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On Natural Transformations of Higher Order Covelocities Functor

O transformacjach naturalnych funktora koprędkości wyższego rzędu

Abstract. In this paper, all natural transformations of the $(2,r)$-covelocities functor T_2^{r*} into the $(1,r)$-covelocities functor T_1^{r*} and T_2^{r*}, are determined. We deduce that all natural transformations of T_2^{r*} into T_1^{r*} form an $(2r+\frac{r(r-1)}{2})$-parameter family linearly generated by the generalized (s,t)-power mixed transformations $A_{s,t}$ for $s=0,1,\dots,r$ and $t=0,1,\dots,r$ with $s+t=1,\dots,r$.

Recently, we have determined in [2] all natural transformations of the r-th order cotangent bundle functor T^{r*} into itself, which constitute the r-parameter family linearly generated by the s-th power natural transformations A_s for $s=1,\dots,r$.

In this paper, we determine all natural transformations of the $(2,r)$-covelocities bundle functor T_2^{r*} into the $(1,r)$-covelocities bundle functor T_1^{r*}. We deduce that all natural transformations of the functor T_2^{r*} into the functor T_1^{r*} form the $(2r+\frac{r(r-1)}{2})$-parameter family linearly generated by the generalized (s,t)-power mixed transformations $A_{s,t}$ or $s=1,\dots,r$ and $t=0,1,\dots,r$ with $s+t=1,\dots,r$.

Moreover, we deduce that all natural transformations of the functor T_2^{r*} into itself form the $2\cdot(2r+\frac{r(r-1)}{2})$-parameter family linearly generated for both components by the generalized (s,t)-power mixed transformations $A_{s,t}$ of T_2^{r*} into T_1^{r*}.

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1. Let M be a smooth n-dimensional manifold. Let $T_k^{r*}M=J^r(M,R^k)_0$ be the space of all r-jets from a manifold M to R^k with target at 0.

A vector bundle $\pi_M: T_k^{r*}M\to M$ with a source r-jet projection is called the (k,r)-covelocities bundle on M.

Every local diffeomorphism $\varphi: M\to N$ is extended into a vector bundles morphism $T_k^{r*}\varphi: T_k^{r*}M\to T_k^{r*}N$ defined by $T_k^{r*}\varphi: j_x^rF\mapsto j_x^r(F\circ\varphi^{-1})$, where φ^{-1} is constructed locally. Hence, the (k,r)-covelocities bundle functor T_k^{r*} is defined on a category $\mathcal{M}f_n$ of smooth n dimensional manifolds with local diffeomorphisms as morphisms and with values in a category $\mathcal{VB}$ of vector bundles.

We have a canonical identification

$$T_k^{r*}M = T_1^{r*}M \times \ldots \times T_1^{r*}M \quad (k\text{–times}) \tag{1.1}$$

of the form $j_x^r F = (j_x^r F^1, \ldots, j_x^r F^k)$ for $F = (F^1, \ldots, F^k)$.

Consider the $(2, r)$–covelocities bundle functor T_2^{r*} and the $(1, r)$–covelocities bundle functor T_1^{r*}.

We have defined in [2] the s–th power natural transformations A_s of the r–th cotangent bundle functor $T^{r*} = T_1^{r*}$ into itself of the form

$$A_s : j_x^r F \mapsto j_x^r (F)^s \tag{1.2}$$

where $(F)^s$ denote the s–th power of F for $s = 1, \ldots, r$.

We define a natural transformations $A_{s,t}$ of the functor T_2^{r*} into the functor T_1^{r*} as a generalization of the power transformations A_s of the functor T_1^{r*} into itself.

Definition 1. A natural transformation $A_{s,t}$ of the $(2, r)$–covelocities functor T_2^{r*} into the $(1, r)$–covelocities functor T_1^{r*} defined by formula

$$A_{s,t} : j_x^r F \mapsto j_x^r (F^1)^s (F^2)^t \tag{1.3}$$

where $F = (F^1, F^2)$ and $(F^k)^p$ denote the p–th power of F^k, is called the generalized (s, t)–power mixed transformation for $s = 0, 1, \ldots, r$ and $t = 0, 1, \ldots, r$ with $s + t = 1, \ldots, r$.

If (x^i) are some local coordinates on M, then we have the induced fibre coordinates $(u_i, \ldots, u_{i_1 \ldots i_r}, v_i, \ldots, v_{i_1 \ldots i_r})$ on $T_2^{r*}M$ (symmetric in all indices) of the form

$$\begin{aligned} u_i(j_x^r F) &= \frac{\partial F^1}{\partial x^i}\Big|_x \\ &\ldots\ldots\ldots\ldots \\ u_{i_1 \ldots i_r}(j_x^r F) &= \frac{\partial^r F^1}{\partial x^{i_1} \ldots \partial x^{i_r}}\Big|_x \\ v_i(j_x^r F) &= \frac{\partial F^2}{\partial x^i}\Big|_x \\ &\ldots\ldots\ldots\ldots \\ v_{i_1 \ldots i_r}(j_x^r F) &= \frac{\partial^r F^2}{\partial x^{i_1} \ldots \partial x^{i_r}}\Big|_x \end{aligned} \tag{1.4}$$

2. In this part first we determine by an induction method all natural transformations of the functor T_2^{r*} into the functor T_1^{r*}.

Theorem 1. *All natural transformations $A : T_2^{r*} \to T_1^{r*}$ of the $(2, r)$–covelocities bundle functor T_2^{r*} into the $(1, r)$–covelocities bundle functor T_1^{r*} form the $(2r + \frac{r(r-1)}{2})$–parameter family of the form*

$$\begin{aligned} A = {} & k_1 A_{1,0} + \ldots + k_r A_{r,0} + l_1 A_{0,1} + \ldots + l_r A_{0,r} + \\ & + m_{1,1} A_{1,1} + \ldots + m_{r-1,1} A_{r-1,1} + \ldots + m_{1,r-1} A_{1,r-1} \end{aligned} \tag{2.1}$$

with any real parameters $k_1, \dots, k_r, l_1, \dots, l_r, m_{1,1}, \dots, m_{r-1,1}, \dots, m_{1,r-1} \in R$ ***and are linearly generated by the generalized*** (s,t)***-power mixed transformations*** $A_{s,t}$ ***for*** $s = 0, 1, \dots, r$ ***and*** $t = 0, 1, \dots, r$ ***with*** $s + t = 1, \dots, r$.

Proof. The $(2,r)$–covelocities bundle functor T_2^{r*} is defined on the category $\mathcal{M}f_n$ of n–dimensional smooth manifolds with local diffeomorphisms as morphisms and is of order r. Then, its standard fibre $S = (T_2^{r*} R^n)_0$ is G_n^r–space, where G_n^r means a group of all invertible r–jets from R^n into R^n with source and target at 0.

According to a general theory, [1], the natural transformations $A : T_2^{r*} \to T_1^{r*}$ are in bijection with G_n^r–equivariant maps of the standard fibres $f : (T_2^{r*} R^n)_0 \to (T_1^{r*} R^n)_0$.

Let $\widetilde{a} = a^{-1}$ denote the inverse element in G_n^r and let $(i_1 \dots i_r)$ denote the symmetrization of indices.

By (1.4) the action of an element $(a^i_j, a^i_{j_1 j_2}, \dots, a^i_{j_1 \dots j_r}) \in G_n^r$ on $(u_i, \dots, u_{i_1 \dots i_r}, v_i, \dots, v_{i_1 \dots i_r}) \in (T_2^{r*} R^n)_0$ and on $(w_i, \dots, w_{i_1 \dots i_r}) \in (T_1^{r*} R^n)_0$ is of the form

$$
\begin{aligned}
(2.2)\qquad & \overline{u}_i = u_j \widetilde{a}^j_i \\
& \overline{u}_{i_1 i_2} = u_{j_1 j_2} \widetilde{a}^{j_1}_{i_1} \widetilde{a}^{j_2}_{i_2} + u_{j_1} \widetilde{a}^{j_1}_{i_1 i_2} \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& \overline{u}_{i_1 \dots i_r} = u_{j_1 \dots j_r} \widetilde{a}^{j_1}_{i_1} \dots \widetilde{a}^{j_r}_{i_r} + \\
& \quad + u_{j_1 \dots j_{r-1}} \frac{r!}{(r-2)!2!} \widetilde{a}^{j_1}_{(i_1} \dots \widetilde{a}^{j_{r-2}}_{i_{r-2}} \widetilde{a}^{j_{r-1}}_{i_{r-1} i_r)} + \\
& \quad + \dots + u_{j_1 j_2} \Big[\frac{r!}{(r-1)!1!} \widetilde{a}^{j_1}_{(i_1} \widetilde{a}^{j_2}_{i_2 \dots i_r)} + \dots \Big] + \\
& \quad + u_{j_1} \widetilde{a}^{j_1}_{i_1 \dots i_r}
\end{aligned}
$$

and is of the same form on coordinates $v_{i_1 \dots i_s}$, $w_{i_1 \dots i_s}$ for $s = 1, \dots, r$.

I. In the first induction step we consider the case $r = 2$. Considering equivariancy of G_n^2–equivariant map $f = (f_i, f_{ij}) : (T_2^{2*} R^n)_0 \to (T_1^{2*} R^n)_0$ in the form

$$
\begin{aligned}
(2.3)\qquad & w_i = f_i(u_i, u_{ij}, v_i, v_{ij}) \\
& w_{ij} = f_{ij}(u_i, u_{ij}, v_i, v_{ij})
\end{aligned}
$$

with respect to homotheties in $G_n^2 : \widetilde{a}^i_j = k\delta^i_j$, $\widetilde{a}^i_{jk} = 0$, we get a homogeneity condition

$$
\begin{aligned}
(2.4)\qquad & k f_i(u_, u_{ij}, v_i, v_{ij}) = f_i(ku_i, k^2 u_{ij}, kv_i, k^2 v_{ij}) \\
& k^2 f_{ij}(u_i, u_{ij}, v_i, v_{ij}) = f_j(ku_i, k^2 u_{ij}, kv_i, k^2 v_{ij})\,.
\end{aligned}
$$

By the homogeneous function theorem, [1], we deduce firstly that f_i is linear in u_i and v_i and is independent on u_{ij} and v_{ij} and is bilinear in u_i, v_j and is quadratic in u_i and v_i.

Using invariant tensor theorem for G_n^1, [1], we obtain f in the form

$$
\begin{aligned}
(2.5)\qquad & f_i = k_1 u_i + l_1 v_i \\
& f_{ij} = k_2 u_i u_j + k_3 u_{ij} + l_2 v_i v_j + l_3 v_{ij} + m_{1,1} u_{(i} v_{j)}
\end{aligned}
$$

with any real parameters $k_1, k_2, k_3, l_1, l_2, l_3, m_{1,1} \in R$.

The equivariancy of f in the form (2.5) with respect to the kernel of the projection $G_n^2 \to G_n^1$: $\widetilde{a}_j^i = \delta_j^i$ and $\widetilde{a}_{jk}^i$ arbitrary, gives relationship for parameters

$$k_3 = k_1 \ , \ l_3 = l_1 \tag{2.6}$$

This gives the 5–parameter family of natural transformations in the form $A = k_1 A_{1,0} + k_2 A_{2,0} + l_1 A_{0,1} + l_2 A_{0,2} + m_{1,1} A_{1,1}$ with any real parametrs $k_1, k_2, l_1, l_2, m_{1,1} \in R$ and proves our theorem for $r = 2$.

II. In the second induction step for $(r-1)$, we assume that G_n^{r-1}–equivariant map $f = (f_i, \ldots, f_{i_1 \ldots i_{r-1}}) : (T_2^{(r-1)*} R^n)_0 \to (T_1^{(r-1)*} R^n)_0$ define the $(2(r-1) + \frac{(r-1)(r-2)}{2})$-parameter family

$$\begin{aligned} A = & k_1 A_{1,0} + \ldots + k_{r-1} A_{r-1,0} + l_1 A_{0,1} + \ldots + l_{r-1} A_{0,r-1} + \\ & + m_{1,1} A_{1,1} + \ldots + m_{r-2,1} A_{r-2,1} + \ldots + m_{1,r-2} A_{1,r-2} \end{aligned} \tag{2.7}$$

with any real parameters $k_1, \ldots, k_{r-1}, l_1, \ldots, l_{r-1}, m_{1,1}, \ldots, m_{r-2,1}, \ldots, m_{1,r-2} \in R$. We assume that G_n^r–equivariant map $\overline{f} : (T_2^{r*} R^n)_0 \to (T_1^{r*} R^n)_0$ is of the form $\overline{f} = (f_i, \ldots, f_{i_1 \ldots i_{r-1}}, f_{i_1 \ldots i_r})$ provided that f is of the form $f = (f_i, \ldots, f_{i_1 \ldots i_{r-1}})$.

Considering equivariancy of $\overline{f}$ with respect to homotheties in G_n^r : $\widetilde{a}_j^i = k\delta_j^i$, $\widetilde{a}_{j_1 j_2}^i = 0, \ldots, \widetilde{a}_{j_1 \ldots j_r}^i = 0$, we obtain for the r-th component $f_{i_1 \ldots i_r}$ a homogeneity condition

$$\begin{aligned} & k^r f_{i_1 \ldots i_r}(u_i, \ldots, u_{i_1 \ldots i_r}, v_i, \ldots, v_{i_1 \ldots i_r}) = \\ & = f_{i_1 \ldots i_r}(k u_i, \ldots, k^r u_{i_1 \ldots i_r}, k v_i, \ldots, k^r v_{i_1 \ldots i_r}) \ . \end{aligned} \tag{2.8}$$

By the homogeneous function theorem and by the invariant tensor theorem, [1], we deduce that $f_{i_1 \ldots i_r}$ is of the general form

$$\begin{aligned} f_{i_1 \ldots i_r} = & p_1 u_{i_1 \ldots i_r} + p_{2,1} u_{(i_1} u_{i_2 \ldots i_r)} + p_{2,2} u_{(i_1 i_2} u_{i_3 \ldots i_r)} + \\ & + \ldots + p_{r-1} u_{(i_1} \ldots u_{i_{r-2}} u_{i_{r-1} i_r)} + p_r u_{i_1} \ldots u_{i_r} + \\ & + q_1 v_{i_1 \ldots i_r} + q_{2,1} v_{(i_1} v_{i_2 \ldots i_r)} + q_{2,2} v_{(i_1 i_2} v_{i_3 \ldots i_r)} + \\ & + \ldots + q_{r-1} v_{(i_1} \ldots v_{i_{r-2}} v_{i_{r-1} i_r)} + q_r v_{i_1} \ldots v_{i_r} + \\ & + n_{1,1} u_{(i_1 \ldots i_{r-1}} v_{i_r)} + \ldots + \overline{n}_{1,1} u_{(i_1} v_{i_2 \ldots i_r)} + \\ & + \ldots + n_{r-2,1} u_{(i_1} \ldots u_{i_{r-2}} v_{i_{r-1} i_r)} + \ldots + \\ & + n_{1,r-2} u_{(i_1 i_2} v_{i_3} \ldots v_{i_r)} + n_{r-1,1} u_{(i_1} \ldots u_{i_{r-1}} v_{i_r)} + \\ & + \ldots + n_{1,r-1} u_{(i_1} v_{i_2} \ldots v_{i_r)} \ . \end{aligned} \tag{2.9}$$

Equivariancy of $\overline{f}$ with respect to the kernel of the projection $G_n^r \to G_n^{r-1}$: $\widetilde{a}_j^i = \delta_j^i$, $\widetilde{a}_{j_1 j_2}^i = 0, \ldots, \widetilde{a}_{j_1 \ldots j_{r-1}}^i = 0$ and $\widetilde{a}_{j_1 \ldots j_r}^i$ arbitrary, gives relationship

$$p_1 = k_1 \ , \ q_1 = l_1 \ . \tag{2.10}$$

Now, consdering equivariancy of $\overline{f}$ with respect to the kernel of the projection $G_n^{r-1} \to G_n^1$: $\widetilde{a}_j^i = \delta_j^i$ and $\widetilde{a}_{j_1 j_2}^i, \ldots, \widetilde{a}_{j_1 \ldots j_{r-1}}^i$ are arbitrary and $\widetilde{a}_{j_1 \ldots j_r}^i = 0$ in G_n^r, we obtain following relationship for parameters

$$(2.11) \qquad p_{2,1} = \frac{r!}{(r-1)!1!} k_2 \, , \; p_{2,2} = \frac{r!}{(r-2)!2!} k_2 \, , \; \ldots \, ,$$

$$p_{r-1} = \frac{r!}{(r-2)!2!} k_{r-1} \, ,$$

$$q_{2,1} = \frac{r!}{(r-1)!1!} l_2 \, , \; q_{2,2} = \frac{r!}{(r-2)!2!} l_2 \, , \; \ldots \, ,$$

$$q_{r-1} = \frac{r!}{(r-2)!2!} l_{r-1}$$

$$n_{1,1} = \frac{r!}{(r-1)!1!} m_{1,1} \, , \; \ldots \, , \; \overline{n}_{1,1} = \frac{r!}{(r-1)!1!} m_{1,1}$$

....

$$n_{r-1,1} = \frac{r!}{(r-2)!2!} m_{r-2,1} \, , \; \ldots \, , \; n_{1,r-2} = \frac{r!}{(r-2)!2!} m_{1,r-2}$$

If we put for parameters $p_r = k_r$, $q_r = l_r$, $n_{r-1,1} = m_{r-1,1}, \ldots, n_{1,r-1} = m_{1,r-1}$, then we obtain A in the form (2.1). This proves our theorem.

At last, we determine all natural transformations of the $(2,r)$ covelocities bundle functor T_2^{r*} into itself.

Using the canonical identification (1.1), $T_2^{r*}M = T_1^{r*}M \times T_1^{r*}M$, any natural transformation $A : T_2^{r*} \to T_2^{r*}$ correspond bijectively to G_n^r–equivariant map $f := (f_{i}, \ldots, f_{i_1 \ldots i_r}; g_{i}, \ldots, g_{i_1 \ldots i_r}) : (T_2^{r*}R^n)_0 \to (T_1^{r*}R^n)_0 \times (T_1^{r*}R^n)_0$. Considering G_n^r–equivariancy of f, we deduce from theorem 1 that both components $(f_{i}, \ldots, f_{i_1 \ldots i_r})$ and $(g_{i}, \ldots, g_{i_1 \ldots i_r})$ define the $(2r + \frac{r(r-1)}{2})$–parameter families of natural transformations $T_2^{r*} \to T_1^{r*}$ of the form (2.1).

Corollary 2. *All natural transformations $A : T_2^{r*} \to T_2^{r*}$ of the $(2,r)$–covelocities bundle functor T_2^{r*} into itself form the $2 \cdot (2r + \frac{r(r-1)}{2})$–parameter family of the form (2.1) for both components and are linearly generated for both componenets by the generalized (s,t)–power mixed transformations $A_{s,t}$ for $s = 0, 1, \ldots, r$ and $t = 0, 1, \ldots, r$ with $s + t = 1, \ldots, r$.*

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STRESZCZENIE

W pracy wyznacza się wszystkie transformacje naturalne funktora $(2,r)$-koprędkości T_2^{r*} w funktory $(1,r)$-koprędkości T_1^{r*} oraz T_2^{r*}. Podstawowymi transformacjami tego typu są uogólnione transformacje (s,t)-potęgowe mieszane $A_{s,t}$ dla $s = 0,1,\ldots,r$ oraz $t = 0,1,\ldots,r$ spełniających $s+t = 1,\ldots,r$.

Wszystkie transformacje funktora T_2^{r*} w T_1^{r*} stanowią $\left(2r + \frac{r(r-1)}{2}\right)$-parametrową rodzinę generowaną liniowo za pomocą uogólnionych transformacji (s,t)-potęgowych mieszanych $A_{s,t}$.

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