ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA

VOL. XLV, 14 SECTIO A 1991

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On Starlike Functions Associated with Parabolic Regions

O funkcjach gwiaździstych związanych z obszarami ograniczonymi parabolą

Abstract. This paper continues the investigations of a class of starlike functions S_p given by the property that $zf'(z)/f(z)$ ranges over a parabolic region. We prove a convolution result for this class and we compute the Koebe constant. We also introduce a generalization of the class S_p and obtain some results for the generalized classes.

1. Introduction. In this paper we shall work within the class S of functions $f(z) = \sum_{k=1}^{\infty} a_k z^k$, analytic and univalent in the unit disk U, and normalized by $f(0) = f'(0) - 1 = 0$. We denote by S_α the class of functions $f \in S$ with the property

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \alpha\,, \quad z \in U\,,\ \ 0 \leq \alpha \leq 1\,. \tag{1.1}$$

This is the classical family of functions starlike of order α. In [4] we introduced a class of starlike functions called S_p given by the property

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \operatorname{Re} \frac{zf'(z)}{f(z)}\,, \quad z \in U\,. \tag{1.2}$$

In the same way as we can say that the functions with the property (1.1) are associated with a halfplane we could say that the functions satisfying (1.2) are associated with a parabolic region, since $|w - 1| = \operatorname{Re} w$ describes a parabola with vertex at $w = \frac{1}{2}$ and $\langle\frac{1}{2}, \infty\rangle$ as symmetry axis. (It is clear that $S_p \subset S_\alpha$ for $0 \leq \alpha \leq 1/2$ and that for $\alpha > 1/2$ the inclusion does not hold.) The class S_p is in a natural way related to the geometrical property *uniform convexity* as introduced by Goodman [1]. A function f is said to be uniformly convex ($\in UCV$) if the image of every circular arc γ contained in U, with center also in U, is convex. We could mention as a remark that in the case that γ is a complete circle within U, then $f(\gamma)$ is convex if f is a convex function in the classical sense ($\in K_0$) (Study [6] and Robertson [3]). So the concept of uniform convexity is a restriction only if γ is a *part* of a circle. The relation between UCV and S_p is given in the theorem below, where also an analytic characterization of UCV is stated.

Theorem A. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S$. Then*
(a) $f \in UCV$ *if and only if*

$$\operatorname{Re}\left\{1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right\} \geq 0\,, \quad (z,\zeta) \in U \times U\,. \tag{1.3}$$

(b) $f \in UCV \iff zf' \in S_p$.

Part (a) is proved in [1] and part (b) is proved in [4].

2. Further properties and generalizations of S_p. The class S_p was introduced in [4] where we found, among other results, a sharp upper bound for the modulus $|f(z)|$, $f \in S_p$ and also some bounds for the coefficients. Improved bounds for the coefficients were given by Ma and Minda [2]. Now we shall prove further results about S_p, and we shall also make a generalization of S_p and adjust some of the results from [4] to the generalized class.

First we prove a result which in particular shows that the class S_p is closed under convolution. This is an application of an important result from convolution theory. We state it in a special version which is sufficient for our purposes.

Lemma 2.1 (**[5, p.54–55]**). *Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ be in $S_{1/2}$. Denote by $f * g$ the Hadamard product $(f*g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k$. Then, for any function $F(z)$ analytic in U, we have for $z \in U$ that*

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \subset \overline{\mathrm{co}}(F(U))\,.$$

($\overline{\mathrm{co}}$ denotes the closed convex hull.)

Theorem 2.2. *Let $f \in S_{1/2}$, $g \in S_p$. Then $f * g \in S_p$.*

Proof. If $g \in S_p$, we have in particular $g \in S_{1/2}$. Assume $f \in S_{1/2}$. Let $zg'(z)/g(z)$ play the role of F in Lemma 2.1, and let $\Omega = \{w \mid |w-1| \leq \operatorname{Re} w\}$. Using the lemma we get for $z \in U$ that

$$\frac{z(f*g)'(z)}{(f*g)(z)} = \frac{f(z) * zg'(z)}{(f*g)(z)} = \frac{f(z) * g(z)\dfrac{zg'(z)}{g(z)}}{(f*g)(z)} \subset \overline{\mathrm{co}}\left(\frac{zg'(z)}{g(z)}\right)_{z\in U} \subset \Omega$$

since Ω is convex and $g \in S_p$. This proves that $f * g \in S_p$.

Let

$$P(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\,. \tag{2.1}$$

If $f \in S_p$ then $zf'(z)/f(z) \prec P(z)$ [4]. Let $k(z)$ be the function, analytic in U, specified by $k(0) = k'(0) - 1 = 0$ and $zk'(z)/k(z) = P(z)$. Ma and Minda [2] proved that this function is extremal for some problems in S_p. (The actually worked with the function $\tilde{k} \in UCV$ related to our k by $k = z\tilde{k}'$. This is of course equivalent due to Theorem A (b).) One of the results in [2] about k is that for $f \in S_p$ we have $f(z)/z \prec k(z)$ and as a consequence of this subordination follows a distortion theorem in UCV which becomes a growth theorem in S_p. We state the result from [2] as

Theorem B. *Assume $f \in S_p$ and $|z| = r < 1$. Then*

$$-k(-r) \leq |f(z)| \leq k(r) \tag{2.2}$$

with equality for $z \neq 0$ only if f is a rotation of k.

The new contribution that we now make, is that we are able to give the upper and lower bounds in (2.2) more explicitely.

Theorem 2.3. *Assume $f \in S_p$ and $|z| = r < 1$. Then*

$$-\frac{8}{\pi^2}\int_0^r \frac{1}{t}(\tan^{-1}\sqrt{t})^2\,dt \leq \log\left|\frac{f(z)}{z}\right| \leq \frac{2}{\pi^2}\int_0^r \frac{1}{t}\left(\log\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)^2 dt \tag{2.3}$$

with equality for $z \neq 0$ only if f is a rotation of k.

Proof. Let $\varphi(z) = zf'(z)/f(z)$ and let $P(z)$ be as in (2.1). Then

$$\log\frac{f(z)}{z} = \int_0^z (\varphi(\xi) - 1)\frac{d\xi}{\xi}$$

and with $z = re^{i\theta}$

$$\log\left|\frac{f(z)}{z}\right| = \int_0^r \mathrm{Re}(\varphi(te^{i\theta}) - 1)\frac{dt}{t}\ .$$

Since $\varphi \prec P$ and P maps $|z| = r$ to a convex curve, symmetric about the x-axis, it follows that

$$P(-t) \leq \mathrm{Re}\,\varphi(te^{i\theta}) \leq P(t)\ .$$

Now the right hand side of (2.3) follows immediately. To get the left hand side note that

$$\left(\log\frac{1+\sqrt{-t}}{1-\sqrt{-t}}\right)^2 = -4(\tan^{-1}\sqrt{t})^2\ .$$

The function k is continuous on $\overline{U}$ [2], so $k(1)$ and $k(-1)$ make sense. This means that the limit as $r \to 1$ in (2.3) exists. Doing that on the right hand side gives the upper bound on $|f(z)|$ which was proved in this way in [4]. Taking the limit of the left hand side we obtain a new result about S_p (covering theorem, Koebe constant).

For a given subclass $\mathcal{F}$ of S, denote by $\mathcal{K}(\mathcal{F})$ the radius of the largest disk contained in $\bigcap_{f\in\mathcal{F}} f(U)$. The number $\mathcal{K}(\mathcal{F})$ is called the Koebe constant in $\mathcal{F}$. It is e.g. well known that $\mathcal{K}(S) = \mathcal{K}(S_0) = 1/4$ and $\mathcal{K}(K_0) = 1/2$.

Starting from (2.3) we get

$$\lim_{r\to 1} -\frac{8}{\pi^2}\int_0^r \frac{1}{t}(\tan^{-1}\sqrt{t})^2 = \lim_{r\to 1} -\frac{4}{\pi^2}\int_0^{2\tan^{-1}\sqrt{r}} \frac{t^2}{\sin t}\,dt$$
$$= -\frac{4}{\pi^2}\int_0^{\pi/2} \frac{t^2}{\sin t}\,dt := \mathcal{I} .$$

This proves

Corollary 2.4.

$$\mathcal{K}(S_p) = -k(-1) = e^{\mathcal{I}} = 0.53399\ldots$$

(The value of $\mathcal{I}$ is found by numerical integration.)

One way to generalize the class S_p could be to introduce a parameter α and define classes $S_p(\alpha)$ by

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \text{Re}\,\frac{zf'(z)}{f(z)} - \alpha . \tag{2.4}$$

We see that (2.4) also defines a region bounded by a parabola. This parabola has its vertex at $w = (1+\alpha)/2$, and when α grows, the parabola becomes narrower until it degenerates for $\alpha = 1$. Our previous class S_p corresponds to $\alpha = 0$, and we see that we get starlike functions $(\text{Re}\, zf'(z))/f(z) \geq 0$ for all α down to $\alpha = -1$. Hence, the functions from $S_p(\alpha)$ are in particular univalent for $\alpha \geq -1$. And also, if we go below -1 with α then $S_p(\alpha)$ must contain non-univalent functions. That is because then the parabola will contain the origin, and for no $f \in S$ can $zf'(z)/f(z) = 0$, $z \in U$.

Hence we have

Theorem 2.5.

$$S_p(\alpha) \subset S_0 \quad \text{for} \quad -1 \leq \alpha < 1 ,$$
$$S_p(\alpha) \not\subset S \quad \text{for} \quad \alpha < -1 .$$

Now, let f and g be functions such that $f = zg'$. Rewriting (2.4) with zg' instead of f we get

$$\left|\frac{zg''(z)}{g'(z)}\right| \leq \text{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} - \alpha . \tag{2.5}$$

In [4] we applied the Minimum principle for harmonic functions to get the connection between (1.2) and (1.3) which is the statement in Theorem A (b). This can be carried out in the same way to see that (2.5) is equivalent to

$$\text{Re}\left\{1 + \frac{(1-\zeta)g''(z)}{g'(z)}\right\} \geq \alpha , \quad (z,\zeta) \in U \times U . \tag{2.6}$$

Let ζ be an arbitrary point in U, and let γ be a circular arc also in U, centered in ζ and with radius r. A point on γ can then be written $z = \zeta + re^{i\theta}$, $\theta \in \langle\theta_1, \theta_2\rangle$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$. Then (2.6) states that

$$\frac{d}{d\theta}\left(\arg\left\{\frac{d}{d\theta}\, g(\zeta + re^{i\theta})\right\}\right) \geq \alpha\,, \quad \theta \in \langle\theta_1, \theta_2\rangle\,. \tag{2.7}$$

This suggests an interpretation of (2.6) which in a natural way gives rise to a concept one could call *uniform convexity of order* α. If we denote by $UCV(\alpha)$ the functions satisfying (2.6), we find the following interesting observation, using Theorem 2.5 and Alexander's theorem ($f \in K_0 \iff zf' \in S_0$).

Theorem 2.6. *If* $-1 \leq \alpha < 1$, *then* $UCV(\alpha) \subset K_0$.

If (2.7) takes a value $\alpha < 0$ for some γ then $g(\gamma)$ is no longer convex, but the value of α in a sense measures how much the tangent of $g(\gamma)$ is allowed to turn back. However, if this α is not less than -1 Theorem 2.6 states that the corresponding function g will still map *complete circles* in U to convex curves.

In the case $\alpha = 0$, which is our former class S_p, the Carethéodory function mapping U onto the parabolic region and 0 to 1 is the function $P(z)$ in (2.1). For $\alpha \neq 0$ we get similarly

$$P_\alpha(z) = 1 + \frac{2(1-\alpha)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2. \tag{2.8}$$

Finally we mention that all the classes $S_p(\alpha)$, $-1 \leq \alpha < 1$, consist only of functions that are bounded in the unit disk. We proved this in the case $\alpha = 0$ in [4] (as mentioned after the proof of Theorem 2.3), and this proof can also be translated to the case $\alpha \neq 0$ without problems using P_α in (2.8) instead of P. The idea of the proof in [4] was to let $r \to 1$ in the right hand side of (2.3). In the same way we can get the Koebe constant in $S_p(\alpha)$. This will give

Theorem 2.7. *Assume* $-1 \leq \alpha < 1$.

(a) If $f \in S_p(\alpha)$ *then*

$$\left|\frac{f(z)}{z}\right| \leq \exp\left(\frac{14(1-\alpha)}{\pi^2}\,\zeta(3)\right)$$

for $|z| < 1$. *The bound is sharp. ($\zeta(t)$ is the Riemann zeta function.)*

(b) $\mathcal{K}(S_p(\alpha)) = e^{(1-\alpha)\mathcal{I}} = (0.53399\ldots)^{1-\alpha}$.

Note as an example that $\mathcal{K}(S_p(-1)) = 0.2852\ldots > \frac{1}{4} = \mathcal{K}(S_0)$ which fits nicely in with the inclusion in Theorem 2.5.

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STRESZCZENIE

W pracy tej kontynuowane są badania funkcji gwiaździstych f klasy S_p, dla których wyrażenie $zf'(z)/f(z)$ zawiera się, przy z należącym do koła jednostkowego, w części prawej półpłaszczyzny ograniczonej parabolą. Dla klasy tej otrzymano pewien rezultat dotyczący splotu oraz wyznaczono stałą Koebego. Wprowadzono również pewne uogólnienie klasy S_p i otrzymano kilka wyników dotyczących tej uogólnionej klasy.

(received February 12, 1992)