 1 = 4\,(M_0(x) - 1)^{-1}\,.$$

Since $(x-1)\, y = x + 3$, we have further

$$(c_2 y^{\alpha_2} - 1)(x-1)^{\alpha_2} = c_2(x+3)^{\alpha_2} - (x-1)^{\alpha_2},$$

which gives the left hand inequality (5.13). Since $c_2(x+3)^{\alpha_2} - (x-1)^{\alpha_2}$ is a decreasing function of x for $1 \leq x \leq 1 + 4.9^{-K}$, we obtain the right hand inequality (5.13).

The proof of (5.11) and (5.12) is similar. Theorem 8 is proved.

6. Inverse functions, Hölder-continuity and Hausdorff dimensions. We obtain an estimate for $q(f^{-1})$ in terms of $q(f) = K$. We write $\mu(K)$ for the solution of $m(y, K) = 2$. Then $\mu(K)$ is a strictly increasing function of K which maps $[1, \infty)$ onto $[2, \infty)$. Thus μ^{-1} is strictly increasing and maps $[2, \infty)$ onto $[1, \infty)$.

Theorem 9. *If f is qs, then*

$$\mu^{-1}(q(f)+1) \leq q(f^{-1}) \leq \mu(q(f)) - 1, \tag{6.1}$$

which is sharp, and

$$\log_2(1 + q(f)^{-1}[\log_2(1 + q(f^{-1})) - 1] \leq 1. \tag{6.1}$$

We have to estimate

$$\frac{f^{-1}(x+t) - f^{-1}(x)}{f^{-1}(x) - f^{-1}(x-t)} \equiv R.$$

We can assume that $x = t = 1$, that $f^{-1}(0) = 0, f^{-1}(1) = 1$ (so that $f \in N(K)$), and that $R \geq 1$. Then $R + 1 = y$, where $f(y) = 2$. Thus $m(y, K) \leq 2$, which gives the right hand inequality (6.1). Equality is possible, since $N(K)$ is compact so that $f(y) = m(y, K)$ is attained for some $f \in N(K)$ and for $y = \mu(K)$. Interchanging f and f^{-1} we obtain the left hand inequality (6.1), which is likewise sharp.

To prove (6.2) we note that by [3], $y^\beta \leq 2^{\beta+1}$ where $\beta = \log_2(1 + q(f)^{-1})$, or $\log_2(1 + q(f)^{-1}) \log_2 y \leq 1 + \log_2(1 + q(f)^{-1})$. This gives (6.2) and Theorem 9 is proved.

6.1. We also determine the best possible exponent of Hölder-continuity.

Theorem 10. *Assume that f is K-qs, $x_1 < x_3$, and write $\alpha_i = \alpha_i(K)$, $i = 1, 2$, $M = f(x_3) - f(x_1)$. Then for $x_1 \leq x_2 \leq x_3$,*

$$(c_1(K)2^{\alpha_1})^{-1}\left(\frac{x_2 - x_1}{x_3 - x_1}\right)^{\alpha_1(K)} \leq [f(x_2) - f(x_1)]M^{-1} \leq (2/c_2(K))\left(\frac{x_2 - x_1}{x_3 - x_1}\right)^{\alpha_2(K)}. \tag{6.3}$$

The exponents $\alpha_1(K)$ and $\alpha_2(K)$ are best possible.

We define $g \in N(K)$ by

$$g(x) = \frac{f(x(x_3 - x_1) + x_1) - f(x_1)}{f(x_3) - f(x_1)}. \tag{6.4}$$

Assume that $x_1 < x_2 \leq x_3$, and write $y = (x_2 - x_1)(x_3 - x_1)^{-1}$. Then by Theorems 2, 5 and 6, and by (2.4) to (2.7) we have

$$g(y) \leq M(y, K) = 1/m(y^{-1}, K) \leq (2/c_2(K))\, y^{\alpha_2(K)}$$

and

$$1/g(y) \leq 1/m(y, K) = M(y^{-1}, K) = \tfrac{1}{2}[M_0(2/y-1)+1] \leq M_0(2/y) \leq c_1 2^{\alpha_1} y^{-\alpha_1}.$$

This gives (6.3), in view of (6.4).

The exponents $\alpha_i(K)$, $i = 1, 2$, are best possible, since the functions

$$f_i(x) = |x|^{\alpha_i(K)} \operatorname{sign} x, \quad i = 1, 2,$$

are K-qs. This proves Theorem 10.

6.2. Proof of Theorem 3. In view of (1.3), Theorem 2, and the remark after it, it suffices to show that $M(x, K)$ is continuous for $x \geq 1$. If $1 \leq x < y$, let $f \in N(K)$ be such that $f(y) = M(y, K)$. Then we deduce as in the proof of Theorem 10 that

$$0 < M(y, K) - M(x, K) \leq f(y) - f(x) \leq A\,(y-x)^{\alpha_2(K)}$$

for some constant A which remains finite as $y - x \to 0$. This proves Theorem 3.

In fact, $M(x, K)$ and the related functions are locally Hölder-continuous with the exponent $\alpha_2(K)$, as the preceding proof shows.

Remark. Let f be a sense-preserving homeomorphism of the unit circle Γ onto itself and assume that there is a point $w_0 \in \Gamma$ such that if $w_i \in \Gamma$, $1 \leq i \leq 3$, and

$$S(w_0, w_1, w_2, w_3) = \frac{w_1 - w_0}{w_1 - w_2}\,\frac{w_3 - w_2}{w_3 - w_0} = -1$$

then

$$K^{-1} \leq |S(f(w_0), f(w_1), f(w_2), f(w_3))| \leq K.$$

By using Möbius transformations together with Theorems 5, 6 and 10, one can show that f is locally Hölder continuous with the exponent $\alpha_2(K)$, also in the neighbourhood of w_0, or more precisely,

$$A_1 |w_1 - w_2|^{\alpha_1(K)} \leq |f(w_1) - f(w_2)| \leq A_2 |w_1 - w_2|^{\alpha_2(K)}$$

for some constants A_1 and A_2, if $|w_1 - w_2|$ is small enough.

6.3. We derive the following crude bounds for the change of $\dim_H A$, the Hausdorff dimension of the set $A \subset R$.

Theorem 11. *If f is K-qs and $A \subset R$, with $\dim_H A = a$, then*

$$a/\alpha_1(K) \leq \dim_H f(A) \leq a/\alpha_2(K). \tag{6.5}$$

Without loss of generality, we may assume that $f \in N(K)$ and $A \subset [0, 1]$. Then as in the proof of Theorem 10 we deduce that

$$A_1(x_2 - x_1)^{\alpha_1} \leq f(x_2) - f(x_1) \leq A_2(x_2 - x_1)^{\alpha_2} \tag{6.6}$$

for $0 \leq x_1 \leq x_2 \leq 1$ and for some constants A_1, A_2.

To prove the right hand inequality (6.5), we can suppose that $a < 1$. Pick b, $a < b \leq 1$, and ϵ, $\delta > 0$. There are intervals $[x_i, y_i] \subset [0, 1]$ covering A such that $y_i - y_i < \delta$ and $\Sigma\,(y_i - x_i)^b < \epsilon$. Thus $\Sigma\,(\,f(y_i) - f(x_i))^{b/\alpha_2} \leq A_2{}^{b/\alpha_2}\ \Sigma\,(y_i - x_i)^b < \epsilon A_2{}^{b/\alpha_2}$, and $f(y_i) - f(x_i) < A_2\delta^{\alpha_2}$ for all i. Thus $\dim_H f(A) \leq b/\alpha_2$. Since b was arbitrary, $a < b \leq 1$, we obtain the right hand inequality (6.5).

The proof of the left hand inequality (6.5), using the left hand inequality (6.6), is similar. This proves Theorem 11.

The functions $h(r) = (\log r^{-1})^{-\eta}$, $\eta > 0$ are such that if M_h is the Hausdorff measure associated with h, then $M_h\,(\,f(A)) = 0$ whenever f is qs and $M_h\,(A) = 0$. More generally, if h is continuous and non-decreasing with $h(0) = 0$, and if for all α, $0 < \alpha < 1$, there are positive numbers r_α and C_α such that $h(r^\alpha) \leq C_\alpha h(r)$, $0 < r < r_\alpha$, then $M_h\,(\,f(A)) = 0$ if f is qs and $M_h\,(A) = 0$.

One can ask whether or not there is a qs function f and a set A with $\dim_H A < 1$ such that $\dim_H f(A) = 1$.

7. Examples. Now we give some examples to illustrate the behaviour of $M_0(x)$ and $m_0^0(x)$.

Let f be a strictly increasing continuous piecewise linear function on $\mathbb{R}$. There are points x_i, $-\infty \leq N_1 \leq i \leq N_2 \leq \infty$, such that $x_i < x_{i+1}$ and f is linear on $[x_i, x_{i+1}]$. We say that the x_i are critical points. If $N_1 > -\infty$ ($N_2 < \infty$), we also count $x = -\infty$ ($x = \infty$) as a critical point. Verifying that the critical point $x = \pm\infty$ has a certain property means checking what happens when $x \to \pm\infty$.

In our examples, the qs functions are piecewise linear. We will leave it to the reader to verify that a given function is indeed K-qs. However, to make this as easy as possible, we give the following result.

Lemma 4. *Let f and the x_i be as above. If*

$$\frac{1}{K} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \equiv R \leq K \tag{7.1}$$

when at least two of the points $x - t$, x, $x + t$ ($t > 0$) are critical, then (7.1) *remains valid for all x and t. Moreover, if the slopes of f are increasing between x_i and x_j, and if the right hand inequality* (7.1) *holds whenever $x - t = x_i$, $x + t \leq x_j$ and x or $x + t$ is critical, or $x + t = x_j$, $x - t \geq x_i$ and x or $x - t$ is critical, or $2x = x_k + x_\ell$, $i \leq k$, $\ell \leq j$ and also $x + t = x_j$ or $x - t = x_i$, then* (7.1) *remains valid for all x and t such that $x_i \leq x - t < < x + t \leq x_j$.*

Note that in the situation of increasing slopes, the left hand inequality (7.1) holds whenever $K \geq 1$ and $x_i \leq x - t < x + t \leq x_j$.

Assume that x is fixed, that $a \leq t \leq a + \delta$ and that for some m and n $x_n \leq x - a - \delta <$

$< x - a \leqslant x_{n+1}$ and $x_m \leqslant x + a < x + a + \delta \leqslant x_{m+1}$. If necessary, we make a smaller and δ larger so that both for $t = a$ and for $t = a + \delta$, at least one of the points $x + t$, $x - t$ is critical.

We write $u = t - a$. Then $0 \leqslant u \leqslant \delta$, and there are slopes s_1, s_2 ($s_i > 0$) such that $f(x+t) = f(x+a) + us_1$, $f(x-t) = f(x-a) - us_2$ for $0 \leqslant u \leqslant \delta$. Thus

$$R = \frac{s_1}{s_2} + \frac{f(x+a) - f(x) - (s_1/s_2)(f(x) - f(x-a))}{f(x) - f(x-a) + us_2}.$$

For $0 \leqslant u \leqslant \delta$, we have $f(x) - f(x-a) + us_2 = f(x) - f(x-t) > 0$. Thus R is constant or a strictly monotonic function of u, so that to find the maximum and minimum of R, it is in any case sufficient to consider only the cases $u = 0$ and $u = \delta$. But then $x + t$ or $x - t$ is critical. Similarly we deduce that one can assume that another point of x, $x \pm t$ is also critical.

The second statement of the lemma can be derived from the following fact. If the slopes are increasing as said, and if x is fixed, $x_i < x < x_j$, then $(f(x+t) - f(x))/t$ is increasing and $(f(x) - f(x-t))/t$ is decreasing as a function of t for $t > 0$ as long as $x_i \leqslant x - t$ and $x + t \leqslant x_j$. This proves Lemma 4.

7.1. Our next result shows which power of K the function $M_0(x, K)$ resembles for a fixed x as $K \to \infty$.

Theorem 12. *Assume that $n \geqslant 0$ and that $3^n < x \leqslant 3^{n+1}$. Then*

$$\lim_{K \to \infty} \frac{\log M_0(x, K)}{\log K} = n + 1. \tag{7.2}$$

By (5.5), $M_0(x, K) \leqslant (2K+1)^{n+1}$, so that

$$\limsup_{K \to \infty} \frac{\log M_0(x, K)}{\log K} \leqslant n + 1. \tag{7.3}$$

To get the opposite inequality, we define f by

$$f(-x) = -f(x),$$

$$f(x) = x, \; 0 \leqslant x \leqslant 1,$$

$$f(x) = (K+1)^n + \tfrac{1}{2} K \, [(K+1)/3]^n \, (x - 3^n), \; 3^n \leqslant x \leqslant 3^{n+1}, \; n > 0.$$

If $K \geqslant 3$, then $f \in N_0(K)$. Thus if $3^n < x \leqslant 3^{n+1}$, then $M_0(x, K) \geqslant f(x) \geqslant \tfrac{1}{2} K^{n+1} 3^{-n} (x - 3^n)$, so that

$$\liminf_{K \to \infty} \frac{\log M_0(x, K)}{\log K} \geqslant n + 1. \tag{7.4}$$

Now (7.2) follows from (7.3) and (7.4) and Theorem 12 is proved.

7.2. Let K be fixed, $K > 1$. The functions $M(x, K)$ and $m(x, K)$ are strictly increasing, and by Theorem 3, they are continuous. Thus they are differentiable at almost every point, even though we do not know any such point. However, we can show that these functions are not differentiable at certain points.

Theorem 13. *We have* $M(2,K) = K+1$, $M(4,K) = (K+1)^2$, $M(5,K) = 2K^2 + 2K + 1$, $M(9,K) = K^3 + 4K^2 + 3K + 1$, *and* $m(2,K) = 1 + 1/K$, $m(4,K) = (1+1/K)^2$. *If* $K > 1$, *then* $M(x,K)$ *is not differentiable at* $x = 2, 4, 5$ *and* $m(x,K)$ *is not differentiable at* $x = 5/4, 4/3, 2$.

By a result of Kelingos [3], we have $M(2^n, K) \leqslant (K+1)^n$ and $m(2^n, K) \geqslant (1+1/K)^n$. By (5.5), $M_0(9,K) \leqslant (2K+1)^2$, so that (2.4) shows that $M(5,K) \leqslant 2K^2 + 2K + 1$. Moreover, $M(9,K) \leqslant M(4,K) + K[M(4,K) - m(-1,K)] \leqslant K^3 + 4K^2 + 3K + 1$.

To get the opposite inequalities, we define f to be the piecewise linear continuous function with $f(0) = 0$ and with the following slopes:

K^2, $-4 \leqslant x \leqslant -3$ or $4 \leqslant x \leqslant 5$,

$K(K+1)/2$, $-3 \leqslant x \leqslant -1$ or $2 \leqslant x \leqslant 4$,

K, $-1 \leqslant x \leqslant 0$ or $1 \leqslant x \leqslant 2$,

1, $0 \leqslant x \leqslant 1$,

$K(K+1)^2/4$, $x \geqslant 5$ or $x \leqslant -4$.

Then $f \in N(K)$, $M(2,K) \geqslant f(2) = K+1$, $M(4,K) \geqslant f(4) = (K+1)^2$, $M(5,K) \geqslant f(5) = 2K^2 + 2K + 1$, $M(9,K) \geqslant f(9) = K^3 + 4K^2 + 3K + 1$.

If $M(x,K)$ is differentiable at $x = 2$, with right hand and left hand derivatives R and L, then $M(x,K) \geqslant f(x)$ shows that $K(K+1)/2 \leqslant R$, $L \leqslant K$ which is impossible since $K > 1$. Similarly we deduce that $M(x,K)$ is not differentiable at $x = 4$ and $x = 5$. Now it follows from Theorem 2 that $m(2,K) = 1 + 1/K$ and that $m(x,K)$ is not differentiable at $x = 5/4, 4/3$ or 2, since $(5/4)^{-1} + 5^{-1} = 1 = (4/3)^{-1} + 4^{-1} = 2^{-1} + 2^{-1}$.

The desired lower bound for $m(4,K)$ is obtained by considering the piecewise linear continuous functions g with $g(0) = 0$ and with the slopes

1, $0 \leqslant x \leqslant 1$

$1/K$, $x \leqslant 0$ or $1 \leqslant x \leqslant 2$

$(K+1)/(2K^2)$, $x \geqslant 2$,

since we have $g \in N(K)$. Theorem 13 is proved.

In view of Theorem 13, it might be interesting to know whether or not $M(x,K)$ (or $m(x,K)$) is completely singular, and what the set A looks like where $M(x,K)$ is not differentiable. For instance, does A contain all rational numbers?

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STRESZCZENIE

Otrzymano szereg twierdzeń o zniekształceniu dla funkcji kwazisymetrycznych i odpowiednio unormowanych.

РЕЗЮМЕ

Получено ряд теорем о искажении для квазисиметрических функций, надлежащим образом нормированных.