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Strong Limit Theorems for the Growth of Increments of Additive Processes in Groups. Part II. Additive Processes in Groups

Abstract. This is a continuation of the paper under the same title published in the previous issue of this journal. In this part we apply maximal symmetrization inequality to obtain sufficient conditions ensuring that given families of neighbourhoods of points are upper or lower classes of sets characterizing the increments of additive processes in groups on asymptotically small and large intervals.

4. Asymptotic and local properties of realizations for additive processes in groups. Let $X = \{X_t,\ t \in R_+^q\}$ be an additive stochastic process taking values in a T_0 topological Abelian group $\mathbb{G}$ with the Baire σ-field $\mathcal{G}$. In this section we are interested in a description of local behaviour of $\Delta X(<0,t))$ as t becomes close to the lower-left boundary ∂R_+^q of the index set R_+^q and asymptotic properties of the same quantities as t grows to infinity. More precisely, we assume that t varies in a set $B = A_1 \times \cdots \times A_q$, where each A_i is a one-dimensional interval $(0,\infty)$, $(0, C_i>$, $<c_i, \infty)$ or $<c_i, C_i>$, but not all of them are equal to $<c_i, C_i>$; say A_i have left endpoints $c_{i(u)} > 0$ for $1 \le u \le p$ and right endpoints $C_{i'(v)} < \infty$ for $1 \le v \le r$, while their remaining endpoints are 0 or ∞ respectively. We shall investigate limiting properties of $\Delta X(<0,t))$ for $t \in B \setminus <T_1^{(n)}, T_2^{(n)}>$ with $T_1^{(n)} \searrow 0$ and $T_2^{(n)} \nearrow \infty$ as $n \to \infty$. In such a situation we write for brevity $t \xrightarrow{B} 0$ or ∞. The behaviour of increments $\Delta X(<0,t))$ may be described by means of limits

$$\liminf_{t \xrightarrow{B} 0 \text{ or } \infty} D_t = \bigcup_{0<T_1<T_2} \bigcap_{\substack{t \notin <T_1,T_2> \\ t \in B}} D_t\,,$$

and

$$\limsup_{t \xrightarrow{B} 0 \text{ or } \infty} D'_t = \bigcap_{0<T_1<T_2} \bigcup_{\substack{t \notin <T_1,T_2> \\ t \in B}} D'_t\,,$$

where $D_t = \{\Delta X(<0,t)) \in U_t(-j)\}$ and $D'_t = \{\Delta X(<0,t)) \notin U_t(-j)\}$ for a fixed $j \ge 1$, and $U_t(-j)$ are globular sets satisfying certain additional conditions. It is easy to see that the above defined limits are random events provided X is a separable process. However, in the frame we are working here such an assumption is a rather nontrivial fact. Therefore, to avoid this restriction we replace below the set B when necessary by a subset B' of the form $B' = A'_1 \times \cdots \times A'_q$, where $A'_i \subset A_i$ are any countable sets satisfying $\inf A_i' = \inf A_i$ and $\sup A_i' = \sup A_i$. Consequently, we investigate probabilities of the events

$$\liminf_{t \xrightarrow{B'} 0 \text{ or } \infty} D_t = \bigcup_{0<T_1<T_2} \bigcap_{\substack{t \notin <T_1,T_2> \\ t \in B'}} D_t \,,$$

and

$$\limsup_{t \xrightarrow{B'} 0 \text{ or } \infty} D_t' = \bigcap_{0<T_1<T_2} \bigcup_{\substack{t \notin <T_1,T_2> \\ t \in B'}} D_t'$$

respectively.

Definition 4.1. Let $\mathcal{U} = \{U_t, t \in R_+^q\}$ be a family of globular sets. We say that $\mathcal{U}$ is regularly varying, if
(i) for all $j \geq 0$ and $t, s \in R_+^q$, $s < t$, the relation

$$U_s(-j) \subseteq U_t(-j)$$

holds.

Theorem 4.2. *Let X be a symmetric additive stochastic process and let $\mathcal{U}$ be a regularly varying family of globular sets. If for a given rectangle $B \subset R_+^q$ as described above,*

$$I_B := \int_B \frac{1}{|t|} \cdot P[\Delta X(<0,t)) \notin U_t(-2q-1)]\, dt < \infty \tag{4.1}$$

then for an arbitrary $\beta \in R_+^q$, $\beta > 1 = (1, \ldots 1)$,

$$P\Big[\liminf_{t \xrightarrow{B'} 0 \text{ or } \infty} (\Delta X(<0,t)) \in U_{\beta t} \Big] = 1 \,, \tag{4.2}$$

where $\beta t = (\beta_1 t_1, \ldots, \beta_q t_q)$.

Proof. Let $a \in R_+$, $1 < a^2 < \beta_0 = \min\{\beta_1, \ldots, \beta_q\}$ be a fixed real number. Define the set $\mathbf{J}$ of indices $k = (k_1, \ldots, k_q) \in \mathbf{Z}^q$ with integer coordinates as follows: $\mathbf{J} = \{k \in \mathbf{Z}^q : < a^k, a^{k+1} > \subset B\}$, where $a^k = (a_1^{k_1}, \ldots, a_q^{k_q})$ and $a = (a, \ldots, a)$. Let Λ be the set of all different projections $R^q \to R^q$ onto various hyperplanes of the system of coordinates. Put $\mathbf{J}' = \{k \in \mathbf{Z}^q : a^{k \pm \lambda 1} \in B$ for some $\lambda \in \Lambda\}$. It suffices to show that for indices of the form $a^k \in B$ we have

$$P\Big[\bigcup_{\substack{j<n \\ j,n \in \mathbf{J}'}} \bigcap_{\substack{t \notin <a^j, a^n> \\ t \in B'}} [\Delta X(<0,t)) \in U_{\beta t}] \Big] = 1 \,.$$

Consider the events

$$A_k = \Big\{ \bigcup_{\substack{0 \leq s \leq v \leq a^k \\ s,v \in B'}} [\Delta X(<s,v)) \notin U_{a^{k+1}}] \Big\} \,, \quad k \in \mathbf{J} \,.$$

On the basis of Corollary 3.3 we obtain

$$P[A_k] \le 4^q P[\Delta X(<0,a^k)) \notin U_{a^{k+1}}(-2q)] \tag{4.3}$$

Suppose $t \in (a^k, a^{k+1} >$ for some $k \in \mathbf{J}$. By hypothesis we have $U_t(-2q) \subseteq U_{a^{k+1}}(-2q)$ and $2P[\Delta X(<0,t)) - \Delta X(<0,a^k)) \in H_{j,t}(-2q)] \ge 1$ for each set $H_{j,t}(-2q)$ corresponding to the globular set $U_t(-2q)$, $j \in J$ (see Definition 2.1). Moreover, $\Delta X(<0,t)) - \Delta X(<0,a^k))$ is independent of $\Delta X(<0,a^k))$. Hence and from (4.3) it follows that

(4.4)

$$P[A_k] \le 4^q \sum_j P[\Delta X(<0,a^k)) \in C_{j,t}(-2q)] \le$$

$$\le 2\cdot 4^q \sum_j P[\Delta X(<0,a^k)) \in C_{j,t}(-2q), \Delta X(<0,t)) - \Delta X(<0,a^k)) \in H_{j,t}(-2q)] \le$$

$$\le 2\cdot 4^q \sum_j P[\Delta X(<0,a^k)) \in C_{j,t}(-2q), \Delta X(<0,t)) \notin U_t(-2q-1)] \le$$

$$\le 2^{2q+1} P[\Delta X(<0,t)) \notin U_t(-2q-1)]\,.$$

Consequently

$$(\ln a)^q P[A_k] = \int_{<a^k,a^{k+1}>} \frac{1}{|t|} P[A_k]\,dt \le \tag{4.5}$$

$$\le 2^{2q+1} \int_{<a^k,a^{k+1}>} \frac{1}{|t|} P[\Delta X(<0,t)) \notin U_t(-2q-1)]\,dt\,,$$

and therefore

$$\sum_{k\in J} P[A_k] \le 2^{2q+1} (\ln a)^{-q} I_B < \infty\,. \tag{4.6}$$

On account of the Borel–Cantelli lemma we conclude that $P[A_k, k \in \mathbf{J} \text{ i.o. }] = 0$, i.e. there exist $\Omega_1 \in \mathcal{F}$, $P[\Omega_1] = 1$ and finite subset $\mathbf{J}_0 = \mathbf{J}(\omega) \subset \mathbf{J}$ such that for every $k \in \mathbf{J} \setminus \mathbf{J}_0$ and $\omega \in \Omega_1$, $\overline{A}_k(\omega)$ holds. Without loss of generality we may replace $\mathbf{J}_0$ by a (random) rectangle $< j+1, n > \subset \mathbf{Z}^q$ with endpoints not necessarily contained in $\mathbf{J}$, having coordinates $j_{i(u)}$ and $n_{i'(v)}$ chosen in such a way that $a^{j_{i(u)}} \le c_{i(u)}$ and $a^{n_{i'(v)}} \ge C_{i'(v)}$, $u = 1,\ldots,p$, $v = 1,\ldots,r$. Observe now that for $t \in < a^{k-1}, a^k > \cap B'$,

$$\Big\{ \bigcup_{t\in<a^{k-1},a^k>\cap B'} [\Delta X(<0,t)) \notin U_{\beta t}] \Big\} \subseteq \Big\{ \bigcup_{t\in<a^{k-1},a^k>\cap B'} [\Delta X(<0,t)) \notin U_{a^{k+1}}] \Big\} \subseteq A_k\,,$$

thus

$$\bigcup_{\substack{k\notin<j+1,n>\\ k\in J}} \Big\{ \bigcup_{t\in<a^{k-1},a^k>\cap B'} [\Delta X(<0,t)) \notin U_{\beta t}] \Big\} \subseteq \bigcup_{\substack{k\notin<j+1,n>\\ k\in J}} A_k\,.$$

Hence

$$\bigcap_{\substack{k\notin<j+1,n>\\ k\in J}} \overline{A}_k \subseteq \bigcap_{\substack{t\notin<a^j,a^n>\\ t\in B'}} [\Delta X(<0,t)) \in U_{\beta t}]\,.$$

Finally

$$P\Big[\bigcup_{\substack{j<n\\ j,n\in J'}}\ \bigcap_{\substack{t\notin<a^j,a^n>\\ t\in B'}} [\Delta X(<0,t))\in U_{\beta t}]\Big] = 1\,,$$

which concludes the proof.

Corollary 4.3. *Let X be a stochastic process satisfying the assumptions of* Theorem 4.2 *but not necessarily symmetric and let $\mathcal{U}=\{U_t, t\in R_+^q\}$ be a regularly varying family of globular sets such that $W_t = U_t - U_t$ for $t\in R_+^q$ are Baire sets (and hence globular) satisfying $W_t(-j) = U_t(-j) - U_t(-j)$, $j\geq 1$. Moreover, assume that* (4.1) *is true. Then there can be found a deterministic function $z: R_+^q \to \mathbb{G}$ such that for an arbitrary $\beta\in R_+^q$, $\beta>1$,*

$$P\Big[\liminf_{t\xrightarrow{B'}0 \text{ or } \infty} [\Delta X(<0,t))\in z(t) + W_{\beta t}]\Big] = 1\,. \tag{4.7}$$

Proof. Let $(\Omega', \mathcal{F}', P')$ be a copy of $(\Omega,\mathcal{F},P)$ and let $X_t(\omega,\omega') = X_t(\omega)$ and $X_t'(\omega,\omega') = X_t(\omega')$ be two independent copies of X on the product probability space $(\Omega\times\Omega', \mathcal{F}\times\mathcal{F}', P\times P')$. Observe that $\mathcal{W}=\{W_t\}$ is a regularly varying family of globular sets and the symmetric process $X - X'$ satisfies (4.1) with $\mathcal{U}$ replaced by $\mathcal{W}$. On the basis of Theorem 4.2 we have

$$P\times P'\Big[\liminf_{t\xrightarrow{B'}0 \text{ or } \infty} [\Delta X(<0,t)) - \Delta X'(<0,t))\in W_{\beta t}]\Big] = 1\,.$$

Applying the well-known Fubini's theorem we conclude that

$$P\Big[\liminf_{t\xrightarrow{B'}0 \text{ or } \infty} [\Delta X(<0,t)) - \Delta X'(<0,t))(\omega') + W_{\beta t}]\Big] = 1$$

for P' - a.a. $\omega'\in\Omega'(\beta)\subseteq\Omega'$, $1<\beta\in R_+^q$. Let $\Omega_1' = \bigcap_{1<\beta\text{-rational}}\Omega'(\beta)$, where the product is taken over all q-tuples $\beta>1$ having rational coordinates. Then the above relation yields (4.7) with $z(t) = \Delta X'(<0,t))(\omega')$ for a fixed $\omega'\in\Omega_1'$.

To formulate further corollaries and provide any examples we consider a class of regularly increasing functions.

A function $g: R_+^q \to R_+$ is called here regularly increasing, if

(a) g is continuous and nondecreasing, i.e. $t<s$ implies $g(t)\leq g(s)$;

(b) $g|_{\partial R_+^q} = 0$, and $g(t)>0$ for $t\notin\partial R_+^q$;

(c) $\displaystyle\lim_{\substack{1\leq\beta\searrow 1\\ \beta\in R_+^q}}\ \sup_{t\in R_+^q\setminus\partial R_+^q}\Big(\frac{g(\beta t)}{g(t)} - 1\Big) = 0\,.$

Corollary 4.4. *Let K be a convex set containing zero and let U be a convex open neighbourhood of zero in a real linear topological Lindelöf space $\mathbb{G}$. Assume that X is a symmetric additive stochastic process in $\mathbb{G}$ such that for each real $\delta>0$,*

$$I_B' := \int_B \frac{1}{|t|}\cdot P[\Delta X(<0,t))\notin(1+\delta)\varphi(t)(K+\rho(t)U)]\,dt<\infty\,, \tag{4.8}$$

where $\varphi, \rho : R_+^q \to R_+$ are certain functions satisfying the conditions: φ and $\varphi \cdot \rho$ are regularly increasing, but ρ alone may be arbitrary. Then, for each $\delta > 0$,

$$(4.9) \qquad P\Big[\liminf_{t \xrightarrow{B'} 0 \text{ or } \infty} \{\Delta X(<0,t)) \in (1+\delta)\varphi(t)(K+\rho(t)U)\}\Big] = 1 .$$

Proof. Observe that $\{U_t = (1+\delta/2)\varphi(t)(K+\rho(t)U),\ t \in R_+^q\}$ is a regularly varying family of globular neighbourhoods of zero. Moreover, from the proof of Proposition 2.2 it follows that we can take $U_t(-2q-1) = (1+\delta/4)\varphi(t)(K+\rho(t)U)$. Therefore, by hypothesis, condition (4.1) is fulfilled. Hence, on the basis of Theorem 4.2, choosing $\beta > 1$ close enough to 1 and using (c) we conclude that (4.9) holds.

Remark. The above Corollary 4.4 for a separable process X and B' replaced by B in (4.9) is a far reaching generalization of Theorem 1 and Remark 1 a), Ch. VI, §6 given by Gikhman and Skorohod (1965) and Theorem 1 by Zinčenko (1979).

Example 5. Let μ_r, $r > 0$ be a centered Gaussian measure with variance parameter $r > 0$ on the Baire σ-field $\mathcal{G}$ in a real locally convex linear topological Lindelöf space $\mathbb{G}$. According to the terminology employed by Borell (1975), (1976), μ is called here a centered Gaussian measure in $\mathbb{G}$ if μ is a Radon measure on $\mathcal{G}$ and for each $f \in \mathbb{G}^*$ – the topological dual of $\mathbb{G}$, $\mu \circ f^{-1}$ is a mean zero normal distribution on $(R, \mathcal{B}(R))$. Let $\mu = \mu_1$ and let $\mathbb{G}^2(\mu)$ be the closure of $\mathbb{G}^*$ in $L^2(\mu)$. Then for every $g \in \mathbb{G}^2(\mu)$ there exists a unique $y \in \mathbb{G}$ such that

$$f(y) = \int_{\mathbb{G}} f(x)g(x)\,d\mu(x) \quad \text{for all } f \in \mathbb{G}^* .$$

In consequence we have a one-to-one linear mapping $\Psi : \mathbb{G}^2(\mu) \to \mathbb{G}$ given by $\Psi g = y$. Denote $\Psi(\mathbb{G}^2(\mu)) = \mathbb{H} \subset \mathbb{G}$. It is easy to see that $\mathbb{H}$ is then a Hilbert space with the scalar product

$$< \Psi g_1, \Psi g_2 > = \int_{\mathbb{G}} g_1(x)g_2(x)\,d\mu(x) .$$

Similarly as in the case $\mathbb{G} = R$ the space $\mathbb{H}$ is said to be the RKHS of μ. Let i be the inclusion map from $\mathbb{H}$ into $\mathbb{G}$ and let γ_r be the canonical cylinder set Gauss measure in $\mathbb{H}$ with variance parameter $r > 0$. We say that μ_r is a centered Gaussian measure in $\mathbb{G}$ with variance parameter $r > 0$ if μ_r is a Radon measure on $\mathcal{G}$ and for every finite-dimensional cylindrical set C in $\mathbb{G}$ of the form $C = \{x \in \mathbb{G} : (f_1(x), \ldots, f_n(x)) \in A\}$, $f_1, \ldots, f_n \in \mathbb{G}^*$, $A \in \mathcal{B}(R^n)$, μ_r and γ_r are associated by the equation $\mu_r(C) = \gamma_r(i^{-1}C)$. In particular, if $< \Psi f_i, \Psi f_j > = \delta_{ij}$, then

$$\mu_r(C) = \gamma_r(i^{-1}C) = \int_A (2\pi r)^{-n/2} \exp\Big\{-(2r)^{-1}\sum_{i=1}^n s_i^2\Big\}\,ds_1 \ldots ds_n .$$

In other words, if ξ is a random element in $(\mathbb{G}, \mathcal{G})$ with distribution $\mu = \mu_1$, then μ_r is the law of $r^{1/2}\xi$. Note that γ_r determines μ_r in a unique manner by virtue of Prohorov's theorem – see Vahania, Tarieladze and Chobanian (1985), Chapter I, Theorem 3.4 and Proposition 3.1. Furthermore, the space $\mathbb{H}_r$ induced by μ_r in an analogous way as $\mathbb{H}$ by μ is identically equal to $\mathbb{H}$, because $\int_{\mathbb{G}} f(x)g(x)d\mu_r(x) = r < \Psi f, \Psi g >$ for $f, g \in \mathbb{G}^*$. Put $\mu_0(D) = \chi(0 \in D)$, $D \in \mathcal{G}$. Then the relation $\gamma_r * \gamma_s = \gamma_{r+s}$, $r, s \geq 0$ implies that $\{\mu_r, r \geq 0\}$ is a convolution semigroup. Let $X = \{X(t),\ t \in R^q_+\}$ be the μ-Brownian surface with values in $\mathbb{G}$, i.e. a stochastic process satisfying the following conditions:

(I) for each $t \in R^q_+$, $X(t)$ is a random element in $(\mathbb{G}, \mathcal{G})$;

(II) the process X has independent increments $\Delta X(V)$ on disjoint bounded rectangles $V =< a, b) \subset R^q_+$;

(III) for a given rectangle $V =< a, b) \subset R^q_+$, $\Delta X(V)$ has distribution $\mu_{\text{vol } V}$, where $\text{vol } V = |b - a|$ is the volume of V, and $X(t) = 0$ if any of the coordinates of $t \in R^q_+$ vanishes.

Observe that under our assumptions the distribution of each finite vector $(X(t_1), \ldots, X(t_n))$, $t_1, \ldots, t_n \in R^q_+$ is a Radon measure on $(\mathbb{G}^n, \mathcal{G}^n)$. Indeed, the problem can be reduced to distributions of vectors $(\Delta X(V_1), \ldots, \Delta X(V_s))$ with independent components, which are obviously Radon measures. Namely, let $V_1, \ldots, V_s$ be the class of disjoint rectangles formed by means of all the hyperplanes parallel to the system ones which contain $t_1, \ldots, t_n$. Then each rectangle $< 0, t_k)$ can be written as the sum $\bigcup_{i \in I_k} V_i$ with appropriately chosen $I_k \subset \{1, \ldots, s\}$. Hence, for $C \in \mathcal{G}^n$ we have

$$[(X(t_1), \ldots, X(t_n)) \in C] = \left[\left(\sum_{i \in I_1} \Delta X(V_i), \ldots, \sum_{i \in I_n} \Delta X(V_i)\right) \in C\right] =$$

$$= [f(\Delta(V_1), \ldots, \Delta X(V_s)) \in C] = [(\Delta X(V_1), \ldots, \Delta X(V_s)) \in f^{-1}(C)]\,,$$

where $f(u_1, \ldots, u_s) = \left(\sum_{i \in I_1} u_i, \ldots, \sum_{i \in I_n} u_i\right)$. Since $\Delta X(V_i)$ on disjoint rectangles $V_i \subset R^q_+$ are independent, we conclude that

$$P[(\Delta X(V_1), \ldots, \Delta X(V_s)) \in f^{-1}(C)] = \sup_{\substack{K \subseteq f^{-1}(C) \\ K-\text{compact}}} P[(\Delta X(V_1), \ldots, \Delta X(V_s)) \in K] =$$

$$= \sup_{\substack{f(K) \subseteq C \\ K-\text{compact}}} P[(X(t_1), \ldots, X(t_n)) \in f(K)] = P[(X(t_1), \ldots, X(t_n)) \in C]$$

and the continuity of f implies that $f(K)$ are compact for K compact. Hence it follows that the distribution of each vector $(X(t_1), \ldots, X(t_n))$, $t_1, \ldots, t_n \in R^q_+$, is a Radon measure on $(\mathbb{G}^n, \mathcal{G}^n)$. Using the well-known Kolmogorov's consistency theorem it is not difficult to construct now the process X on appropriate product space $\mathbb{G}^{R^q_+}$.

Denote by $\mathbb{K}$ the closed unit ball in $\mathbb{H}$ centered at zero and for every $s > 0$ put $Ls = \max(|\log|s||, 1)$, $LLs = L(Ls)$ and so on. Take an arbitrary open convex neighbourhood U of zero in $\mathbb{G}$ and consider sets of the form $U_t = (1 + \delta/2)(2q\ |t|\ LL|t|)^{1/2} \cdot (\mathbb{K} + \varepsilon(|t|)U)$, where $\delta > 0$ and $\varepsilon : R_+ \to R_+$ is a function specified below. Namely, suppose that $(|t|LL|t|)^{1/2}\varepsilon(|t|)$ is regularly increasing on R^q_+, but $(LLs)^{1/2}\varepsilon(s)$ is non-increasing as $s \in R_+$ increases. Then $\{U_t\}$ is a regularly varying family of globular sets and we can take $U_t(-2q - 1) = (1 + \delta/4)(2q|t|LL|t|)^{1/2}(\mathbb{K} + \varepsilon(|t|)U)$. Moreover, introduce a function $\alpha : R_+ \to R$ defined by $\alpha(s) = \Phi^{-1}(\mu(sU))$ for $s > 0$, where Φ

stands for the standard normal distribution function. Applying the main Theorem 3.1 by Borell (1975) and using some familiar estimates for the tail of Φ we obtain

(4.10)

$$\begin{aligned}
&P[\Delta X(<0,t)) \notin U_t(-2q-1)] = 1 - \mu((1+\delta/4)(2qLL|t|)^{1/2}(\mathbf{K}+\varepsilon(|t|)U)) \le \\
&\le 1 - \Phi\{(1+\delta/4)(2qLL|t|)^{1/2} + \alpha((1+\delta/4)(2qLL|t|)^{1/2}\varepsilon(|t|))\} \le \\
&\le C\exp\{-[(1+\delta/4)(2qLL|t|)^{1/2} + \alpha((1+\delta/4)(2qLL|t|)^{1/2}\varepsilon(|t|))]^2/2\} = \\
&= C\exp\{-(1+\delta/4)^2 qLL|t| - (1+\delta/4)(2qLL|t|)^{1/2}\alpha((1+\delta/4)(2qLL|t|)^{1/2}\varepsilon(|t|)) - \\
&- \alpha^2((1+\delta/4)(2qLL|t|)^{1/2}\varepsilon(|t|))/2\}\ .
\end{aligned}$$

If in addition $\varepsilon(s) \searrow 0$ as $s \nearrow \infty$ in such a way that for some $0 < \delta' < \delta$ and large s,

$$-(1+\delta/4)(2qLLs)^{1/2}\alpha((1+\delta/4)(2qLLs)^{1/2}\varepsilon(s)) - \frac{1}{2}\,\alpha^2((1+\delta/4)(2qLLs)^{1/2}\varepsilon(s)) \le$$

$$\le [\delta'/2 + (\delta'/4)^2]qLLs\ , \tag{4.11}$$

then taking $B = <a,\infty)^q \subset R_+^q$ with a fixed $0 < a < \infty$, we see that (4.1) is satisfied. Thus, based on Corollary 4.4, for an arbitrary $\delta > 0$ we have

$$P\Big[\liminf_{t \xrightarrow{B'} 0 \text{ or } \infty}\ [\Delta X(<0,t)) \in (1+\delta)(2q|t|LL|t|)^{1/2}(\mathbf{K}+\varepsilon(|t|)U]\Big] = 1\ . \tag{4.12}$$

Clearly, this happens provided $(LLs)^{1/2}\varepsilon(s)$ is bounded or tends to zero slowly enough. Note that in the last situation the speed of convergence to $-\infty$ of $\alpha(\cdot)$ in (4.11) depends on the behaviour of $\mu(sU)$ for $s > 0$ near zero. In general this behaviour is not known, but using the Minkowski's functional of U the problem can be reduced to the lower tail of a Gaussian seminorm in R^∞, and then on account of results by Hoffmann-Jørgensen, Shepp and Dudley (1979) it is seen that lower bounds for $\log\mu(sU)$, $s > 0$ constitute a broad spectrum of functions. Let us consider a special case discussed also by Goodman and Kuelbs (1988), namely $\log\mu(sU) \ge -Cs^{-\vartheta}$ for some positive constants C, ϑ–cf. Example 4.5 in Hoffmann-Jørgensen, Shepp and Dudley (1979). Then the inequality $\exp\{-Cs^{-\vartheta}\} \le \mu(sU) = \Phi(\alpha(s)) \le \exp\{-\alpha^2(s)/2\}$ valid for $s > 0$ near zero implies that $\alpha^2(s) \le 2Cs^{-\vartheta}$, i.e. $0 > \alpha(s) \ge -(2C)^{1/2}s^{-\vartheta/2}$. Hence, in this case the best possible rate of convergence is of order $\varepsilon(|t|) = d(LL|t|)^{-(2+\vartheta)/2\vartheta}$ for a constant $d = d(C,q,\delta) > 0$ sufficiently close to zero. The same rate of convergence was obtained by Goodman and Kuelbs (1988) in their Theorem 4 for the functional LIL in a separable Banach space and $q = 1$.

On the basis of our results we are able to describe as well the rate of convergence in the local LIL. Let

$$U_t = (1+\delta/2)(2q|t|LL|t|)^{1/2}\cdot(\mathbf{K}+\varepsilon(1/|t|)U)$$

and let $\varepsilon'(s) = \varepsilon(1/s)$ satisfy (4.11) in a neighbourhood of zero. Then, replacing ε by ε', for a fixed $0 < b < \infty$ and $B = <0,b)^q \subset R_+^q$ we get (4.1), whence again (4.12) with $\varepsilon = \varepsilon'$ follows. This result seems to be new even for $q = 1$. Furthermore, we can treat in an analogous way sets for which some coordinates of points may grow to infinity, while the other tend to zero. For simplicity, consider the case $q=2$ and $B=<0,b>\times<a,\infty)$. Put

$$U_t=(1+\delta/2)(2q|t|LL(t_2/t_1))^{1/2}\cdot(\mathbb{K}+\varepsilon(t_2/t_1)U)$$

and assume (4.11) with $LL(t_2/t_1)$ and $\varepsilon(t_2/t_1)$. It can be easily verified that now

$$\int_B\frac{1}{|t|}\cdot P[\Delta X(<0,t))\notin(1+\delta/4)(2q|t|LL(t_2/t_1))^{1/2}(\mathbb{K}+\varepsilon(t_2/t_1)U)]dt<\infty\ ,$$

and thus

$$P\Big[\liminf_{t_1\xrightarrow{B'}0,\ t_2\xrightarrow{B'}\infty}[\Delta X(<0,t))\in(1+\delta)(2q|t|LL(t_2/t_1))^{1/2}(\mathbb{K}+\varepsilon(t_2/t_1)U)]\Big]\ .$$

Observe that in the last case $|t|=t_1\cdot t_2$ may stay bounded. Under the additional supposition $\log\mu(sU)\geq -Cs^{-\vartheta}$ the best possible rate of convergence in the local LIL and "mixed" LIL is the same as previously (with $|t|$ replaced by t_2/t_1 in the last case). Evidently, letting $\delta\to 0$ through an arbitrary countable set of numbers we infer that

$$P[\mathcal{C}(\Delta X(<0,t))/(2q|t|LL|t|)^{1/2})\subseteq\mathbb{K}]=1\ ,$$

where $\mathcal{C}(\cdot)$ denotes the set of cluster points as $t\xrightarrow{B'}0$ or ∞.

Remarks. 1) It is not known if there exists a separable version of the process X. However, it is possible to construct the continuous multiparameter Brownian sheet with values in a locally pseudoconvex metric linear space.

2). Since we consider μ-Brownian surface generated by a Gaussian Radon measure μ in $\mathbb{G}$, the Lindelöf property of $\mathbb{G}$ is in fact a superfluous assumption (cf. the Remark concluding Section 3).

Taking into account the above results the considered so far families of globular sets may be viewed as upper classes of sets for increments $\Delta X(<0,t))$. The introduced thus notion of an upper class of sets is a natural generalization of an upper class of functions.

As a corollary arising from Theorem 4.2 we can obtain too a more precise description of the set of cluster points of $\Delta X(<0,t))$. For this purpose we shall investigate how often the increments $\Delta X(<0,t))$ visit neighbourhoods of various points.

Definition 4.5. Let $\mathcal{U}=\{U_t,t\in R^q_+\}$ be a regularly varying family of globular neighbourhoods of zero in a T_0 topological Abelian group $\mathbb{G}$. We say that $\mathcal{U}$ is absolutely regularly varying, if it satisfies the condition

(i) for all $j\geq 0$ and $t,s\in R^q_+$, $s<t$,

$$U_s(-j)\subseteq U_t(-j)\ ,$$

of Definition 4.1, and in addition

(ii) for every $j\geq 0$ and $0<\varepsilon<1$, $\varepsilon\in R^q_+$ there exists $0<\rho<\varepsilon$, $\rho\in R^q_+$, such that

$$\pm U_{\rho t}(-j)\pm U_{\rho t}(-j)\subseteq U_{\varepsilon t}(-j)\text{ for all }t\in R^q_+\setminus\partial R^q_+\ ,$$

and

(iii) for each $j \geq 0$, $0 < \varepsilon < 1$ and $\beta > 1$, $\varepsilon, \beta \in R^q_+$ there can be found $a = (a, \ldots, a) \in R^q_+$, $a > 1$ so large that

$$U_{\beta(\lambda t + \lambda^c a t)}(-j) \subseteq U_{\varepsilon a t}(-j)$$

for all $t \in R^q_+ \setminus \partial R^q_+$ and $\lambda \in \Lambda$, $\lambda \neq 0$.

(The symbol Λ denotes here, as before, the set of all projections in R^q onto various hyperplanes of the system of coordinates, and $\lambda^c = \mathrm{Id} - \lambda$).

Corollary 4.6. *Let X be a symmetric additive stochastic process and let $\mathcal{U}$ be an absolutely regularly varying family of globular neighbourhoods of zero in $\mathbb{G}$. Assume that X and $\mathcal{U}$ are chosen in such a way that (4.1) is fulfilled. Moreover, suppose that there exists a mapping $z : R^q_+ \to \mathbb{G}$ satisfying the condition: for each $0 < \rho \in R^q_+$ and sufficiently large $a \in R^q_+$, say $a \geq a_\rho$,*

$$\sum_{k \in \mathbb{J}} P[\Delta X(< a^k, a^{k+1})) \in z(a^{k+1}) + U_{\rho a^{k+1}}] = \infty \,. \tag{4.13}$$

Then for every $0 < \eta \in R^q_+$ we have

$$P\Big[\limsup_{t \xrightarrow{B'} 0 \text{ or } \infty} [\Delta X(< 0, t)) \in z(t) + U_{\eta t}]\Big] = 1 \,, \tag{4.14}$$

where $B' = \{a^{k+1} : k \in \mathbb{J}\}$.

Proof. The proof is immediate. For a given $\eta > 0$, $\eta \in R^q_+$ choose $0 < \rho \in R^q_+$ such that $\pm U_{\rho t} \pm \cdots \pm U_{\rho t}$ (2^q times) $:= \pm 2^q U_{\rho t} \subseteq U_{\eta t}$. Next fix $\beta > 1$, $\beta \in R^q_+$ and find $a = (a, \ldots, a) > 1$, satisfying (iii) with $j = 0$ and $\varepsilon = \rho$. By Theorem 4.2 we get $\Delta X(< 0, \lambda a^k + \lambda^c a^{k+1})) \in U_{\beta(\lambda a^k + \lambda^c a^{k+1})} \subseteq U_{\rho a^{k+1}}$ a.s. for all but finitely many $k \in \mathbb{J}$. Since the events $C_k = [\Delta X(< a^k, a^{k+1})) \in z(a^{k+1}) + U_{\rho a^{k+1}}]$ are independent, on account of the Borel-Cantelli lemma $P[C_k \text{ i.o. } k \in \mathbb{J}] = 1$. Hence it follows that with probability 1, $\Delta X(< 0, a^{k+1})) = \Delta X(< a^k, a^{k+1})) - \sum_{\lambda \neq 0} (-1)^{\operatorname{sgn} \lambda} \cdot \Delta X(< 0, \lambda a^k + \lambda^c a^{k+1})) \in z(a^{k+1}) \pm 2^q U_{\rho a^{k+1}} \subseteq z(a^{k+1}) + U_{\eta a^{k+1}}$ for infinitely many $k \in \mathbb{J}$, where $\operatorname{sgn} \lambda = p$ for $\lambda t = (0, \ldots, 0, t_{i_1}, 0, \ldots, 0, t_{i_p}, 0, \ldots, 0)$.

To describe the behaviour of Brownian surfaces discussed in Example 5 we need a simple estimate, which is a modification of Lemma 3.2 by Goodman (1988).

Lemma 4.7. *Let μ be a centered Gaussian measure with variance parameter 1 generated by the RKHS $\mathbb{H}$ in a locally convex linear topological space $\mathbb{G}$. Furthermore, let U be an open convex neighbourhood of zero in $\mathbb{G}$ and let p be the Minkowski's functional of U. Then for an arbitrary vector $h \in \mathbb{H}$, $h \neq 0$ and every $\varepsilon > 0$ there exists a constant $0 < C(\varepsilon) < \infty$ such that*

$$\mu(h + \varepsilon U) \geq C(\varepsilon) \Big\{ \Phi(-|h|[1 - \varepsilon/2p(ih)]) - \Phi(-|h|[1 + \varepsilon/2p(ih)]) \Big\} \,.$$

Proof. Let $e = h/|h|$, $Px =< e, x > ie$ and $Qx = x - Px$, where $< e, \cdot >= \Psi^{-1}e \in \mathbb{G}^2(\mu)$ (see Example 5). The generalized Cameron–Martin formula (cf. Corollary 2.1 by Borell (1976)) yields

$$\mu(h + \varepsilon U) = \int_{\varepsilon U} \exp\{|h| < e, x > -|h|^2/2\} d\mu(x) \geq$$

$$\geq \int_{\{x: p(Px) < \varepsilon/2\} \cap \{x: p(Qx) < \varepsilon/2\}} \exp\{|h| < e, x > -|h|^2/2\} d\mu(x) .$$

Observe that Px and Qx are Gaussian random elements on $(\mathbb{G}, \mathcal{G}, \mu)$ taking values in $\mathbb{G}$ with distributions μ_P and μ_Q generated by the canonical Gauss cylinder set measures on orthogonal subspaces of $\mathbf{H}$, namely $\mathbf{H}_e = \mathrm{Lin}\{e\}$ and $\mathbf{H}_e^\perp = \{h \in \mathbf{H} : h \perp e\}$ respectively. Hence it follows that Px and Qx are independent, and consequently

$$\mu(h + \varepsilon U) \geq \mu\{x : p(Qx) < \varepsilon/2\} \int_{\{x: p(Px) < \varepsilon/2\}} \exp\{|h| < e, x > -|h|^2/2\} d\mu(x) .$$

Since $< e, \cdot >$ is a standard normal random variable on $(\mathbb{G}, \mathcal{G}, \mu)$ and $\{x : p(Px) < \varepsilon/2\} = \{x : | < e, x > | p(ih)/|h| < \varepsilon/2\}$, the last integral is equal to

$$\int_{\{-\varepsilon|h|/2p(ih) < s < \varepsilon|h|/2p(ih)\}} (2\pi)^{-1/2} \exp\{-(s - |h|)^2/2\} ds =$$

$$= \Phi(-|h| \, [1 - \varepsilon/2p(ih)]) - \Phi(-|h| \, [1 + \varepsilon/2p(ih)]) .$$

Moreover, for each $h \in H_e^\perp$, $\mu_Q(\cdot - h)$ is a Gaussian measure absolutely continuous with respect to μ_Q and $\mu_Q(\mathbf{H}_e^\perp + (\varepsilon/2)U) = 1$. Therefore $\mu\{x : p(Qx) < \varepsilon/2\} = \mu_Q\{(\varepsilon/2)U\} \geq C(\varepsilon) > 0$.

Example 6. Let X be a μ–Brownian surface in a real locally convex linear topological (Lindelöf) space $\mathbb{G}$ as described in Example 5. Choose an arbitrary vector $h \in \mathbf{K}$, $h \neq 0$ and an open convex neighbourhood U of zero in $\mathbb{G}$. Put $z(t) = (2q|t|LL|t|)^{1/2}h$ and

$$U_t = (1 + \delta/2)(2q|t|LL|t|)^{1/2} \cdot (\mathbf{K} + \varepsilon(|t|)U) , \quad \text{where } \varepsilon(s) \equiv \theta > 0 .$$

It can be easily seen that now $\alpha((1+\delta/4)(2qLLs)^{1/2} \cdot \varepsilon(s)) \to \infty$ as $s \to \infty$, i.e. (4.11) is fulfilled, and thus also (4.1) with the same $U_t(-2q-1)$ as in Example 5. Observe next that in view of Corollary 4.1 by Borell (1976), $\mathbf{K} \subset \eta U$ for some $0 < \eta < \infty$, because there always exists an open convex set $V \ni 0$ such that $V - V \subseteq U$, and plainly $\mu(V) > 0$. Hence

$$(1 + \delta/2)\theta(2q|t|LL|t|)^{1/2}U \subseteq U_t \subseteq d(2q|t|LL|t|)^{1/2}U$$

for some $0 < d < \infty$, and thus to verify (4.13) it is enough to show that for every $0 < \rho \in R_+^q$ and $P_k = P[\Delta X(< a^k, a^{k+1})) \in (2q|a^{k+1}|LL|a^{k+1}|)^{1/2}h +$

$|\rho|(2q|a^{k+1}|LL|a^{k+1}|)^{1/2}U]$, we have $\sum_{k\in J} P_k = \infty$ provided $a \geq a_\rho$ is sufficiently large. Now by Lemma 4.7,

$$\begin{aligned} P_k &= \mu((a/a-1)^{q/2}(2qLL|a^{k+1}|)^{1/2}h + |\rho|(a/a-1)^{q/2}(2qLL|a^{k+1}|)^{1/2}U) \geq \\ &\geq C(\rho,a,q)\{\Phi(-|h|(a/a-1)^{q/2}(2qLL|a^{k+1}|)^{1/2}[1-|\rho|/2p(ih)]) \\ &\quad - \Phi(-|h|(a/a-1)^{q/2}(2qLL|a^{k+1}|)^{1/2}[1+|\rho|/2p(ih)])\}\ . \end{aligned}$$

Using the inequality $\Phi(-a+\delta) - \Phi(-a-\delta) \geq C\exp\{-(a-\delta)^2/2\}/(a+\delta)$ valid for $0 < a, \delta \to \infty$, we conclude that

$$\begin{aligned} P_k &\geq C'\exp\{-|h|^2(a/a-1)^q q(1-|\rho|/2p(ih))^2 LL|a^{k+1}|\}/(LL|a^{k+1}|)^{1/2} \geq \\ &\geq C''/(\log|k_1+\cdots+k_q+q|)^{1/2}\cdot|k_1+\cdots+k_q+q|^{q'}, \end{aligned}$$

where $q' = |h|^2(a/a-1)^q q(1-|\rho|/2p(ih))^2$ and $|k_1+\cdots+k_q+q| \geq e$. Notice next that $\sum_{i=r}^{\infty} 1/(i+j)^p \geq C(p)/(r+j)^{p-1}$, $p>1$. Thus, taking $a=(a,\dots,a)$ so large that $q' < q$ we obtain $\sum_{k\in J} P_k = \infty$, whenever $B = <a,\infty)^q$ or $<0,b>^q$. For the "mixed" LIL we must use $LLa^{|k_1+1|+\cdots+|k_q+1|}$ instead of $LL|a^{k+1}|$ to get the same conclusion, but in any case B cannot have a coordinate bounded away from 0 and ∞ simultaneously. On account of Corollary 4.6, choosing appropriate $0 < \varepsilon \in R_+^q$ for a given $\tau > 0$ so that $d(2q|\varepsilon t|LL|\varepsilon t|)^{1/2} \leq \tau(2q|t|LL|t|)^{1/2}$ we obtain

$$P[\Delta X(<0,t))/(2q|t|LL|t|)^{1/2} \in h + \tau U \text{ i.o. } t\in B'] = 1\ ,$$

$B' = \{a^{k+1} : k \in \mathbf{J}\}$. If $h=0$ then the same assertion is true as well, because then $P_k \geq C(\rho,a,U) > 0$.

In the case $|h| < 1$ we can obtain even a better result. Namely, let $\varepsilon(|t|) = \theta(LL|t|)^{-\kappa}$, where $\theta > 0$ and $0 < \kappa < 1/2$ are constants. Put

$$P_k' = P[\Delta X(<a^k, a^{k+1})) \in z(a^{k+1}) + |\rho|(2q|a^{k+1}|)^{1/2}((LL|a^{k+1}|)^{-\kappa+1/2}U]\ .$$

Then we have

$$\begin{aligned} P_k' &\geq C'\exp\{-|h|^2(a/a-1)^q q(1-|\rho|/2p(ih)(LL|a^{k+1}|)^\kappa)^2 LL|a^{k+1}|\}/(LL|a^{k+1}|)^{1/2} = \\ &= C'\exp\{-|h|^2(a/a-1)^q q(1-|\rho|/p(ih)(LL|a^{k+1}|)^\kappa + \\ &\qquad + |\rho|^2/4p^2(ih)(LL|a^{k+1}|)^{2\kappa})LL|a^{k+1}|\}/(LL|a^{k+1}|)^{1/2} \geq \\ &\geq C''\exp\{-|h|^2(a/a-1)^q qLL|a^{k+1}|\}/(LL|a^{k+1}|)^{1/2}\ . \end{aligned}$$

Taking a so large that $|h|^2(a/a-1)^q < 1$, we conclude that $\sum_{k\in J} P_k' = \infty$. Hence, in view of Corollary 4.6 the assertion

$$P[\Delta X(<0,t))/(2q|t|LL|t|)^{1/2} \in h + \tau[\mathbf{K} + U/(LL|t|)^\kappa] \text{ i.o. } t\in B'] = 1$$

for $B' = \{a^{k+1} : k\in \mathbf{J}\}$ follows, provided $0 < \eta \in R_+^q$ is chosen so that

$$(1+\delta/2)(2q|\eta t|LL|\eta t|)^{1/2}\cdot(\mathbf{K} + \varepsilon(|\eta t|)U) \subseteq \tau(2q|t|LL|t|)^{1/2}\cdot(\mathbf{K} + U/(LL|t|)^\kappa)\ .$$

The similar argument can be applied to the "mixed" LIL. Obviously, the Remark 2 following Example 5 relates also to the case considered in Example 6.

Remark. The results of Examples 5 and 6 can be treated as an extension to the law of the iterated logarithm, which was known till now only for random elements or stochastic processes with values in metrizable linear spaces (c.f. LePage (1972), Kuelbs and LePage (1973), Borell (1976) § 10, and Jurlewicz (1987)).

Now we turn to the problem of necessity of our integral test for statements like (4.2). The investigations are continued ab contrario, i.e. we seek for conditions ensuring that

$$P[\liminf_{t \xrightarrow{B'} 0 \text{ or } \infty} (\Delta X(<0,t)) \in U_t(-j))] = 0 \,,$$

or equivalently

$$P[\limsup_{t \xrightarrow{B'} 0 \text{ or } \infty} (\Delta X(<0,t)) \notin U_t(-j))] = 1 \,.$$

On that occasion we find an analogue of the lower class of functions for increments of the process X. As was indicated by Goodman and Kuelbs (1988) in the discussion following their Corollary 2, about the "lower class statements" concerning strong limit theorems in infinite dimensional spaces very little is known except of certain very special cases. However, we are able to supply some information. The first straightforward theorem for general additive processes gives a result almost opposite to (4.2), but under the assumption which is rather uncomparable with (4.1). We have to impose in addition somewhat more restrictive conditions on the family U.

As above, let $\lambda^c = \mathrm{Id} - \lambda$ for a given projection $\lambda \in \Lambda$ of R_+^q into a hyperplane of the system of coordinates. Note that for $s < t$, the points $\lambda s + \lambda^c t$, $\lambda \in \Lambda$ represent various vertices of the rectangle $< s,t)$.

Definition 4.8. An extended regularly varying family of globular sets is a family $\mathcal{U} = \{U_{<s,t)}, < s,t) \subseteq R_+^q\}$ of globular sets indexed by all bounded rectangles of the form $< s,t) \subseteq R_+^q$, which satisfies the following conditions:

(i)′ $< u,v) \subseteq < s,t)$ implies that $U_{<u,v)}(-j) \subseteq U_{<s,t)}(-j)$ for every $j \geq 0$, and

(ii)′ given any q-tuple $\alpha \in R_+^q$, $0 < \alpha < 1$ and $j \geq 0$, there can be selected $\beta \in R_+^q$, $\beta > 1$ such that for each $t \in R_+^q \setminus \partial R_+^q$

$$\sum_{\lambda \in \Lambda} (-1)^{\operatorname{sgn} \lambda} U_{\lambda t + \lambda^c \alpha \beta t}(-j) \subseteq U_{<t,\beta t)}(-j) \,, \tag{4.15}$$

where $\operatorname{sgn} \lambda = p$ for $\lambda t = (0,\ldots,0,t_{i(1)},0,\ldots,0,t_{i(p)},0,\ldots,0)$.

(To simplify the writing $U_{<0,t)}$ is denoted here and in the sequel by U_t.)

Observe that if $0 < \alpha' < \alpha < 1$, then (4.15) is fulfilled for α' with the same β, but if $\alpha < \alpha'' < 1$, then (4.15) need not hold for α'' and β. Moreover, $\beta = \beta(\alpha, j)$ satisfying condition (ii)′ of Definition 4.8 is not specified uniquely. Let $E_\alpha(j)$ be the set consisting of all these q-tuples $\beta = \beta(\alpha, j)$ for which (4.15) is true.

Let us consider the parameter set $B = A_1 \times \cdots \times A_q \subseteq R_+^q$ being the product of one-dimensional intervals $A_i \subseteq R_+$. Suppose that B is the rectangle determined by the lower-left vertex T and upper-right vertex S. Then, given any $a = (a_1,\ldots,a_q) \in$

R^q_+, $a > 1$, denote by $B(a)$ the rectangle in R^q_+ having lower-left vertex $T/a = (T_1/a_1, \ldots, T_q/a_q)$ and upper-right vertex $Sa = (S_1a_1, \ldots, S_qa_q)$. It can be easily noted that if $< a^k, a^{k+1} > \cap B \neq \emptyset$ for some $k \in \mathbb{Z}$, then $\lambda a^k + \lambda^c a^{k+1} \in B(a)$ for every $\lambda \in \Lambda$. Next, by analogy to B', define the set of indices $B'(a) = \{a^k : a^k \in B(a)\}$. In contradiction with the case of lim inf, investigating lim sup D_t for some events D_t we do not require that $t \in B'$, but $t \in B'(a)$. Thus the limit $\limsup_{t \xrightarrow{B'(a)} 0 \text{ or } \infty} D_t$ is given by

$$\limsup_{t \xrightarrow{B'(a)} 0 \text{ or } \infty} D_t = \bigcap_{0<T_1<T_2} \bigcup_{\substack{t \notin <T_1, T_2> \\ t \in B'(a)}} D_t .$$

Theorem 4.9. *Let X be a symmetric additive stochastic process taking values in a T_0 topological Abelian group $\mathbb{G}$ with the σ-field $\mathcal{G}$ and let $\mathcal{U}$ be an extended regularly varying family of globular sets. If for some $0 < \alpha < 1$ there exists $a \in E_\alpha(2q)$ such that*

$$(4.16) \qquad J_B := \int_B \frac{1}{|t|} \cdot \inf_{0 \leq s < t} P[\Delta X(< s, t)) \notin U_{<s, at)}] \, dt = \infty ,$$

then for each $0 < \alpha' \leq \alpha$,

$$(4.17) \qquad P\Big[\limsup_{t \xrightarrow{B'(a)} 0 \text{ or } \infty} [\Delta X(< 0, t)) \notin U_{\alpha' t}(-2q)]\Big] = 1 .$$

Proof. Obviously, it suffices to prove (4.17) for $\alpha' = \alpha$. For a fixed $a \in E_\alpha(2q)$ satisfying (4.16), define the sets of indices $\mathbf{J} = \{k \in \mathbf{Z}^q :< a^k, a^{k+1} > \cap B \neq \emptyset\}$ and $\mathbf{J}' = \{k \in \mathbf{Z}^q : a^k \in B(a)\}$. Then $B'(a) = \{a^k : k \in \mathbf{J}'\}$. First we will show that

$$(4.18) \qquad P\Big[\bigcap_{\substack{j<n \\ j,n \in \mathbf{J}'}} \bigcup_{\substack{t \notin <a^j, a^n> \\ t \in B'(a)}} [\Delta X(< 0, t)) \notin U_{\alpha t}(-2q)]\Big] = 1 .$$

It will be shown below that (4.17) follows easily from (4.18). Consider the independent events $B_k = \{\Delta X(< a^k, a^{k+1})) \notin U_{<a^k, a^{k+1})}(-2q)\}$. On account of Lemma 3.2 and condition (i)' of Definition 4.8, for $t \in (a^k, a^{k+1} >$ we have

$$(4.19) \qquad \begin{aligned} P[B_k] &\geq 4^{-q} P[\Delta X(< a^k, t)) \notin U_{<a^k, a^{k+1})}] \geq \\ &\geq 4^{-q} P[\Delta X(< a^k, t)) \notin U_{<a^k, at)}] \geq \\ &\geq 4^{-q} \inf_{0 \leq s < t} P[\Delta X(< s, t)) \notin U_{<s, at)}] . \end{aligned}$$

Hence

$$(4.20) \qquad \begin{aligned} (\ln a)^q P[B_k] &= \int_{<a^k, a^{k+1}>} \frac{1}{|t|} P[B_k] \, dt \geq \\ &\geq 4^{-q} \int_{<a^k, a^{k+1}>} \frac{1}{|t|} \inf_{0 \leq s < t} P[\Delta X(< s, t)) \notin U_{<s, at)}] \, dt , \end{aligned}$$

and in consequence, by (4.16),

$$\sum_{k\in \mathbf{J}} P[B_k] \geq (4\ln a)^{-q} J_B = \infty \,. \tag{4.21}$$

Since B_k are independent events, from the Borel–Cantelli lemma it follows now that $P[B_k \text{ i.o. } k \in \mathbf{J}] = 1$. In other words,

$$P\Big[\bigcap_{\substack{j<n\\ j,n\in \mathbf{J}}} \bigcup_{\substack{k\notin <j,n>\\ k\in \mathbf{J}}} B_k\Big] = 1 \,.$$

Notice next that in view of (4.15) the event B_k implies that at least one of the events $\{\Delta X(<0, \lambda a^k + \lambda^c a^{k+1})) \notin U_{\lambda a^k + \lambda^c \alpha a^{k+1}}(-2q)\}$, $\lambda \in \Lambda$ holds. Furthermore, the points of the form $\lambda a^k + \lambda^c a^{k+1}$, $\lambda \in \Lambda$, are various vertices of the rectangle $< a^k, a^{k+1})$, i.e. $\lambda a^k + \lambda^c a^{k+1} = a^{k'}$ for some $k' = k'(k, \lambda) \in \mathbf{J}'$ provided $k \in \mathbf{J}$. Hence we infer that with probability 1 the events $C_k = \{\Delta X(<0, a^k)) \notin U_{\alpha a^k}(-2q)\}$ occur for infinitely many indices $k \in \mathbf{J}'$, because $U_{\lambda \alpha a^k + \lambda^c \alpha a^{k+1}}(-2q) \subseteq U_{\lambda a^k + \lambda^c \alpha a^{k+1}}(-2q)$. The last assertion implies easily (4.18).

Finally, note that for any events D_t and arbitrary $T_1, T_2 \in R^q_+$, $0 < T_1 < T_2$ we can find $j, n \in \mathbf{J}'$, $j < n$, such that

$$< T_1, T_2 >^c \cap B'(a) \supseteq < a^j, a^n >^c \cap B'(a) \,.$$

Consequently,

$$\bigcup_{\substack{t\notin <T_1,T_2>\\ t\in B'(a)}} D_t \supseteq \bigcup_{\substack{t\notin <a^j,a^n>\\ t\in B'(a)}} D_t \supseteq \bigcap_{\substack{j<n\\ j,n\in \mathbf{J}'}} \bigcup_{\substack{t\notin <a^j,a^n>\\ t\in B'(a)}} D_t \,,$$

and thus

$$\bigcap_{\substack{j<n\\ j,n\in \mathbf{J}'}} \bigcup_{\substack{t\notin <a^j,a^n>\\ t\in B'(a)}} D_t \subseteq \bigcap_{0<T_1<T_2} \bigcup_{\substack{t\notin <T_1,T_2>\\ t\in B'(a)}} D_t \,.$$

Therefore, the relation $P\Big[\bigcap_{\substack{j<n\\ j,n\in \mathbf{J}'}} \bigcup_{\substack{k\notin <j,n>\\ k\in \mathbf{J}'}} C_k\Big] = 1$ implies (4.17) via (4.18), which concludes the proof.

To present an example of an extended regularly varying family of globular sets we introduce the notion of a function with regularly varying increments. We say that $g : R^q_+ \to R_+$ has regularly varying increments, if g satisfies conditions (a), (b) of a regularly increasing function, and moreover

(d) $\Delta g(< s, t)) > 0$ for every nonempty rectangle $< s, t) \subset R^q_+$;

(e) $\lim_{\substack{1 \geq \alpha \nearrow 1\\ \alpha\in R^q_+}} \inf_{t\in R^q_+ \setminus \partial R^q_+} \left(\frac{g(\alpha t)}{g(t)} - 1\right) = 0$;

and

(f) for each $\alpha \in R^q_+$, $0 < \alpha < 1$ there exists $\beta \in R^q_+$, $\beta > 1$ such that for all $t \in R^q_+ \setminus \partial R^q_+$,

$$\sum_{\lambda\in\Lambda} \Delta g(< 0, \lambda t + \lambda^c \alpha \beta t)) \leq \Delta g(< t, \beta t)) \,.$$

The simplest example of a function satisfying the above requirements is a function which generates the Lebesgue measure, namely $g(t) = |t|$. Evidently, conditions (a), (b), (d) and (e) are then satisfied, so that it is enough to verify (f). Since $g|_{\partial R^q_+} = 0$ and the vertices of the rectangle $< t, \beta t)$ are of the form $\lambda t + \lambda^c \beta t$, (f) will follow easily if we prove that

$$2 \sum_{\lambda \neq 0} g(\lambda t + \lambda^c \beta t) + g(\alpha \beta t) \leq g(\beta t) ,$$

or

$$2 \sum_{\lambda \neq 0} |\lambda 1 + \lambda^c \beta| \cdot |t| \leq |\beta|(1 - |\alpha|)|t| .$$

If $\beta = (\beta, \dots, \beta)$, then the above inequality is a consequence of the relation $2 \cdot 2^q \cdot \beta^{q-1} \leq \beta^q(1 - |\alpha|)$, thus it suffices to take $\beta \geq 2 \cdot 2^q/(1 - |\alpha|)$. A similar argument shows that $g(t) = |t|^r$, $r > 0$ possesses regularly varying increments. Furthermore, since

$$\frac{\partial^q(|t|LL|t|)^{1/2}}{\partial t_1 \dots \partial t_q} \geq \frac{\partial^q |t|^{1/2}}{\partial t_1 \dots \partial t_q} > 0$$

for $t \geq b = (b, \dots, b) \in R^q_+$ sufficiently large, $g(t) = (|t|LL|t|)^{1/2}$ has also regularly varying increments on $B = < b, \infty)^q$.

Example 7. Let U be a symmetric open convex neighbourhood of zero in a linear topological Lindelöf space $\mathbb{G}$ and let p be the Minkowski's functional for U. Take an arbitrary function $g : R^q_+ \to R_+$ with regularly varying increments and put

$$U_{<s,t)} = \{x \in \mathbb{G} : p(x) < \Delta g(< s, t))\} , \quad < s, t) \subset R^q_+ .$$

Then it can be easily seen that $\mathcal{U} = \{U_{<s,t)}, < s, t) \subset R^q_+\}$ is an extended regularly varying family of globular sets. Moreover, assuming that the process X for our family of globular sets satisfies (4.16) with a number $a \in E_\alpha(2q)$, for each $\varepsilon \in R_+$ such that $0 < \varepsilon < f(\alpha) = \inf\{g(\alpha t)/g(t) : t \in R^q_+ \setminus \partial R^q_+\}$, we have

$$P\Big[\limsup_{t \xrightarrow{B'(a)} 0 \text{ or } \infty} [p(\Delta X(< 0, t)) \geq \varepsilon g(t)]\Big] = 1 .$$

Indeed, by Theorem 4.9, (4.17) holds for $U_{\alpha t}(-2q)$. Moreover, in the proof of Proposition 2.2 $U_{\alpha t}(-2q)$ can be chosen in such a way that for an arbitrarily fixed $0 < b < 1$, $\{x \in \mathbb{G} : p(x) < bg(\alpha t)\} \subseteq U_{\alpha t}(-2q)$ for all $t \in R^q_+ \setminus \partial R^q_+$. Using property (e) of g we get $g(\alpha t) \geq f(\alpha)g(t) > g(t)(f(\alpha) + \varepsilon)/2$ for all $t \notin \partial R^q_+$, and thus taking $b \geq 2\varepsilon/(f(\alpha) + \varepsilon)$ we obtain the desired assertion.

By analogy to Corollary 4.3 we shall formulate also a "desymmetrized" version of Theorem 4.9.

Corollary 4.10. *Let X be a stochastic process satisfying the assumptions of Theorem 4.9 but not necessarily symmetric and let $\mathcal{U} = \{U_{<s,t)} :< s, t) \subset R^q_+\}$ be an extended regularly varying family of globular sets such that $W_{<s,t)} = U_{<s,t)} - U_{<s,t)}$ are Baire sets (and hence globular) satisfying $W_{<s,t)}(-j) = U_{<s,t)}(-j) - U_{<s,t)}(-j)$,*

$< s,t) \subset R^q_+$, $j \geq 1$. Furthermore, assume that for some $0 < \alpha < 1$ there exists $a \in E_\alpha(2q)$ such that

$$(4.22)\quad J'_B := \int_B \frac{1}{|t|} \inf_{0 \leq s < t} P \times P'[\Delta X(< s,t)) - \Delta X'(< s,t)) \notin W_{<s,at)}]\, dt = \infty .$$

Then there can be found a deterministic function $z : R^q_+ \to G$ such that for an arbitrary $\alpha' \in R^q_+$, $0 < \alpha' \leq \alpha$,

$$(4.23)\qquad P\Big[\limsup_{t \xrightarrow{B'(s)} 0 \text{ or } \infty} [\Delta X(< 0,t)) \notin z(t) + W_{\alpha' t}(-2q)]\Big] = 1 .$$

Proof. The proof of (4.23) goes along the lines that of Corollary 4.3 with an application of Theorem 4.9 instead of Theorem 4.2, so we omit it.

It can be seen that our Theorem 4.2 was inspired by Theorems 1, 5 and Remark 1 a) of Chapter VI, §6 given by Gikhman and Skorohod (1965) (which are due in fact to Khintchine and Gnedenko). Therefore one may ask a question why there are not obtained any generalizations of Theorems 2, 6 and Remark 1 b), Chapter VI, §6 in Gikhman and Skorohod's (1965) book concerning the lower class of functions? It is worth mentioning here that we cannot provide analogous statements, because their proofs are incorrect. To see this, consider the below example.

Example 8. (A counterexample) Let ξ be a real valued one–parameter standard Brownian motion. Choose $1 < a \in R_+$ so close to 1 that

$$P[\xi(1) > 1/\sqrt{a-1}] \leq (1 - \Phi(1))/4 ,$$

where Φ is the standard normal distribution function. Next, for a fixed $k \in \{1, 2, \ldots\}$ define $g(a^k) = \varepsilon > 0$ with $\varepsilon < \sqrt{a} - 1$, $g(a^{k+1}) = \sqrt{a^{k+1}}$ and $B_k = \{\xi(a^{k+1}) - \xi(a^k) > g(a^{k+1}) - g(a^k)\}$. Estimating $P[B_k]$ by analogy to the method employed by Gikhman and Skorohod (1965), Chapter VI, §6, we would obtain

$$P[B_k] \geq \int_{-\infty}^{g(a^k)} P[\xi(a^{k+1}) - z > g(a^{k+1}) - g(a^k)] P[\xi(a^k) \in dz] \geq$$
$$\geq P[\xi(a^{k+1}) > g(a^{k+1})] P[\xi(a^k) < \varepsilon] \geq (1 - \Phi(1))/2 .$$

On the other hand we have

$$P[B_k] = P[\xi(a^k(a-1)) > \sqrt{a^{k+1}} - \varepsilon] \leq P[\xi(1) > 1/\sqrt{a-1}] \leq (1 - \Phi(1))/4 ,$$

which leads to a contradiction. Thus the lower bound for $P[B_k]$ given by Gikhman and Skorohod is not valid. This is a consequence of the fact that $\xi(a^{k+1}) - z$ and $\xi(a^k) - \xi(0)$ are not independent.

Conditions (4.1) and (4.16) describing upper and lower classes of sets for increments of an additive group–valued process X are not comparable. From this point of view (4.16) does not seem completely satisfactory. However, assuming homogeneity of increments of the process X we are able to give lower class statement under an assumption that is almost alternative to (4.1). First we formulate certain regularity conditions which have to satisfy globular sets.

Definition 4.11. Let $\mathcal{U} = \{U_t, t \in R_+^q\}$ be a family of globular neighbourhoods of zero in G. We say that $\mathcal{U}$ is completely regularly varying, if it satisfies conditions (i) and (ii) of Definitions 4.1 and 4.5 respectively, and
(iv) for every $j \geq 1$ and $\alpha \in R_+^q$, $0 < \alpha < 1$ there exist $\alpha' \in R_+^q$, $\alpha < \alpha' < 1$ and $\epsilon' \in R_+^q$, $\epsilon' > 0$ such that

$$U_{\epsilon' t}(-j) + U_{\alpha t}(-j) \subseteq U_{\alpha' t}(-j) \text{ for all } t \in R_+^q \setminus \partial R_+^q .$$

We can now investigate only the case when all the coordinates of t tend simultaneously either to 0 or ∞. Therefore the index set B should be slightly changed. Namely, suppose that B is of the form $\bigcup_{i\in \mathbf{Z}} < T^{(i)}, S^{(i)} >$, where $0 < T^{(i)} < S^{(i)}$, $T^{(i)} < T^{(i+1)}$ and $S^{(i)} < S^{(i+1)}$ for each $i \in \mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, $T^{(i)}, S^{(i)} \to 0$ as $i \to -\infty$ and $T^{(i)}, S^{(i)} \to \infty$ as $i \to \infty$. Evidently, if the case $t \to 0$ is only treated then B may be bounded from above, and for $t \to \infty$ the set B may be bounded away from zero. Moreover, because of the form of B, we introduce the companion parameter set $B_\alpha = \bigcup_{i\in \mathbf{Z}} < T^{(i)}, S^{(i)}/\alpha >$, taking $1 > \alpha = (\alpha, \dots, \alpha) \in R_+^q$, $S^{(i)}/\alpha = (S_1^{(i)}/\alpha, \dots, S_q^{(i)}/\alpha)$. As previously, we consider the behaviour of X only along the discrete set of indices of the form

$$B'_{\alpha,a} = \{a^{n+N(k+1)} = (a^{n_1+N_1(k_1+1)}, \dots, a^{n_q+N_q(k_q+1)}) \in B_\alpha : n, N \in \mathbf{N}^q,\ k \in \mathbf{Z}^q\}$$

with a constant $a = (a, \dots, a) \in R_+^q$, $a > 1$ depending on α, and examine the limit

$$\limsup_{t \xrightarrow{B'_{\alpha,a}} 0 \text{ or } \infty} D_t = \bigcap_{0<T_1<T_2} \bigcup_{\substack{t\notin <T_1,T_2> \\ t\in B'_{\alpha,a}}} D_t$$

for some events D_t to be specified later.

We must impose also certain conditions on distributions of the process considered. Assume the following:

1°. For each $\lambda \in \Lambda$, $\lambda \neq 0$ and a fixed $s \in R_+^q$,

$$\Delta X(< 0, \lambda t + \lambda^c s)) \to 0 \text{ in probability as } t \to 0 \; ; \tag{4.24}$$

2°. For each $j \geq 0$ and any fixed $\delta \in R_+^q$, $0 < \delta < 1$ there exists a constant $\eta > 0$ such that

$$P[\Delta X(< 0, t)) \in U_{\delta t}(-j)] \geq \eta > 0 \tag{4.25}$$

for all $t \in R_+^q$ sufficiently close to zero ;

3°. For every $\lambda, \lambda' \in \Lambda$, $\lambda < \lambda'$ and fixed arbitrarily $s, \delta \in R_+^q$, $\delta > 0$,

$$P[\Delta X(< 0, \lambda t + \lambda^c s)) \in \pm U_{\delta(\lambda' t + \lambda'^c s)}(-j)] \to 1 \text{ as } t \to \infty \, , \tag{4.26}$$

where $j \geq 0$ and $\lambda < \lambda'$ means that $\lambda^c \circ \lambda' \neq 0, \lambda^c = \mathrm{Id} - \lambda$;

4°. For each $j \geq 0$ and a fixed $\delta \in R_+^q$, $0 < \delta < 1$, there exists a constant $\eta > 0$ such that

$$P[\Delta X(<0,t)) \in U_{\delta t}(-j)] \geq \eta > 0 \tag{4.27}$$

for all sufficiently large $t \in R_+^q$.

These conditions are applied in fact only for $j = 2q$. Furthermore, it is clear that for $t \to 0$ conditions (4.26)–(4.27) are not needed, and if $t \to \infty$ then (4.24)–(4.25) are superfluous.

Theorem 4.12. *Let $X = \{X(t), t \in R_+^q\}$ be a stationary symmetric additive stochastic process taking values in a T_0 topological Abelian group $\mathbf{G}$ with the σ-field $\mathcal{G}$ and let $\mathcal{U}$ be a completely regularly varying family of globular sets. If*

$$S_B := \int_B \frac{1}{|t|} \cdot P[\Delta X(<0,t)) \notin U_t]\, dt = \infty\ , \tag{4.28}$$

then for an arbitrary $\alpha \in R_+^q$, $0 < \alpha < 1$, there exists $a \in R_+^q$, $a > 1$ sufficiently close to 1, such that

$$P\Big[\limsup_{\substack{B'_{\alpha,a}\\ t \to 0 \text{ or } \infty}} [\Delta X(<0,t)) \notin U_{\alpha t}(-2q)]\Big] = 1\ . \tag{4.29}$$

Proof. For a given $\alpha \in R_+^q$, $0 < \alpha < 1$, fix $\alpha' \in R_+^q$, $\sqrt{\alpha} < \alpha' < 1$ and $0 < \varepsilon' \in R_+^q$ satisfying condition (iv) of Definition 4.11 with α replaced by $\sqrt{\alpha} = (\sqrt{\alpha_1}, \dots, \sqrt{\alpha_q})$. Next choose $a > 1$, $a \in R_+^q$ so close to 1 that $\alpha' a \leq 1$ and then select $N_0 \in \mathbf{N}^q = \{1, 2, \dots\}^q$ so large that $\sqrt{\alpha} a^N / (a^N - 1) = (\sqrt{\alpha_1} a^{N_1} / (a^{N_1} - 1), \dots, \sqrt{\alpha_q} a^{N_q} / (a^{N_q} - 1)) \leq \sqrt{\alpha} a < 1$ for all $N \geq N_0$. Define the set of indices $\mathbf{J} = \{k \in \mathbf{Z}^q :< a^k, a^{k+1} > \cap B \neq \emptyset\}$ similarly as in the proof of Theorem 4.9. First we prove that

$$\sum_{k \in \mathbf{J}} P[\Delta X(<0, a^{k+1})) \notin U_{\alpha' a^{k+1}}(-2q)] = \infty\ .$$

Notice that $\alpha' a^{k+1} \leq t$ whenever $t \in < a^k, a^{k+1} >$, so that $U_t^c(-2q) \subseteq U^c_{\alpha' a^{k+1}}(-2q)$. Hence, by Lemma 3.2,

$$\begin{aligned} P[\Delta X(<0, a^{k+1})) \notin U_{\alpha' a^{k+1}}(-2q)] &\geq P[\Delta X(<0, a^{k+1})) \notin U_t(-2q)] \geq \\ &\geq 4^{-q} P[\Delta X(<0,t)) \notin U_t]\ . \end{aligned} \tag{4.30}$$

Consequently

$$(\ln a)^q P[\Delta X(<0, a^{k+1})) \notin U_{\alpha' a^{k+1}}(-2q)] \geq 4^{-q} \int_{<a^k, a^{k+1}>} \frac{1}{|t|} \cdot P[\Delta X(<0,t)) \notin U_t]\, dt\ ,$$

and thus

$$\sum_{k \in \mathbf{J}} P[\Delta X(<0, a^{k+1})) \notin U_{\alpha' a^{k+1}}(-2q)] \geq (4 \ln a)^{-q} S_B = \infty\ . \tag{4.31}$$

From (4.31) it follows that for an arbitrarily fixed $N \in \mathbf{N}^q$, $N \geq N_0$ there can be found $n \in \mathbf{N}^q$, $1 \leq n \leq N$ such that

$$\sum_{k \in \mathbf{J}_n} P[\Delta X(< 0, a^{n+Nk})) \notin U_{\alpha' a^{n+Nk}}(-2q)] = \infty ,$$

where $\mathbf{J}_n = \{k \in \mathbf{Z}^q : n - 1 + Nk \in \mathbf{J}\}$. The q-tuple N will be specified later. Given any $N \in \mathbf{N}^q$, define

$$A_{N,k}^{(n)} = a^{n+Nk}/(a^N - 1) = (a^{n_1+N_1k_1}/(a^{N_1} - 1), \dots, a^{n_q+N_qk_q}/(a^{N_q} - 1))$$

and

$$E_k^{(N)} = [\Delta X(< A_{N,k}^{(n)}, A_{N,k+1}^{(n)})) \notin U_{\alpha' a^{n+Nk}}(-2q)], \ k \in \mathbf{J}_n .$$

Since $A_{N,k+1}^{(n)} - A_{N,k}^{(n)} = a^{n+Nk}$, taking into account stationarity of X we obtain

$$\sum_{k \in \mathbf{J}_n} P[E_k^{(N)}] = \sum_{k \in \mathbf{J}_n} P[\Delta X(< 0, a^{n+Nk})) \notin U_{\alpha' a^{n+Nk}}(-2q)] = \infty .$$

However, $E_k^{(N)}$ are independent, thus with probability 1 infinitely many events $E_k^{(N)}$, $k \in \mathbf{J}_n$ occur. To simplify the notation and avoid writing subscripts we denote by $\{k\}$ an infinite subsequence for which $E_k^{(N)}$ hold a.s., $k \to -\infty$ or $k \to \infty$. In the next step of the proof we shall select from $\{k\}$ further subsequences having appropriate properties.

Let $\varepsilon, \rho \in R_+^q$, $\varepsilon, \rho > 0$ be chosen in such a way that $\pm 2^q U_{\varepsilon t}(-2q) \subseteq U_{\varepsilon' t}(-2q)$ and $\pm 2^q U_{\rho t}(-2q) \subseteq U_{\varepsilon t}(-2q)$ for all $t \in R_+^q \setminus \partial R_+^q$. Suppose that $k(j) \in \{k\}$ is already fixed. On account of (4.24) we can find $k(j+1) < k(j)$, $k(j+1) \in \{k\}$ such that for each $\lambda, \mu \in \Lambda$, $\lambda, \mu \neq 0$,

$$P[\Delta X(< 0, A_{N,\lambda k(j+1)+\lambda^c k(j)+1-\mu 1}^{(n)})) \notin \pm U_{\rho a^{n+Nk(j)}}(-2q)] \leq 1/2^j .$$

Then for an arbitrary $\mu \in \Lambda$, $\mu \neq 0$ we have

$$\sum_{j \geq 1} \sum_{\lambda \neq 0} P[\Delta X(< 0, A_{N,\lambda k(j+1)+\lambda^c k(j)+1-\mu 1}^{(n)})) \notin \pm U_{\rho a^{n+Nk(j)}}(-2q)] \leq$$
$$\leq 2^q \sum_{j \geq 1} 1/2^j < \infty ,$$

and hence with probability 1 for $j \geq j_1 = j_1(\omega)$ and each $\lambda, \mu \neq 0$,

$$\Delta X(< 0, A_{N,\lambda k(j+1)+\lambda^c k(j)+1-\mu 1}^{(n)})) \in \pm U_{\rho a^{n+Nk(j)}}(-2q) . \tag{4.32}$$

Similarly we can select an increasing sequence $m(j) \subset \{k\}$ with analogous properties. Indeed, let us fix $N \in \mathbf{N}^q$, $N \geq N_0$ so large that $1/(a^N - 1) = (1/(a^{N_1} - 1), \dots, 1/(a^{N_q} - 1)) \leq \rho$ and take $\delta \in R_+^q$, $\delta > 0$ satisfying the condition $\delta a^N \leq 1$. Then

$$\delta A_{N,\lambda m(j)+\lambda^c m(j+1)+1-\mu 1}^{(n)} \leq \delta A_{N,\mathrm{Id}(m(j+1))+1-\mu 1}^{(n)} \leq \rho a^{n+Nm(j+1)} ,$$

so that

$$\pm U_{\delta A^{(n)}_{N,1d(m(j+1))+1-\mu 1}}(-2q) \subseteq \pm U_{\rho a^{n+Nm(j+1)}}(-2q) .$$

In consequence, by (4.26),

$$P[\Delta X(<0, A^{(n)}_{N,\lambda m(j)+\lambda^c m(j+1)+1-\mu 1})) \notin \pm U_{\rho a^{n+Nm(j+1)}}(-2q)] \leq$$
$$\leq P[\Delta X(<0, A^{(n)}_{N,\lambda m(j)+\lambda^c m(j+1)+1-\mu 1})) \notin U_{\rho A^{(n)}_{N,1d(m(j+1))+1-\mu 1}}(-2q)] \to 0$$

as $m(j+1) \to \infty$. Therefore, given any $m(j) \in \{k\}$ there can be found $m(j+1) > m(j)$, $m(j+1) \in \{k\}$ such that for every $\lambda, \mu \in \Lambda$, $\lambda, \mu \neq 0$,

$$P[\Delta X(<0, A^{(n)}_{N,\lambda m(j)+\lambda^c m(j+1)+1-\mu 1})) \notin \pm U_{\rho a^{n+Nm(j+1)}}(-2q)] \leq 1/2^j .$$

Hence as previously we conclude that with probability 1 for $j \geq j_2 = j_2(\omega)$ and each $\lambda, \mu \in \Lambda$, $\lambda, \mu \neq 0$,

$$\text{(4.33)} \qquad \Delta X(<0, A^{(n)}_{N,\lambda m(j)+\lambda^c m(j+1)+1-\mu 1})) \in \pm U_{\rho a^{n+Nm(j+1)}}(-2q) .$$

Consider the events

$$B^{\mu}_j = [\Delta X(<A^{(n)}_{N,k(j+1)+1-\mu 1}, A^{(n)}_{N,k(j)+1-\mu 1})) \in U_{\rho a^{n+Nk(j)}}(-2q)] ,$$

and

$$C^{\mu}_j = [\Delta X(<A^{(n)}_{N,m(j)+1-\mu 1}, A^{(n)}_{N,m(j+1)+1-\mu 1})) \in U_{\rho a^{n+Nm(j+1)}}(-2q)]$$

for $\mu \in \Lambda$, $\mu \neq 0$. Observe that

$$A^{(n)}_{N,k(j)+1-\mu 1} - A^{(n)}_{N,k(j+1)+1-\mu 1} = a^{n+Nk(j)}\varphi(N,j,\mu) ,$$

where

$$\varphi(N,j,\mu) = a^{N-\mu N}[1-a^{-N(k(j)-k(j+1))}]/(a^N-1) \leq \rho a^N .$$

Consequently

$$U_{\delta\varphi(N,j,\mu)a^{n+Nk(j)}}(-2q) \subseteq U_{\rho a^{n+Nk(j)}}(-2q) \text{ for } \delta a^N \leq 1 .$$

Based on (4.25) and stationarity of X we infer that for large enough j,

$$P[B^{\mu}_j] = P[\Delta X(<0, a^{n+Nk(j)}\varphi(N,j,\mu))) \in U_{\rho a^{n+Nk(j)}}(-2q)] \geq$$
$$\geq P[\Delta X(<0, a^{n+Nk(j)}\varphi(N,j,\mu))) \in U_{\delta\varphi(N,j,\mu)a^{n+Nk(j)}}(-2q)] \geq \eta > 0 .$$

By analogy we have

$$A^{(n)}_{N,m(j+1)+1-\mu 1} - A^{(n)}_{N,m(j)+1-\mu 1} = a^{n+Nm(j+1)}\psi(N,j,\mu) ,$$

where

$$\psi(N,j,\mu) = a^{N-\mu N}[1-a^{-N(m(j+1)-m(j))}]/(a^N-1) \leq \rho a^N ,$$

so that

$$U_{\delta\psi(N,j,\mu)a^{n+Nm(j+1)}}(-2q) \subseteq U_{\rho a^{n+Nm(j+1)}}(-2q).$$

Hence, by (4.27),

$$P[C_j^\mu] \geq P[\Delta X(<0, a^{n+Nm(j+1)}\psi(N,j,\mu))) \in U_{\delta\psi(N,j,\mu)a^{n+Nm(j+1)}}(-2q)] \geq \eta > 0$$

for all sufficiently large j. Thus in both cases we obtain

$$\sum_{j'\geq 1} P[B_{j'}^\mu] = \infty \quad \text{and} \quad \sum_{j''\geq 1} P[C_{j''}^\mu] = \infty,$$

where $\{j'\}$ and $\{j''\}$ are arbitrary infinite subsequences of $\{j\}$. Since $\{B_j^\mu\}$ and $\{C_j^\mu\}$ for a fixed $\mu \neq 0$ are families of independent events, there exist two subsequences $\{j'\}$ and $\{j''\}$ of $\{j\}$ such that $B_{j'}^\mu$ and $C_{j''}^\mu$ hold with probability 1 for all j', j'' and every $\mu \in \Lambda$, $\mu \neq 0$. Hence and from (4.32)–(4.33) it follows that either for all $j' \geq j_1$ and $\mu \neq 0$,

$$\begin{aligned}
&\Delta X(<0, A^{(n)}_{N,k(j')+1-\mu 1})) = \Delta X(<A^{(n)}_{N,k(j'+1)+1-\mu 1}, A^{(n)}_{N,k(j')+1-\mu 1})) \\
&- \sum_{\lambda\neq 0}(-1)^{\operatorname{sgn}\lambda}\Delta X(<0, A^{(n)}_{N,\lambda k(j'+1)+\lambda^c k(j')+1-\mu 1})) \in \pm 2^q U_{\rho a^{n+Nk(j')}}(-2q) \subseteq \\
&\subseteq U_{\varepsilon a^{n+Nk(j')}}(-2q) \quad \text{a.s.},
\end{aligned}$$

or for all $j'' \geq j_2$ and $\mu \neq 0$,

$$\begin{aligned}
&\Delta X(<0, A^{(n)}_{N,m(j''+1)+1-\mu 1})) = \Delta X(<A^{(n)}_{N,m(j'')+1-\mu 1}, A^{(n)}_{N,m(j''+1)+1-\mu 1})) \\
&- \sum_{\lambda\neq 0}(-1)^{\operatorname{sgn}\lambda}\Delta X(<0, A^{(n)}_{N,\lambda m(j'')+\lambda^c m(j''+1)+1-\mu 1})) \in \pm 2^q U_{\rho a^{n+Nm(j''+1)}}(-2q) \subseteq \\
&\subseteq U_{\varepsilon a^{n+Nm(j''+1)}}(-2q) \quad \text{a.s.}
\end{aligned}$$

Notice now that in the first case for $j' \geq j_1$ we get

$$\begin{aligned}
&\sum_{\mu\neq 0}(-1)^{\operatorname{sgn}\mu}\Delta X(<0, A^{(n)}_{N,k(j')+1-\mu 1})) = \Delta X(<A^{(n)}_{N,k(j')}, A^{(n)}_{N,k(j')+1})) \\
&- \Delta X(<0, A^{(n)}_{N,k(j')+1})) \in \pm 2^q U_{\varepsilon a^{n+Nk(j')}}(-2q) \subseteq U_{\varepsilon' a^{n+Nk(j')}}(-2q),
\end{aligned}$$

and on the other hand

$$\Delta X(<A^{(n)}_{N,k(j')}, A^{(n)}_{N,k(j')+1})) \notin U_{\alpha' a^{n+Nk(j')}}(-2q)$$

with probability 1, because for $k = k(j')$, $E_k^{(N)}$ occurs. However, by the choice of ε' and α' we have $\alpha A^{(n)}_{N,k(j')+1} \leq \sqrt{\alpha}a^{n+Nk(j')}$, and hence

$$U_{\varepsilon' a^{n+Nk(j')}}(-2q) + U_{\alpha A^{(n)}_{N,k(j')+1}}(-2q) \subseteq U_{\alpha' a^{n+Nk(j')}}(-2q).$$

Therefore

$$\Delta X(< 0, A^{(n)}_{N,k(j')+1})) \notin U_{\alpha A^{(n)}_{N,k(j')+1}}(-2q)$$

for all $k(j')$, $j' \geq j_1$ with probability 1 , since otherwise we would get

$$\{\Delta X(< A^{(n)}_{N,k(j')}, A^{(n)}_{N,k(j')+1}))-\Delta X(< 0, A^{(n)}_{N,k(j')+1}))\} + \Delta X(< 0, A^{(n)}_{N,k(j')+1})) \in$$
$$\in U_{\alpha' a^{n+Nk(j')}}(-2q) ,$$

which leads to a contradiction. Arguing similarly in the second case, we see that with probability 1 for all $m(j''+1)$, $j'' \geq j_2$,

$$\Delta X(< 0, A^{(n)}_{N,m(j''+1)+1})) \notin U_{\alpha A^{(n)}_{N,m(j''+1)+1}}(-2q) .$$

Finally we conclude that

$$P\Big[\bigcap_{0<T_1<T_2} \bigcup_{t\notin<T_1,T_2>, t\in B'_{\alpha,a}} [\Delta X(< 0,t)) \notin U_{\alpha t}(-2q)]\Big] = 1 .$$

Remarks. 1) In the proof of Theorem 4.12 we have shown that for infinitely many indices of the form $A^{(n)}_{N,k'+1} \doteq a^{n+N(k'+1)}/(a^N-1)$ we have

$$\Delta X(< 0, A^{(n)}_{N,k'+1})) \notin U_{\alpha A^{(n)}_{N,k'+1}}(-2q) \quad \text{a.s.} ,$$

where $n-1+Nk' \in \mathbb{J} = \{k \in \mathbb{Z}^q :< a^k, a^{k+1} > \cap B \neq \emptyset\}$. Let us observe that $a^{n+N(k'+1)}/(a^N-1) \geq a^{n+N(k'+1)}/a^N = a^{n+Nk'}$, and on the other hand $a^{n+N(k'+1)}/(a^N-1) = (a^N/(a^N-1))a^{n+Nk'} \leq a^{n+Nk'+1}$ for $N \geq N_0$. Since $< a^{n+Nk'-1}, a^{n+Nk'} > \cap B \neq \emptyset$ and $\sqrt{\alpha}a < 1$, we conclude that $a^{n+Nk'}$ and $a^{n+Nk'+1} \in B_\alpha$, i.e. $A^{(n)}_{N,k'+1} \in B_\alpha$. However, the process X should be indexed by a larger parameter set than B_α because we consider increments of X on various rectangles, and they are well-defined provided all the vertices of these rectangles are in the domain of X.

2) If the parameter set B is bounded from above, then in (4.29) the lim sup as $t \xrightarrow{B'_{\alpha,a}} 0$ should be considered, and if B is bounded away from 0 but unbounded from above, then in (4.29) $t \xrightarrow{B'_{\alpha,a}} \infty$ should be written.

Corollary 4.13. *Let X be a stochastic process satisfying all the assumptions of* Theorem 4.12 *but not necessarily symmetric and let $\mathcal{U} = \{U_t, t \in R^q_+\}$ be a completely regularly varying family of globular sets such that $W_t = U_t - U_t$ are Baire sets (and hence globular) satisfying $W_t(-j) = U_t(-j) - U_t(-j)$, $t \in R^q_+$, $j \geq 1$. Furthermore, assume that conditions* (4.24) - (4.27) *are true, and*

$$(4.34) \qquad S'_B := \int_B \frac{1}{|t|} \cdot P \times P'[\Delta X(< 0,t)) - \Delta X'(< 0,t)) \notin W_t]\, dt = \infty .$$

Then there exists a deterministic function $z : R^q_+ \to \mathbb{G}$, such that for an arbitrary $\alpha \in R^q_+$, $0 < \alpha < 1$, there can be found $a \in R^q_+$, $a > 1$ sufficiently close to 1 satisfying

$$(4.35) \qquad P\Big[\limsup_{t \xrightarrow{B'_{\alpha,a}} 0 \text{ or } \infty} [\Delta X(<0,t)) \notin z(t) + W_{\alpha t}(-2q)]\Big] = 1 \, .$$

Proof. Observe that under our hypotheses conditions (4.24) – (4.27) with X replaced by $X - X'$, P by $P \times P'$, $U_t(-j)$ by $W_t(-j)$ and η by η^2 are fulfilled as well. Hence, based on Theorem 4.12 we infer that for an arbitrary $\alpha \in R^q_+$, $0 < \alpha < 1$, there can be chosen $a \in R^q_+$, $a > 1$, such that

$$P \times P'\Big[\limsup_{t \xrightarrow{B'_{\alpha,a}} 0 \text{ or } \infty} [\Delta X(<0,t)) - \Delta X'(<0,t)) \notin W_{\alpha t}(-2q)]\Big] = 1 \, .$$

To conclude the proof we use the similar argument as in Corollary 4.3.

In order to formulate an example we introduce the notion of a completely regularly increasing function.

A function $g : R^q_+ \to R_+$ is called here completely regularly increasing, if it satisfies conditions (a), (b), (e) of a function with regularly varying increments, and (g) for each $\varepsilon \in R^q_+$, $0 < \varepsilon < 1$ there can be found $\rho \in R^q_+$, $0 < \rho < \varepsilon$ such that

$$2g(\rho t) \le g(\varepsilon t) \text{ for all } t \in R^q_+ \setminus \partial R^q_+ \, ;$$

(h) for each $\alpha \in R^q_+$, $0 < \alpha < 1$ there exist $\alpha' \in R^q_+$, $\alpha < \alpha' < 1$ and $\varepsilon' \in R^q_+$, $\varepsilon' > 0$ such that

$$g(\varepsilon' t) + g(\alpha t) \le g(\alpha' t) \text{ for all } t \in R^q_+ \setminus \partial R^q_+ \, .$$

It can be easily seen that $g(t) = |t|^r$, $r > 0$ and $g(t) = (|t| LL|t|)^{1/2}$ are completely regularly increasing functions.

Example 9. Let U be a symmetric convex open neighbourhood of zero in a linear topological Lindelöf space $\mathbb{G}$ and let p be the Minkowski's functional for U. Furthermore, let $X = \{X(t), t \in R^q_+\}$ be a symmetric stationary additive stochastic process taking values in $\mathbb{G}$. Denote $U_t = \{x \in \mathbb{G} : p(x) < g(t)\}$, $t \in R^q_+$, where $g : R^q_+ \to R_+$ is a completely regularly increasing function and assume that for the defined so family of globular sets the process X satisfies conditions (4.24)–(4.27). Then for each $\varepsilon \in R^q_+$, $0 < \varepsilon < 1$ there exist $\alpha \in R^q_+$, $0 < \alpha < 1$ and $a \in R^q_+$, $1 < a$ sufficiently close to 1 such that

$$(4.36) \qquad P\Big[\limsup_{t \xrightarrow{B'_{\alpha,a}} 0 \text{ or } \infty} [p(\Delta X(<0,t))) \ge \varepsilon g(t)]\Big] = 1 \, .$$

Indeed, on the basis of Proposition 2.2 and properties of g we infer that $\mathcal{U} = \{U_t, t \in R^q_+\}$ is a completely regularly varying family of symmetric globular sets in $\mathbb{G}$. Moreover, applying Theorem 4.12 we see that (4.29) is true for every $\alpha \in R^q_+$, $0 < \alpha < 1$ and

$U_{\alpha t}(-2q)$, along with $a \in R_+^q$, $1 < a$ depending on α. Note now that $U_{\alpha t}(-2q)$ in the proof of Proposition 2.2 can be chosen in such a way that for a fixed arbitrarily $b \in R_+^q$, $0 < b < 1$, $\{x \in \mathbb{G} : p(x) < bg(\alpha t)\} \subseteq U_{\alpha t}(-2q)$ for all $t \in R_+^q \setminus \partial R_+^q$. Choose $\alpha \in R_+^q$ and $b \in R_+$ so that $1 > bf(\alpha) \geq \varepsilon$, where $f(\alpha) = \inf\{g(\alpha t)/g(t) : t \in R_+^q \setminus \partial R_+^q\}$. Then $\{x \in \mathbb{G} : p(x) < \varepsilon g(t)\} \subseteq \{x \in \mathbb{G} : p(x) < bg(\alpha t)\}$, which implies (4.36).

Remark. The last example shows that our Theorem 4.12 is a far reaching generalization of Theorem 2 by Zinčenko (1979).

Part III of this paper, devoted to limit theorems for additive processes in torus, will appear soon in the next issue of the Annales UMCS.

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