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The Peak Sets

Abstract. This is a survey article on the set $M(\Phi)$ of points where a "derivative" Φ attains local maxima. A typical example of Φ is the Bloch derivative $F_f(z)=(1-|z|^2)|f'(z)|$ of f holomorphic in the unit disk. The components of $M(F_f)$ are classified into the three: isolated points; simple analytic arcs ending nowhere in the disk; analytic Jordan curves. The remaining Φ which are mainly studied are the spherical derivative $|f'|/(1+|f|^2)$ of f meromorphic in a domain in the complex plane and the minus of the Gauss curvature of a minimal surface in the Euclidean space with the parameter in a domain in the plane. Parts of this article were presented on October 21, 1992, at the meeting of the Minisemester: "Functions of One Complex Variable" (in the Semester on Complex Analysis) held at Stefan Banach International Mathematical Center in Warsaw, Poland.

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1. Introduction. We shall study the set $M(\Phi)$, in a domain in the complex plane $\mathbf{C} = \{|z| < \infty\}$, set where the "derivatives", symbolically denoted by Φ, attain local maxima. We call $M(\Phi)$ the peak set of Φ. Most of the results in the present paper are extracted from [26, 27, 28] and notation is partially different from that in the cited papers.

We shall be mainly concerned with the peak sets of the following three types of Φ:

(BD) *The Bloch derivative*:

$$F_f(z) = (1-|z|^2)|f'(z)|$$

of f holomorphic in the disk $D = \{|z| < 1\}$.

(SD) *The spherical derivative*:

$$f^{\#} = |f'|/(1+|f|^2)$$

of f meromorphic in a domain $G \subset \mathbf{C}$.

(GC) *The minus of the Gauss curvature*: $-K$ of a regular minimal surface $x : G \to \mathcal{R}^3$ in the Euclidean space $\mathcal{R}^3$.

Suppose that Φ is considered in a domain $G \subset \mathbf{C}$. Let $M(\Phi)$ be the set of points $z_0 \in G$ such that $\Phi(z_0) \geq \Phi(z)$ in a disk $\{|z - z_0| < \delta(z_0)\} \subset G$ ($\delta(z_0)$ depends on

z_0) and let $M^*(\Phi)$ be the set of points $z_0 \in G$ such that $\Phi(z_0) \geq \Phi(z)$ for all $z \in G$. Thus $M^*(\Phi) \subset M(\Phi)$ is immediate.

In all the described cases, except for the trivial ones, the connected components of the peak set $M(\Phi)$ are classified into three types:

(1) *isolated points;*

(2) *simple analytic curves ending nowhere in G;*

(3) *analytic Jordan curves.*

Since Φ is shown to be constant on curves of types (2) and (3) we have the same classification of the set $M^*(\Phi)$. Let $M_k(\Phi)$ be the set of components of $M(\Phi)$ of type (k) explained in the above, $k = 1, 2, 3$. Similarly for $M_k^*(\Phi)$.

We shall study geometric properties of $M(\Phi)$ for Φ of (BD) or (SD). A typical one is that if $c \in M_2(F_f) \cup M_3(F_f)$, then the slope of the tangent at each $z \in c$ to c is $-\tan\{\Theta(z)/2\}$, where $\Theta(z)$ is the argument of the Schwarzian derivative $(f''/f')' - 2^{-1}(f''/f')^2$ of f at z.

In conjunction with (BD) we shall consider the density of the Poincaré metric in Section 5. The results in this section are not explicitly stated in any paper of [26, 27, 28]. Applications of the case (SD) are to know behaviour of solutions of a nonlinear elliptic partial differential equation and to know behaviour of the Gauss curvature of graphs of harmonic functions. These are summarized in Sections 7 and 8.

Suppose that a $c \in M_3(\Phi)$ exists and let Δ be the Jordan domain bounded by c. Here we assume that $\Delta \subset G$ in cases (SD) and (GC). In case (BD), the non-Euclidean area of Δ is expressed by the number of the zeros of f' in Δ. In case (SD) the spherical area of the Riemann image surface (the Riemannian image, for short) of Δ by f is expressed by the number of the zeros and poles of f' in Δ. Finally, in case (GC) the total curvature of the subsurface with parameter restricted to Δ is expressed by the number of the zeros and poles of the derivative g' in Δ, where g is the Gauss map of the whole surface.

2. The Bloch derivative. We begin with case (BD). The Bloch derivative at z of a function f holomorphic in D is

$$F_f(z) \equiv (1 - |z|^2)|f'(z)| = \lim_{w \to z} (|f(w) - f(z)|/\pi(w, z))$$

where $\pi(w, z) = \tanh^{-1}(|z - w|/|1 - \overline{w}z|)$ with $\tanh^{-1} x = (1/2)\log\{(1 + x)/(1 - x)\}$, $0 \leq x < 1$, is the Poincaré distance of w and z in D. The Bloch derivative appears in the proof of the Bloch theorem:

There exists a universal constant $c_B > 0$, called the Bloch constant, such that if f is holomorphic in D and $f'(0) = 1$, then the Riemannian image of D by f over C *contains an open one-sheeted disk of (Euclidean) radius c_B. See* [13].

We nowadays call f Bloch if F_f is bounded in D. This term "Bloch function" prevails, among recent papers, ignoring R. M. Robinson's earlier paper [19].

If f is nonconstant and holomorphic in D, then $1/F_f$ is subharmonic in D minus the zeros of f'; actually, $\Delta \log(1/F_f(z)) = 4/(1 - |z|^2)^2 > 0$ there, and $1/F_f = \exp[\log(1/F_f)]$. Thus, F_f has "trivial" local minimum at each zero of f' and has no local minimum at any other point of D.

We begin with the theorem essentialiy due to J.A. Cima, W.R. Wogen [5], S. Ruscheweyh and K.-J. Wirths [20] (they all actually suppose that f is Bloch; see [26] and also [4, 21, 24]).

Theorem 2.1. *Suppose that $M(F_f)$ is nonempty for f nonconstant and holomorphic in D. Then the components of $M(F_f)$ are at most countable and they consist of the three types* (1), (2), (3). *Furthermore, the isolated points of $M(F_f)$ accumulate nowhere in D.*

For g nonconstant and meromorphic in G we denote $\lambda(g) = g''/g'$, the logarithmic derivative of g'. Then the meromorphic function $\sigma(g) = \lambda(g)' - 2^{-1}\lambda(g)^2$ is the Schwarzian derivative of g. We observe that if $z \in M(F_f)$ for nonconstant f, then $f'(z) \neq 0$ and

$$0 = (\partial/\partial z)\log F_f(z) = 2^{-1}\lambda(f)(z) - \overline{z}/(1-|z|^2),$$

so that, $\overline{z} = H_f(z)$, where

$$H_f(z) = \lambda(f)(z)/(z\lambda(f)(z)+2)\,; \tag{2.1}$$

here, as usual,

$$2(\partial/\partial z) = (\partial/\partial x) - i(\partial/\partial y)\,, \quad 2(\partial/\partial \overline{z}) = (\partial/\partial x) + i(\partial/\partial y)\,, \quad z = x+iy\,.$$

A core of our proof of Theorem 2.1 consequently is an analysis of the closed set

$$\Sigma(H) = \{z \in G;\ \overline{z} = H(z)\}\,,$$

where H is meromorphic in G. Such a function H is called the Schwarz function of $\Sigma(H)$ by P.J. Davis [6] *under the condition that $\Sigma(H)$ is a curve.* We have

Lemma 2.2. [20, Lemma 1]. *If $a \in G$ is an accumulation point of $\Sigma(H)$, then there is an open disk $U(a) \subset G$ of center a such that $\Sigma(H) \cap U(a)$ is a simple analytic arc with both terminal points on the circle $\partial U(a)$. In particular, isolated points of $\Sigma(H)$ accumulate nowhere in G.*

With the aid of Lemma 2.2 we can easily observe that if $\Sigma(H)$ is nonempty, then each component of $\Sigma(H)$ is one of types (k), $k = 1,2,3$, described in Section 1. We let $\Sigma_k(H)$ be the set of the components of type (k), $k = 1,2,3$. A detailed analysis then yields

Theorem 2.3. [26, Theorem 3]. *For f nonconstant and holomorphic in D with nonempty $M(F_f)$ and for H_f in* (2.1) *we have*

$$M_1(F_f) \subset \Sigma_1(H_f)\,; \quad M_2(F_f) = \Sigma_2(H_f)\,; \quad M_3(F_f) = \Sigma_3(H_f)\,.$$

3. The Schwarzian derivative, geodesics, and $M_3(F_f)$. Let f be nonconstant and meromorphic in G. In case $G = D$, the function

$$N_f(z) = 2^{-1}(1-|z|^2)^2|\sigma(f)(z)|\,,$$

which is called the Nehari derivative of f at $z \in D$, is significant in Univalent Function Theory. Namely, if the Nehari condition

$$\text{(N)} \qquad \sup_{z \in D} N_f(z) \leq 1$$

holds, then f is univalent in the whole D; the constant 1 is the best possible [16, 9]. We shall show that N_f also plays a role in our study of the peak set $M(F_f)$.

By a geodesic in D we mean the intersection of D with a circle or a straight line orthogonal to ∂D. By a geodesic segment in D we mean an arc on a geodesic, arc both terminal points of which are included.

Theorem 3.1. [26, Theorem 1]. *Suppose that f is nonconstant and holomorphic in D with the nonempty peak set $M(F_f)$. Then we have the following:*

$$\text{(3.1)} \qquad \sup_{z \in M(F_f)} N_f(z) \leq 1 \,.$$

(3.2) *If $N_f(z) < 1$ at $z \in M(F_f)$, then $\{z\} \in M_1(F_f)$.*
(3.3) *Suppose that $c \in M_2(F_f) \cup M_3(F_f)$. (Then $N_f(z) = 1$ at each $z \in c$ by (3.2).) Then the tangent to c at $z \in c$ is $\{z + te^{-i\Theta(z)/2};\ t \in \mathcal{R}\}$, where $\Theta(z) = \arg \sigma(f)(z)$. Furthermore, there exits a geodesic segment $\Lambda \equiv \{\psi(t);\ -\tau \leq t \leq \tau\}$ orthogonal to c at $z = \psi(0)$ such that $(d^2/dt^2)F_f(\psi(t)) < 0$ for $|t| \leq \tau$.*

The function $F_f(\psi(t))$ consequently attains the maximum at $t = 0$ in the strict sense. The part $\{(x, y, F_f(z));\ z = x + iy \in M(F_f)\}$ of the graph $\{(x, y, F_f(z));\ z = x + iy \in D\}$ in $\mathcal{R}^3$ thus symbolically consists of summits, ridges, and sommas (mountains around a crater).

Let $\mathbf{A}$ be the family of functions $a \log((1 + \mu)/(1 - \mu)) + b$, where $a \neq 0$ and b are complex constants, and μ runs over all the Möbius transformations mapping D onto D. For $g(z) = a \log((1 + z)/(1 - z)) + b \in \mathbf{A}$, the set $M^*(F_g) = M(F_g)$ is the real diameter of D. As a result, $M(F_f)$ for $f \in \mathbf{A}$ is a geodesic because $F_f = F_g \circ \mu$ by $f = g \circ \mu$.

Theorem 3.2. [26]. *Suppose that the Nehari condition* (N) *holds for f holomorphic in D. Then $M(F_f)$ is the empty set, a one-point set or $f \in \mathbf{A}$ (hence $M(F_f)$ is a geodesic.)*

We can apparently replace $M(F_f)$ by $M^*(F_f)$ in Theorem 3.2. Under condition (N) for meromorphic f, F.W. Gehring and C. Pommerenke [8] proved that $f(D)$ is either a Jordan domain in $\mathbf{C} \cup \{\infty\}$ or the Möbius image (namely, the image by a Möbius transformation) of a band. Theorem 3.2 gives a further analysis in case $f(D)$ ($\subset \mathbf{C}$) is a Jordan domain in $\mathbf{C} \cup \{\infty\}$.

We know that if f is meromorphic and univalent in D and further if $f(D)$ is the Möbius image in $\mathbf{C} \cup \{\infty\}$ of a convex domain in $\mathbf{C}$, then (N) holds. Furthermore we know that the equality in (N) holds for each $f \in \mathbf{A}$. See [14, p. 63]. We next consider $M_3(F_f)$ in

Theorem 3.3. [26]. *Suppose that f is nonconstant and holomorphic in D. Suppose further that $c \in M_3(F_f)$ exists and let Δ be the Jordan domain bounded by c. Then,*

$$\int\int_{\Delta} (1-|z|^2)^{-2}\,dxdy = (\pi/2)\nu_{\Delta}(f') \quad (z = x+iy) , \tag{3.4}$$

where $\nu_{\Delta}(f')$ is the total number of the zeros of f' in Δ, the multiplicities being counted.

The left-hand side of (3.4) is the non-Euclidean hyperbolic area of D. It follows from Theorem 3.3 that if f' never vanishes in D, then $M_3(F_f)$ is empty.

We note here that if $M_3(F_f)$ is nonempty, then $M_3(F_f)$ consists of just one element, say, c. Furthermore, $M_2(F_f)$ is empty and isolated points of $M(F_f)$ are finite in number and are contained in the Jordan domain bounded by c. See [26, Theorem B] for example.

4. Determination of f with preassigned $M(F_f)$. Given a simple analytic curve c in D, can we find an f such that $M(F_f) = c$? We consider the case where c is the intersection of D with a circle or a straight line [26]. The functions are somewhat complicated even in this very simple case. In this section $A \neq 0$ and B are always complex constants.

(I) *A complete circle*: $c = \{|z-a| = r\}$; $a \in D$, $0 < r < 1-|a|$. We have $M(F_f) = c$ if and only if

$$\left(\frac{N}{N+2}\right)^{1/2} = (2r)^{-1}[1-|a|^2+r^2-\{(1-|a|^2+r^2)^2-4r^2\}^{1/2}] ,$$

where N is a natural number. Under the above condition we have

$$f(z) \equiv A[(z-b)/(1-\bar{b}z)]^{N+1} + B ,$$

where

$$b = 2a[1+|a|^2-r^2+\{(1-|a|^2+r^2)^2-4r^2\}^{1/2}]^{-1} .$$

(II) *An oricycle*: $c = \{|z-pe^{i\alpha}| = 1-p\}$; α real, $0 < p < 1$. We have $M(F_f) = c$ if and only if

$$f(z) = A\exp\left[\frac{2(p-1)}{p(1-e^{i\alpha}z)}\right] + B .$$

(III) *A hypercycle*: $c = \{|z-pe^{i\alpha}| = r\}$; α real, $p, r > 0$, $|1-p| < 1 < 1+p$. We have $M(F_f) = c$ if and only if

$$f(z) = A\int_0^{ze^{i\alpha}} \left(\exp\left[\int_0^w \frac{-2p\zeta+2(p^2-r^2)}{p\zeta^2+(r^2-p^2-1)\zeta+p}\,d\zeta\right]\right)dw + B .$$

(IV) *A rectilinear segment*: $c = \{e^{i\alpha}(\cos\beta + iy);\ -\sin\beta < y < \sin\beta\} \cap D$; α real, $0 < \beta \le \pi/2$. We have $M(F_f) = c$ if and only if

$$f(z) = A\int_0^{ze^{i\alpha}} \left(\exp\left[\int_0^w \frac{4\cos\beta - 2\zeta}{1 - 2\zeta\cos\beta + \zeta^2}\, d\zeta\right]\right) dw + B\ .$$

5. The Poincaré density. Recall that the Bloch derivative has a relation with the Poincaré density. We call a subdomain G of $\mathbb{C}$ hyperbolic if $\mathbb{C} \setminus G$ contains at least two points. In this section G is always a hyperbolic domain in $\mathbb{C}$. Then, G has the Poincaré metric $P_G(z)|dz|$. The density function, or the Poincaré density, P_G is defined in G by the identity $P_G(z) = 1/F_\varphi(w)$, $z = \varphi(w)$, $w \in D$, where φ is a holomorphic universal covering projection from D onto G, in notation, $\varphi \in \mathrm{Proj}(G)$. The definition is independent of the specified choice of φ and w as far as the equality $z = \varphi(w)$ is satisfied. In particular, $1/P_D(z) = 1 - |z|^2$ and $\pi(w, z)$ in Section 2 is the integral of $P_D(\zeta)|d\zeta|$ from w to z along the geodesic segment. See [1] and [14, pp. 147-149] for general theory of $P_G(z)|dz|$ (see also [30] for some sharp estimates of P_G in geometrical terms); note that $2P_G(z)|dz|$ instead of $P_G(z)|dz|$ is adopted in [1]. Now, $\log P_G$ is subharmonic in G because $\Delta \log P_G(z) = 4P_G(z)^2 > 0$, $z \in G$, and hence $P_G = \exp(\log P_G)$ is subharmonic in G. Hence P_G has no local maximum in G. Let $M(1/P_G)$ be the set of points $z \in G$ where P_G attains local minima: $P_G(z) \le P_G(w)$ in $\{|w - z| < \delta(z)\} \subset C$. Then, $M(1/P_G) = \varphi(M(F_\varphi))$ for each $\varphi \in \mathrm{Proj}\,(G)$. Since φ' never vanishes in D, the set $M_3(F_\varphi)$ is empty by Theorem 3.3. Since φ is locally univalent, there is a one-to-one correspondence between a part of $\Sigma(H_\varphi)$ and a part of $M(1/P_G)$. Applying Lemma 2.2, we consequently obtain

Theorem 5.1. *If $M(1/P_G)$ is nonempty, then each component of $M(1/P_G)$ is one of the three types* (1), (2), (3). *The isolated points of $M(1/P_G)$ accumulate nowhere in G.*

We can further show that $M(1/P_G)$ in Theorem 5.1 may be replaced by the set $M^*(1/P_G)$ of points where P_G attains the global minimum. Let $M_k(1/P_G)$ be the set of the components of type (k), $k = 1, 2, 3$. We observe that the three types actually exist. With a slight misuse of notation we shall sometimes denote $M_k(\Phi)$ $(k = 1, 2, 3)$ instead of the union $\bigcup_{c \in M_k(\Phi)} c$ if there is no confusion. This remark is available also to the sets $M_k(1/P_G)$, $k = 1, 2, 3$.

(I) $M(1/P_G) = M_1(1/P_G)$. Examples of G are many. As a typical one of nonconvex bounded domains we choose the interior of the cardioid $\mathcal{C} = \{w + w^2/2;\ w \in D\}$. Then, $M(1/P_\mathcal{C}) = \{7/18\}$ follows from

$$1/P_\mathcal{C}(z) = (1 - |(1 + 2z)^{1/2} - 1|^2)|1 + 2z|^{1/2}\ .$$

Here, $\mathcal{C}$ is not a Möbius image of the band

$$\mathcal{B} = \{-\pi/2 < \mathrm{Im}\, z < \pi/2\}\ .$$

(II) $M(1/P_G) = M_2(1/P_G)$. For $\mathcal{B}$ we know that $M(1/P_\mathcal{B})$ is just the real axis because $1/P_\mathcal{B}(z) = 2\cos(\mathrm{Im}\, z)$.

(III) $M(1/P_G) = M_3(1/P_G)$. For the ring domain

$$R = \{e^{-\pi/2} < |z| < e^{\pi/2}\}$$

we have $M(1/P_R) = \{|z| = e^{\pi/4}\}$ because

$$1/P_R(z) = 2|z|\cos(\log|z|)\ .$$

Here, it is interesting that for

$$\varphi(w) = \exp(i\log\{(1+w)/(1-w)\} \in \mathrm{Proj}\,(R)$$

we have $M_3(1/P_R) = \varphi(M_2(F_\varphi))$, where

$$M(F_\varphi) = M_2(F_\varphi) = \{|z+i| = \sqrt{2}\} \cap D\ .$$

In all the above examples, we always have $M^*(1/P_G) = M(1/P_G)$.

Set $\delta(G) = \sup_{z\in D} N_\varphi(z)$ for a $\varphi \in \mathrm{Proj}\,(G)$. The supremum is independent of the particular choice of φ. Theorem 3.2 actually has the following version.

Theorem 5.2. *If $\delta(G) \le 1$, then $M(1/P_G) = M^*(1/P_G)$. Further, $M(1/P_G)$ is the empty set a one-point set or a straight line.*

The peak set $M(1/P_G)$ under $\delta(G) \le 1$ is a straight line if and only if $G = f(D)$ for an $f \in \mathbf{A}$. The condition $\delta(G) \le 1$ in Theorem 5.2 cannot be relaxed. For $R(a) = \{e^{-\pi a/2} < |z| < e^{\pi a/2}\}$ $(a > 0)$ we observe that

$$1/P_{R(a)}(z) = 2|z|\cos(a^{-1}\log|z|)\ , z \in R(a)\ .$$

Hence $M^*(1/P_{R(a)}) = M(1/P_{R(a)}) = M_3(1/P_{R(a)})$ is the circle $\{|z| = \exp(a\mathrm{Arctan}\,a)\}$ and $\delta(R(\alpha)) = 1 + a^2$.

See also [29, Theorem 2] for a specified case.

6. The spherical derivative. For f meromorphic in a domain $G \subset \mathbf{C}$ and for $z \in G$ we set

$$f^\#(z) = \begin{cases} |f'(z)|/(1+|f(z)|^2) & \text{if } f(z) \ne \infty\ ; \\ |(1/f)'(z)| & \text{if } f(z) = \infty\ . \end{cases}$$

The chordal distance of a and b in $\mathbf{C} \cup \{\infty\}$ is

$$X(a,b) = |a-b|(1+|a|^2)^{-1/2}(1+|b|^2)^{-1/2}$$

with the obvious convention in case $a = \infty$ or $b = \infty$. Then,

$$f^\#(z) = \lim_{w\to z} X(f(w), f(z))/|w-z|\ .$$

Note that $f^\#(z) \ne 0$ if and only if z is a simple pole of f or $f(z) \ne \infty$ with $f'(z) \ne 0$, or $f^\#(z) = 0$ if and only if z is a pole of $\sigma(f)$. If f is nonconstant and meromorphic in G, then $1/f^\#$ is subharmonic in G minus the zeros of $f^\#$; actually,

$\Delta\log(1/f^{\#}(z)) = 4f^{\#}(z)^2 > 0$ there, and $1/f^{\#} = \exp[\log(1/f^{\#})]$. Thus, $f^{\#}$ has "trivial" local minimum at each zero of $f^{\#}$ and has no local minimum at any other point of G.

In contrast with the holomorphic case: $\Phi = F_f$, a difficulty arises at the poles of f. If $z \in M(f^{\#})$ and $f(z) \neq \infty$, then a calculation shows that

$$0 = (\partial/\partial z)\log f^{\#}(z) = 2^{-1}\lambda(f)(z) - \overline{f(z)}f'(z)/(1+|f(z)|^2) ,$$

whence

$$\overline{f(z)} = h_f(z) , \quad h_f = \lambda(f)/(2f' - f\lambda(f)) .$$

Thus, roughly speaking, a core of our study is an analysis of the set

$$\Sigma(g,h) = \{z \in G;\ \overline{g(z)} = h(z)\} ,$$

where g and h are holomorphic and meromorphic in G, respectively. Ruscheweyh and Wirths's lemma, Lemma 2.2 in Section 2, needs an unessential change.

Lemma 6.1. *If $a \in G$ is an accumulation point of $\Sigma(g,h)$ and if $g'(a) \neq 0$, then there exists an open disk $U(a) \subset G$ of center a such that $\Sigma(g,h) \cap U(a)$ is a simple analytic arc with both terminal points on the circle $\partial U(a)$.*

The condition on g implies the local univalency of g at a. Hence this case is reduced to the case $g(z) = z$. We cannot drop the condition $g'(a) \neq 0$ in Lemma 6.1. For example, if $G = \mathbb{C}$, $a = 0$, $g(z) = h(z) = z^n$ $(n \geq 2)$, then $\Sigma(g,h)$ consists of n half lines issuing from the origin.

Theorem 6.2. [28, Theorem 1]. *Suppose that $M(f^{\#})$ is nonempty for f nonconstant and meromorphic in G. Then, components of $M(f^{\#})$ are at most countable and each component is one of the three types* (1), (2), (3).

A conjecture is therefore that the isolated points of $M(f^{\#})$ accumulate at no point of G. This is reduced to considering the case $g'(a) = 0$ in Lemma 6.1.

We note that Theorem 6.2 depends on a local property of $f^{\#}$, namely, that of an appropriate pair, g, h, described in Lemma 6.1. We observe, as a result, the following: If a quantity in G is defined in terms of $f^{\#}$, where f is defined in a suitable neighbourhood of every point of G, then the obvious type of Theorem 6.2 for this quantity is true. We shall return to this topic in detail in Section 7 where the quantity is $\omega = \log(2a^{-1}(f^{\#})^2)$ with $a > 0$ a constant.

An analogue of Theorem 3.1 is the following, where we set

$$N_f^*(z) = 2^{-1}f^{\#}(z)^{-2}|\sigma(f)(z)| , \quad z \in G .$$

Theorem 6.3. [28, Theorem 2]. *Suppose that $M(f^{\#})$ is nonempty for f nonconstant and meromorphic in G. Then, we have the following:*

$$\sup_{z \in M(f^{\#})} N_f^*(z) \leq 1 . \tag{6.1}$$

(6.2) *If* $N_f^*(z) < 1$ *at* $z \in M(f^\#)$, *then* $\{z\} \in M_1(f^\#)$.

(6.3) *Suppose that* $c \in M_2(f^\#) \cup M_3(f^\#)$. (Then, $N_f^*(z) = 1$ at each $z \in c$ by (6.2).) *Then* $\{z + te^{-i\Theta(z)/2}; t \in \mathcal{R}\}$ *is the tangent to* c *at* $z \in c$, *where* $\Theta(z) = \arg \sigma(f)(z)$. *Furthermore, there exits a* $\tau > 0$ *such that the function* $f^\#(z + ite^{-i\Theta(z)/2})$ *of* $t \in (-\tau, \tau)$ *has the strictly negative second derivative at each* t.

The set $\{(x, y, f^\#(z));\ z = x + iy \in c\}$ for $c \in M_2(f^\#) \cup M_3(f^\#)$ is again a ridge or a somma.

We can easily find f with the nonempty $M_3(f^\#)$. Actually, for $f(z) = z^n$ $(n > 1)$ in $\mathbb{C}$ we observe that the set $M^*(f^\#) = M_3(f^\#)$ is the circle $\{|z| = ((n-1)/(n+1))^{1/(2n)}\}$. Apparently, for $f(z) = z$ in $\mathbb{C}$, we have $M(f^\#) = \{0\}$. A novelty in the meromorphic case is the following result on $M_3(f^\#)$.

Theorem 6.4. [28, Theorem 3]. *Suppose that* $c \in M_3(f^\#)$ *exists for* f *meromorphic in* G. *Suppose further that the Jordan domain* Δ *bounded by* c *is contained in* G. *Then,*

$$\iint_\Delta f^\#(z)^2 dx\, dy = (\pi/2)(\nu_\Delta(f') + \mu_\Delta(f') - 2n)\ , \tag{6.4}$$

where $\nu_\Delta(f')$ *and* $\mu_\Delta(f')$ *are the total number of the zeros and poles of* f' *in* Δ, *the multiplicities being counted, and* n *is the number of the distinct poles of* f' *in* Δ.

The integral in the left-hand side of (6.4) is the spherical area of the Riemannian image of Δ by f. As a result, if $f^\#$ never vanishes in G, then G does not contain any Jordan domain bounded by a curve of $M_3(f^\#)$.

7. A partial differential equation. Let a real function ω defined in a domain $G \subset \mathbb{C}$ be a solution of the nonlinear elliptic partial differential equation

$$(\partial^2/\partial z \partial \bar{z})\omega + ae^\omega = 0 \quad \text{in } G\ , \tag{7.1}$$

where $a > 0$ is a constant. If f is meromorphic with nonvanishing $f^\#$ in G, then

$$\omega = \log(2a^{-1}(f^\#)^2) \tag{7.2}$$

is a solution. Conversely, if G is simply connected, then J. Liouville [15] proved that for each solution ω of (7.1) there exists f meromorphic in G such that (7.2) is valid; see [2, pp. 27–28], [23] and see also [3]. We consequently obtain the formula (7.2) locally for each solution ω in a general G. In view of the remark after Theorem 6.2 we thereby have the classification of the components of the "peak" set $M(\omega)$ of points in G where ω has local maxima as well as of the set $M^*(\omega)$ of points in G where ω has the global maximum.

We suppose, in general, the boundary condition

$$\lim_{z \to \zeta} \omega(z) = 0 \tag{7.3}$$

at each boundary point ζ of G in $\mathbb{C} \cup \{\infty\}$. We then have [28]

Theorem 7.1. *Suppose that ω is a solution of (7.1) under condition (7.3) for a simply connected G. Then $M^*(\omega)$ is a finite set.*

First, $M^*(\omega) \subset M(\omega)$. Theorem 6.4, on the other hand, shows that $M_3(\omega)$ is empty. Also $M_2(\omega)$ is empty by (7.3) because $f^\#$ is constant on $M_2(\omega)$ and ω is a positive, nonconstant, superharmonic function in G. Since ω is constant (= the maximum) on $M^*(\omega)$, it follows that $M^*(\omega)$ consists of isolated points. These points cannot accumulate at any point of G. In fact, $f^\#$ never vanishes in G, and a local consideration with the aid of Lemma 6.1 shows that $M^*(\omega)$ has no accumulation point in G.

As a final remark we note that condition (6.1) reads

$$|(\partial^2/\partial z^2)\omega(z) - 2^{-1}((\partial/\partial z)\omega(z))^2| \le ae^{\omega(z)}, \quad z \in M(\omega) ,$$

because

$$\sigma(f)(z) = (\partial^2/\partial z^2)\omega(z) - 2^{-1}((\partial/\partial z)\omega(z))^2 ;$$

see [2, p. 29] and [3, p. 231].

8. The Gauss curvature. Let a real-valued function $h : G \to \mathcal{R}$ be nonconstant. Consider the graph of h, or the set $\Gamma(h)$ of points $P \equiv P(x,y) = (x, y, h(x,y)) \in \mathcal{R}^3$, where $z = x + iy \in G$. Suppose that $\Gamma(h)$ has the unit normal vector $\mathbf{n} = \mathbf{n}(P)$ at a P. Suppose further that the intersection of $\Gamma(h)$ with each plane π_θ parallel to $\mathbf{n}$ and containing P, is, near P, a curve passing through P with the vector expression $\mathbf{c}_\theta(s)$ in terms of the arc length s, so that $\mathbf{c}_\theta(s_\theta)$ always expresses P. Note that the suffix θ naming the planes π_θ ranges over $0 \le \theta < 2\pi$. The Gauss curvature of $\Gamma(h)$ at P is the product of the maximum and the minimum of inner products:

$$\mathbf{n}(P) \cdot \{(\partial^2/\partial s^2)\mathbf{c}_\theta(s)\}_{s=s_\theta} , \quad 0 \le \theta < 2\pi .$$

In general, given a twice continuously differentiable $h : G \to \mathcal{R}$, we define *a priori* the Gauss curvature at $P(x,y)$ of $\Gamma(h)$ as the value of the function

$$K = (h_{xx}h_{yy} - h_{xy}^2)/(1 + h_x^2 + h_y^2)^2 \quad \text{at } z = x + iy .$$

The Gauss curvature explained in the preceding paragraph, in particular, coincides with $K(z) = K(x,y)$. A calculation yields

$$K/4 = [(\partial^2 h/\partial z\partial \bar{z})^2 - |(\partial^2 h/\partial z^2)|^2]/[1 + 4|(\partial h/\partial z)|^2] .$$

As a typical example, let u be a harmonic function in G. Then, for $\Gamma(u)$ we have $K = -f^{\#2}$, where $f = 2(\partial u/\partial z)$ is a holomorphic function in G. We thus have the classification of the components of the peak set $M(-K)$ of points where K has local minima. For relating subjects we refer the reader to [7, 10, 11, 12, 22, 25].

9. The Gauss curvature of a minimal surface in $\mathcal{R}^3$. We call a mapping $x : G \to \mathcal{R}^3$ with $x = (x_1, x_2, x_3)$ a regular minimal surface in $\mathcal{R}^3$ if the following hold:

(HA) Each x_k is harmonic in G, $k = 1, 2, 3$.

(IS) The parameter $w = u + iv \in G$ is isothermal in the sense that

$$\sum_{k=1}^{3} (\partial x_k/\partial w)^2 \equiv 0 \quad \text{in } G \, .$$

(RE) The function

$$\sum_{k=1}^{3} |\partial x_k/\partial w|^2$$

never vanishes in G.

See [17, 18] for general theory of minimal surfaces.

Suppose that a regular minimal surface $x : G \to \mathcal{R}^3$ is contained in no plane in the sense that there is no plane π with $x(w) \in \pi$ for all $w \in G$. Then $f = 2((\partial x_1/\partial w) - i(\partial x_2/\partial w))$ is holomorphic and not identically zero in G and the Gauss map is $g = 2(\partial x_3/\partial w)/f$, that is, g is meromorphic in G and the unit normal $\mathbf{n}(w)$ at $x(w)$ is given by the formula

$$\mathbf{n}(w) = \Big(\frac{2\mathrm{Re}\, g(w)}{|g(w)|^2 + 1}, \frac{2\mathrm{Im}\, g(w)}{|g(w)|^2 + 1}, \frac{|g(w)|^2 - 1}{|g(w)|^2 + 1} \Big) \, .$$

We have a neighborhood $U(w_0)$ of each $w_0 \in G$ such that the subsurface $\{x(w);\ w \in U(w_0)\}$ is just the graph $\Gamma(h)$ of a suitable $h : V(w_0) \to \mathcal{R}$, where $V(w_0)$ is a domain in $\mathbb{C}$; see [18, p.7, Lemma 1.2] for example. The Gauss curvature of $\Gamma(h)$ at the point corresponding to $x(w_0)$ is just $K(w_0)$, where

$$\text{(9.1)} \qquad K(w) = -\Big(\frac{4g^{\#}(w)}{|f(w)|(1 + |g(w)|^2)} \Big)^2 \, , \quad w \in G \, .$$

We may thus consider $M(\Phi)$ and $M^*(\Phi)$ for $\Phi = -K$.

Since $-(\partial^2/\partial z \partial \bar{z}) \log \sqrt{-K} = 2g^{\#2}$ except for the zeros of $g^{\#}$, it follows that $1/\sqrt{1 - K}$ is subharmonic in G minus the zeros of $g^{\#}$, so that K has no local maximum at any point of G except for the zeros of $g^{\#}$. The set $M(-K)$ consists of the points $z \in G$ where K attains local minima.

Following the lines as in the cases of F_f and $f^{\#}$, we have

Theorem 9.1. **[27, Theorem 1].** *Let $x : G \to \mathcal{R}^3$ be a regular minimal surface contained in no plane and with nonempty $M(-K)$. Then, components of $M(-K)$ are at most countable and each component is one of the three types* (1), (2), (3).

The proof depends on the expression of K in (9.1), together with Lemma 6.1, so that, again, a conjecture is that the isolated points of $M(-K)$ accumulate nowhere in G. As before, we can replace $M(-K)$ by $M^*(-K)$.

We set

$$Q = \frac{1}{2}\left(\frac{g''}{g'} - \frac{f'}{f}\right), \quad H = \frac{Q}{2g' - Qg}\, ; Q_1 = Q - \frac{2g'}{g}\, , \; H_1 = \frac{-g^2 Q_1}{2g' + Q_1 g}\, .$$

Suppose that $w \in M(-K)$. If $g(w) \neq \infty$ and $g'(w) \neq 0$, then we observe that $w \in \Sigma(g, H)$, while if w is a simple pole of g, then we observe that $w \in \Sigma(1/g, H_1)$. Since $g^{\#}(w) \neq 0$ at w, these are the whole possible cases. We give here typical examples of $x : G \to \mathcal{R}^3$ for which $M^*(-K) = M(-K) = M_j(-K)$, $j = 1, 2, 3$.

(I) *Enneper's surface:* $x : \mathbb{C} \to \mathcal{R}^3$, where

$$x_1(w) = (1/2)(u - u^3/3 + uv^2)\, ,$$
$$x_2(w) = (1/2)(-v + v^3/3 - u^2 v)\, ,$$
$$x_3(w) = (1/2)(u^2 - v^2)\, .$$

We then have $M^*(-K) = M(-K) = M_1(-K) = \{0\}$.

(II) *Helicoid:* $x : \mathbb{C} \to \mathcal{R}^3$, where

$$x_1(w) = \sinh u \cos v\, ,$$
$$x_2(w) = \sinh u \sin v\, ,$$
$$x_3(w) = v\, .$$

We then have $M^*(-K) = M(-K) = M_2(-K) = \{\mathrm{Re}\, w = 0\}$.

(III) *Catenoid:* $x : \mathbb{C} \setminus \{0\} \to \mathcal{R}^3$, where

$$x_1(w) = (-u/2)[1 + (u^2 + v^2)^{-1}]\, ,$$
$$x_2(w) = (v/2)[1 + (u^2 + v^2)^{-1}]\, ,$$
$$x_3(w) = (1/2)\log(u^2 + v^2)\, .$$

We then have $M^*(-K) = M(-K) = M_3(-K) = \{|w| = 1\}$.

Finally in this section we propose [27, Theorem 2]:

Theorem 9.2. *Let $x : G \to \mathcal{R}^3$ be a regular minimal surface contained in no plane. Suppose that $c \in M_3(-K)$ exists and further that the Jordan domain Δ bounded by c is contained in G. Then,*

$$-T(\Delta) = \pi\{\nu_\Delta(g') + \mu_\Delta(g') - 2n\}\, .$$

Here,

$$T(\Delta) = 2\iint_\Delta K \sum_{k=1}^{3} |\partial x_k/\partial w|^2 du\, dv$$

is the total curvature of the subsurface $x : \Delta \to \mathcal{R}^3$ and we consequently have

$$-T(\Delta) = 4\iint_\Delta g^{\#2} du\, dv\, .$$

Here, $\nu_\Delta(g')$ and $\mu_\Delta(g')$ are the total number of the zeros and poles of g' in Δ, respectively, the multiplicities being counted, and n is the total number of the distinct poles of g' in Δ.

There does exist x for which $\Delta \subset G$ actually happens as described in Theorem 9.2. A simple example is $x : \mathbb{C} \to \mathcal{R}^3$, with the Gauss map $g(w) = w^2$, defined by the Weierstrass–Enneper formulae:

$$x_1(w) = (1/2)\mathrm{Re} \int_0^w (1 - g(\zeta)^2) d\zeta ,$$

$$x_2(w) = (1/2)\mathrm{Re} \int_0^w i(1 + g(\zeta)^2) d\zeta ,$$

$$x_3(w) = \mathrm{Re} \int_0^w g(\zeta) d\zeta .$$

We then observe that $M^*(-K) = M(-K)$ is the circle $\{|w| = 7^{-1/4}\}$ which surrounds the disk $\Delta \subset \mathbb{C}$.

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