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On a Measure of Noncompactness in the Space of Continuous Functions

O pewnej mierze niezwartości w przestrzeni funkcji ciągłych

Abstract. In this note we propose a new definition of a measure of noncompactness in the space of continuous functions. Our measure $p(\cdot)$ is comparable with two classical ones; the Kuratowski measure $\alpha(\cdot)$ and a Hausdorff measure $\chi(\cdot)$.

1. Introduction. The measure of noncompactness α was introduced by K. Kuratowski in 1930 [4]. For any bounded set X in a metric space, $\alpha(X)$ is defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of sets of diameter smaller than r. Another the most commonly used measure $\chi(X)$ is named after Hausdorff and defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of balls of radii smaller than r. Obviously for any set we have

$$\chi(X) \le \alpha(X) \le 2\chi(X) .$$

The Hausdorff measure is often more convenient that Kuratowski measure since in many spaces there are formulae allowing to calculate or evaluate its values ([1], [2]) while the methods of evaluating values of Kuratowski measure are practically unknown.

Such situation can be illustrated in the spaces of continuous functions. Let $C = C([0,1], \mathbf{R})$ denotes the Banach space of continuous real valued functions defined on $[0,1]$ with the standard norm "supremum". For any bounded set $X \subset C$ we have [3], [2]

$$\chi(X) = \frac{1}{2}\,\omega_0(X)$$

where

$$\omega_0(X) = \lim_{h\to 0}\ \sup_{x\in X} \sup\{|x(t) - x(s)| : |t - s| \le h,\ t, s \in [0,1]\} .$$

Thus we have

$$\frac{1}{2}\,\omega_0(X) \le \alpha(X) \le \omega_0(X) .$$

This paper is an attempt to find a stronger evaluation of the measure α than the one above.

2. The definition of $p(X)$ and its properties. First we prove the following lemma:

Lemma . *Let X be a bounded set in the space $C([0,1],\mathbf{R})$. Then*

$$\alpha(X) \ge p(X)$$

where

$$p(X) = \sup_{t_0\in[0,1]} \limsup_{h\to 0} \sup_{x\in X} \sup\{|x(t)-x(t_0)| : |t-t_0| \le h,\ t\in[0,1]\}$$

Proof. Suppose that $X \subset \bigcup_{i=1}^{k} A_i$. Pick an $\varepsilon > 0$. From the definition of $p(X)$ we can choose $t_0 \in [0,1]$ and sequences $\{x_n\} \subset X$, $\{s_n\} \subset [0,1]$, $(n \in \mathbf{N})$ so that

$$|t_0 - s_n| \le \frac{1}{n} \text{ and } |x_n(t_0) - x_n(s_n)| \ge p(X) - \varepsilon .$$

Let $I \subset \mathbf{N}$ denotes such an infinite set that $x_n \in A_j$ for every $n \in I$, $j \in \{1,\dots,k\}$ is fixed (existing such A_j follows from the fact that a number of sets A_i is finite). It is enough to show that diam $A_j \ge p(X) - \varepsilon$. Consider the set $\{x_n(t_0) : n \in I\}$. It is bounded, so there exists an infinite set $J \subset I \subset \mathbf{N}$ and $n_0 \in J$ such that

$$|x_n(t_0) - x_m(t_0)| < \varepsilon \quad \text{for every} \quad n,m \ge n_0,\ n,m \in J .$$

Since the function x_{n_0} is continuous, there exists $\delta > 0$ such that

$$|x_{n_0}(t) - x_{n_0}(t_0)| < \varepsilon \quad \text{for} \quad |t - t_0| < \delta .$$

Take $n \in J$ so great that $|t_0 - s_n| \le \frac{1}{n} < \delta$. Thus we have

$$|x_n(t_0) - x_n(s_n)| \ge p(X) - \varepsilon \text{ and } |x_{n_0}(t_0) - x_{n_0}(s_n)| < \varepsilon .$$

Hence

$$|x_n(s_n) - x_{n_0}(s_n)| \ge |x_n(s_n) - x_n(t_0)| - |x_n(t_0) - x_{n_0}(t_0)| - |x_{n_0}(t_0) - x_{n_0}(s_n)|$$
$$\ge p(X) - 3\varepsilon .$$

Thus for every $\varepsilon > 0$ we can find such A_j that

$$\operatorname{diam} A_j \ge |x_n(s_n) - x_{n_0}(s_n)| \ge p(X) - 3\varepsilon .$$

Hence there exists such A_{j_0} that diam $A_{j_0} \ge p(X)$ so $\alpha(X) \ge p(X)$.

Proposition . *The function $p(\cdot)$ defined on the class of all bounded subsets of $C([0,1],\mathbf{R})$ is a regular measure of noncompactness (in the sense of definition contained in [2]) i.e. has the following properties hold:*

1. $p(X) = 0 \iff X$ *is compact*

2. $p(\overline{X}) = p(X)$
3. $X \subset Y \Longrightarrow p(X) \leq p(Y)$
4. $p(\text{conv } X) = p(X)$
5. $p(\lambda X + (1-\lambda)Y) \leq \lambda p(X) + (1-\lambda)p(Y)$ *for* $\lambda \in [0,1]$
6. *if* X_n *is bounded* $X_n = \overline{X}_n$ *and* $X_{n+1} \subset X_n$ *for* $n = 1, 2, \ldots$ *and if* $\lim_{n\to\infty} p(X_n) = 0$, *then* $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$
7. $p(X \cup Y) = \max\{p(X), p(Y)\}$
8. $p(\lambda X) = |\lambda| p(X)$
9. $p(X + Y) \leq p(X) + p(Y)$

Proof. It is easy to check that $\omega_0(X) \leq 2p(X)$. Thus we have

$$\frac{1}{2}\,\omega_0(X) \leq p(X) \leq \alpha(X)$$

and properties (1), (6) follows from the fact that ω_0 and α are regular measures. The proof of the other properties is standard.

3. Examples. In this section we illustrate differences among $p(X)$, $\alpha(X)$ and $\omega_0(X)$.

Example 1. Let $K = \{x \in C : \|x\| \leq 1\}$ denotes the unit ball in the space of continuous functions. We have $p(K) = 2$ and $\omega(K) = 2$ so immediately $\alpha(K) = 2$. (More general fact, that $\alpha(K) = 2$ in every infinitely dimensional Banach space E $\alpha(K) = 2$ in every infinitely dimensional Banach space E

Example 2. Let $0 < a < 1$ and

$$X_a = \{x \in C : a \leq x(t) \leq 1 \text{ for } 0 \leq t < \tfrac{1}{2},\ x(\tfrac{1}{2}) = a,$$
$$-1 \leq x(t) \leq a \text{ for } \tfrac{1}{2} < t \leq 1\}$$

We have $\omega_0(X_a) = 2$ and instantly $1 \leq \alpha(X_a) \leq \text{diam } X_a = 1 + a$. Using the measure p, we obtain $p(X_a) = 1 + a$ and $\alpha(X_a) = 1 + a$.

In these examples there is $\alpha(X) = p(X)$. But it is not true in general. Let us consider the following example.

Example 3. Let

$$X = \{x_n \in C : x(0) = 0,\ x(\tfrac{1}{n}) = 1,\ x(\tfrac{2}{n}) = -1\ ,$$
$$x(t) = -1 \ \text{ for } \ \frac{2}{n} < t \leq 1 \text{ and } x_n \text{ is linear besides}, n = 3, 4, \ldots\}$$

We have $p(X) = 1$ and $\omega_0(X) = 2$. We show that $\alpha(X) = 2$. Suppose that $X \subset \bigcup_{i=1}^{k} A_i$. There exists such A_j that $x_n \in A_j$ for every $n \in I$ and $I \subset \mathbf{N}$ is finite. It is enough to choose such $n, m \in I$ so that $\frac{1}{n} \geq \frac{2}{m}$. Then diam $A_j \geq |x_n(\frac{1}{n}) - x_m(\frac{1}{n})| = 2$.

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STRESZCZENIE

W pracy tej zdefiniowano nową miarę niezwartości $p(\cdot)$ w przestrzeni funkcji ciągłych. Jest ona porównywalna z dwoma klasycznymi miarami; miarą Kuratowskiego $\alpha(\cdot)$ i miarą Hausdorffa $\chi(\cdot)$.