ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA

LUBLIN – POLONIA

VOL. XXXVI/XXXVII, 3 SECTIO A 1982/1983

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Criteria for Local Variations for Slit Mappings

Kryteria dla wariacji lokalnych odwzorowań na obszary bez punktów zewnętrznych

Условия для локальных вариаций отображения на области без внешних точек

Let S be the family of all analytic functions f on the unit disc $U = \{z: |z| < 1\}$ normalized by $f(0) = 1 - f'(0) = 0$. Consider a domain $f(U)$, for f in S, having a piecewise smooth boundary which contains piecewise analytic slits. In short note we obtain criteria for this type of domain to be locally varied on its boundary to produce a domain of larger mapping radius. We also apply these criteria to domains bounded by certain 'spiralled' slits.

We shall apply the Julia variational formula as formulated (based on J. Krzyż's work) and described in detail by the author in his Transactions AMS paper [2]. Let f be in S with $D = f(U)$ and ∂D be piecewise smooth and contain an analytic slit L. Let f also designate the extended function to $\overline{U}$ where appropriate. We obtain a varied function $\mathring{f}$ back in S by making a sufficiently small local perturbation of the slit L. For ϵ sufficiently small, let $\epsilon\phi(w)$ be a twice continuously differentiable variation (zero on $\partial D \backslash L$) in the direction of the unit normal, $n(w)$, of the point $w = f(z)$ for $z = e^{i\theta}$, θ real. The change in mapping radius, denoted by Δm.r., is given up to $o(\epsilon)$ terms by (continuity arguments enable us to drop the $o(\epsilon)$ terms in the remaining argument):

$$\mathring{f}'(0) - f'(0) = \frac{\epsilon}{2\pi i} \int_{\partial D} \frac{\phi(w)\, n(w)\, dw}{[zf'(z)]^2}$$

In this manner we can vary the slit L locally by moving the 'sides' of the slit an equal amount but in opposite directions along their respective interiors and exterior normals to produce another slit L'. Let ϕ be 0 on the ∂D except on an arc $\widehat{w_0 w_2}$ of L were $w_0 = f(z_0) = f(\zeta_0)$ with corresponding preimages ℓ_1, ℓ_2 on ∂U, i.e. let $f(\ell_1) = \widehat{w_0 w_2}$, $f(\ell_2) = \widehat{w_2' w_0}$ with z in ℓ_1 and ζ in ℓ_2. Then the Δm.r. is given by

$$\frac{\epsilon}{2\pi i}\left\{\int_{w_0}^{w_2}\frac{\phi(w)\,n(w)\,dw}{[zf'(z)]^2}-\int_{w_2}^{w_0}\frac{\phi(w)\,n(w)\,dw}{[\zeta f'(\zeta)]^2}\right\}=$$

$$=\frac{\epsilon}{2\pi}(w_2-w_0)\left\{\frac{1}{w_2-w_0}\int_{w_0}^{w_2}\left[\frac{n[f(z)]}{i[zf'(z)]^2}+\frac{n[f(\zeta)]}{i[\zeta f'(\zeta)]^2}\right]\right\}dw$$

We note that $n[f(z)] = zf'(z)/|zf'(z)|$ and that $\arg \zeta_0 f'(\zeta_0) = \arg z_0 f'(z_0) - \pi$.

Letting $z \to z_0$ and $\zeta \to \zeta_0$ along ℓ_1 and ℓ_2 resp., we obtain as $w \to w_0$ between w_2 and w_0 along L,

$$\lim_{w\to w_0}\frac{\epsilon}{2\pi}|w-w_0|e^{i\arg(w-w_0)}\cdot\left\{\phi(w_0)\left[\frac{1}{i[z_0f'(z_0)]\,|z_0f'(z_0)|}+\right.\right.$$

$$\left.\left.+\frac{1}{i[\zeta_0f'(\zeta_0)]\,|\zeta_0f'(\zeta_0)|}\right]\right\}=\frac{\epsilon}{2\pi}\lim_{w\to w_0}|w_1-w_0|e^{i\arg[iz_0f'(z_0)]}(-i)\phi(w)\cdot$$

$$\cdot\left[\frac{e^{-i\arg z_0f'(z_0)}}{|z_0f'(z_0)|^2}+\frac{e^{i[\pi-\arg z_0f'(z_0)]}}{|\zeta_0f'(\zeta_0)|^2}\right]=$$

$$=\frac{\epsilon}{2\pi}\lim_{w\to w_0}\left|\frac{w-w_0}{z-z_0}\right|\lim_{z\to z_0}|z-z_0|e^{i\arg z_0f'(z_0)}\phi(w)\cdot$$

$$\cdot\left[\frac{1}{|z_0f'(z_0)|^2}-\frac{1}{|\zeta_0f'(\zeta_0)|^2}\right]=$$

$$=\frac{\epsilon}{2\pi}\lim_{z\to z_0}|z-z_0|\,\phi(w)\,[1-|z_0f'(z_0)|^2/|\zeta_0f'(\zeta_0)|^2]/|z_0f'(z_0)| \qquad (*)$$

Let $w_0 = f(z_0) = f(\zeta_0)$ be a point on a slit L were $f'(z_0)$ and $f'(\zeta_0)$ exist (finitely) and are nonzero. It follows from (*) and a standard continuity argument that if $|f'(z_0)| \neq |f'(\zeta_0)|$ then a sufficiently short arc $\widehat{w_1w_0}$ of L can be chosen such that with an appropriate $\phi(w)$ the mapping radius can always be increased. We have shown:

Theorem. *A domain $f(U)$, $f \in S$, with a piecewise smooth boundary having an analytic slit which contains a point where the opposing normal exist and are of unequal moduli can always be varied locally on the boundary to produce a domain of strictly larger mapping radius.*

Using this result as a basis we shall be able to conclude that any domain as described above having a piecewise analytic slit in this boundary that contains corners with one of the angle openings equal to $\alpha\pi$, $1 < \alpha \leqslant 2$, or two angle openings $\beta\pi$ and $\beta'\pi$ where $0 < \beta < \beta' \leqslant 1$ can be varied as above. (The case $\alpha = 2$ does not include the tip of a slit, i.e. requires the point where the corner occurs to have at least two distinct preimages on $|z| = 1$.) We claim a corner of opening $\alpha\pi$, $1 < \alpha \leqslant 2$ will have a neighborhood on the slit containing points where the opposing normals have unequal moduli. We can conclude this from the following Lemma modyfying a result of Tsuji's.

Lemma. *Let G be a region on the w plane with boundary curve C passing through $w = 1$. The part of C which lies in a small neighborhood of $w = 1$ is divided by $w = 1$ into two parts C_1 and C_2, which we assume are analytic Jordan arcs making an interior angle $\alpha\pi$, $(0 < \alpha \leqslant 2)$ at $w = 1$. We let $w = w(z)$ conformally map U onto G such that $w(0) = 0$ and $w(1) = 1$. Let $\Delta = \left\{z\colon 1 < |z - 1| \leqslant \rho, |z| \leqslant 1\right\}$ be a 'half neighborhood' of 1. Then, if $\rho > 0$ is sufficiently small*

$$A\,|z-1|^{\alpha-1} \leqslant \left|\frac{dw}{dz}\right| \leqslant B\,|z-1|^{\alpha-1},\ z \in \Delta,$$

where $A > 0$, $B > 0$ are constants.

Proof. The result follows directly from Tsuji's result [5, pp. 365] by a linear transformation taking the upper half plane into U and by noting that the argument is a local argument hence the sufficiency of piecewise analyticity in a neighborhood of $w = 1$. Also the map $w^{1/\alpha}$ when $\alpha = 2$ satisfies the required properties for the proof to be valid in this case.

To apply this result we consider the corners as separate superimposed boundary arcs. Also the analytic extensions across these boundary arcs are considered as lying on their superimposed, but distinct, Riemann surfaces with their analytic properties being described by the limiting behaviour from inside the domain $f(U)$. A given point may have any finite integer greater than one of corners with their appropriate edges superimposed. It is clear from the geometry of the corners that if there existsa corner at w_0 of opening $\alpha\pi$, $1 < \alpha \leqslant 2$ then the other one or more corners at w_0 will have opening $\beta\pi, 0 \leqslant \beta < 1$. From the Lemma the derivative will approach 0 from inside and along the edges of the corner at w_0 of opening $\alpha\pi$, $1 < \alpha \leqslant 2$ while the derivative will approarch ∞ from inside and along the edges of the other corners at w_0 of opening $\beta\pi, 0 < \beta < 1$. The case when $\beta = 0$ does not follow directly from the Lemma. The case can occur in two ways. Either with an adjacent corner of opening $\beta'\pi$, $0 < \beta' < 1$ which can be handled as above, or the original corner has opening $\alpha\pi$, $\alpha = 2$ where $f'(z) \to 0$ along the boundary. In this case, of an interior cusp of opening 0π, the guaranteed locally univalent extensions (on its distinct Riemann surface) in neighborhood of the analytic edges would contradict the property of any analytic extension across an analytic arc where $f'(\zeta) \to 0$ must be locally nonunivalent (i.e., the existence of an asymptotic value (approached along on analytic arc) of zero for the derivative assures nonconformality).

Our claim as to the existence of points in a neighborhood along the slit of a corner of opening $\alpha\pi$, $1 < \alpha \leqslant 2$ where the opposing normals have unequal moduli will then follow by continuity. In the case when there exist corners at w_0 with angle openings $\beta\pi$ and $\beta'\pi$, $0 < \beta < \beta' \leqslant 1$ the lemma may again be applied (noting the different rates the derivatives would have to change) along with continuity to obtain opposing normals with unequal moduli near w_0.

We observe that for a function f in S is a boundary slit $L = f(\ell_1) = f(\ell_2)$, for ℓ_1, ℓ_2 subarcs of ∂U, has the property that $|f'(z)| = |f'(\zeta)|$ whenever $f(z) = f(\zeta) \in L$, then the lenght, $|\ell_1|$ of ℓ_1 equals the length, $|\ell_2|$ of ℓ_2. This follows since

$$|\ell_1| = \int_{\ell_1} |dz| = \int_{\ell_1} \left| \frac{df^{-1}(w)}{dw} \right| |dw(z)| = \int_{\ell_1} \frac{|dw(z)|}{|f'(z)|} = \int_{\ell_2} \frac{|dw(\zeta)|}{|f'(\zeta)|} =$$

$$= \int_{\ell_2} |d\zeta| = |\ell_2|.$$

We then observe from V. R. Kü hnau's [3] extension of Löwner's Lemma that if a slit L is the only boundary of a domain $f(U)$ for f in S and has the property that the two 'sides', say L_1 and L_2, do not change curvature (i.e. from positive to negative) then their preomages $\ell_1 = f^{-1}(L_1)$ and $\ell_2 = f^{-1}(L_2)$ are of equal length iff L is a geodesic. In our case L must be a half line pointing of the origin. We note that the only half line for f in S with the above property is a radial line. Although this is clear from the geometry of the sphere we also note that the extended mapping function f in S determining a straight line $L = f(\ell_1) = f(\ell_2)$ is given by (up to rotation) $f(z) = z\,[1-(xz)]/(1-z)^2$ where $x = (1+e^{\theta})/2$, $0 \leq \theta < 2\pi$. Since $f'(z) = [1+(1-2x)\,z]/(1-z)^3$ it follows that the endpoints of ℓ_1 and ℓ_2 are 1 and $e^{-i\theta}$. So that $|\ell_1| = |\ell_2|$ iff $\theta = \pi$.

Consider a domain $f(U)$, for f in S, bounded by an analytic slit L with 'sides' that do not change curvature. It follows from our arguments that this type of domain can be locally varied in such a way as to increase the mapping radius unless L is a radial line. Although a similar and indeed more general result would follow from the properties of circular symmetrization the local nature of the observations in this paper will hopefully add to the generality of their applications.

We also note here that a domain may be bounded by an arbitrary number of radial slits so arranged as to have the opposing normals at every point on each slit be of equal moduli. A typical example would be the k-symmetric form $f(z) = z\,(1-z^k)^{-2/k}$, while a non-2-symmetric form would be $f(z) = z/(1 - 2z\cos t + z^2)$. However a standard Goluzin type variation or a variation on the jumps as was done in [4] can be used to control the size of the mapping radius. Another method by which this control can be obtained is a local variation using the Löwner differential equation representation for slit mappings to locally extend and contract the tips of slits as was done in [1].

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STRESZCZENIE

Otrzymuje się warunki jakie powinien spełniać obszar jednospójny o brzegu kawałkami gładkim i zawierającym kawałkami analityczne nacięcia, aby przez lokalną deformację brzegu można było zeń otrzymać obszar o większym konforemnym promieniu wewnętrznym.

РЕЗЮМЕ

Получаются условия которые должна выполнить односвязная область с кусочно гладкой границей, к которой принадлежат кусочно аналитические разрезы, чтобы локальной деформацией ее берега получить область, имеющая большой внутренний конформный радиус.