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Spectral Values and Eigenvalues of a Quasicircle

Abstract. This paper aims at generalizing the notion of Fredholm eigenvalues of a Jordan curve Γ in the case when Γ is a quasicircle. The properties of spectral values and eigenvalues of a quasisymmetric automorphism of the unit circle $\mathbb{T}$ and of a quasicircle, introduced here, are studied. In particular the eigenvalues existence problem for a quasisymmetric automorphism and a quasicircle is considered. This paper improves and completes results from [19] and proves theorems quoted in [12].

0. Introduction. Let Γ be a smooth Jordan curve in the finite plane $\mathbb{C}$. Many important problems in conformal mapping and the potential theory can be reduced to the solution of a linear integral equation of Fredholm type with the Neumann-Poincaré kernel (or its transposition)

$$(0.1)\qquad k(\zeta, z) = -\frac{1}{\pi}\frac{\partial}{\partial \vec{n}_\zeta}\log|\zeta - z|\,, \quad \zeta, z \in \Gamma\,, \quad \zeta \neq z\,,$$

where $\dfrac{\partial}{\partial \vec{n}_\zeta}$ denotes the derivative along the interior normal of Γ at the point ζ. For details see e.g. [4], [6]. If $\Gamma \in C^3$ then the kernel k has a continuous extension on $\Gamma \times \Gamma$, where $k(z,z) = \frac{1}{2\pi}\kappa(z)$ and $\kappa(z)$ is the curvature of Γ at $z \in \Gamma$. A real number λ is called a Fredholm eigenvalue of Γ if it is an eigenvalue of the kernel k, i.e. the homogeneous integral equation

$$(0.2)\qquad \mu(z) = \lambda \int_\Gamma k(\zeta, z)\mu(\zeta)|d\zeta|\,, \quad \text{for a.e. } z \in \Gamma$$

has a non-constant real-valued solution μ integrable on Γ. The set of all Fredholm eigenvalues of Γ, denoted by $\Lambda_k(\Gamma)$, plays an important role as far as the existence of solutions of Fredholm integral equation with the kernel k is concerned. In particular the smallest positive Fredholm eigenvalue λ_0 of Γ determines the rate of convergence of the Neumann series for the kernel k. The basic results in the theory of Fredholm eigenvalues of a Jordan curve were established by Bergman and Schiffer for analytic Jordan curves in [2]. Later on Schiffer, cf. [24], was able to prove many interesting properties of Fredholm eigenvalues of a Jordan curve Γ under the assumption $\Gamma \in C^3$. Ahlfors, cf. [1], has proved that $\lambda_0 \geq \frac{K+1}{K-1}$ if Γ admits a K-quasiconformal reflection. Moreover, under some assumptions Γ admits the unique extremal K-quasiconformal reflection, i.e. the reflection with the smallest maximal dilatation, as pointed out by Kühnau in [14] and Krzyż in [8]. Then $\lambda_0 = \frac{K+1}{K-1}$. Another interesting relations between Fredholm eigenvalues and some extremal problems involving conformal mappings with quasiconformal extension were obtained by Kühnau in [13], [14], [15] and by Schiffer in [25]. The new approach to the topic of Fredholm eigenvalues $\Lambda_k(\Gamma)$ of a Jordan curve Γ being a quasicircle was presented in [19]. The main idea depends on restricting the consideration to the unit circle $\mathbf{T}$ where all information about Γ is stored in a sense-preserving homeomorphic self-mapping γ of $\mathbf{T}$, being a welding homeomorphism or a conformal parametrization of Γ. Then, as shown in [19, Theorem 5.3 (vii)], all Fredholm eigenvalues of Γ are strictly related to eigenvalues of a linear operator R_γ assigned to γ, in the case of Γ being an analytic Jordan curve. In fact, this approach has its roots in a clever idea due to Krzyż [7] who expressed a Fredholm eigenvalue by means of a pair of functions analytic in the complementary domains of Γ. The present paper completes and develops results obtained in [19], so it can be regarded as its continuation, or its second part. Most results obtained here were presented, without proofs, in a special issue to memory of Glenn Schober, cf. [12], so the reader may consult this paper for proofs. The detailed references to papers [19] and [12] will be given while dealing with respective results. We now give some preliminaries concerning the present paper.

Let $\mathbb{Q}_{\mathbf{T}}(K)$, $1 \leq K < \infty$, be the class of all homeomorphic self-mappings of the unit circle $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ which admit a K-quasiconformal extension to the unit disc $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ and let $\mathbb{Q}_{\mathbf{T}} = \bigcup_{1\leq K<\infty} \mathbb{Q}_{\mathbf{T}}(K)$. For any homeomorphism $\gamma \in \mathbb{Q}_{\mathbf{T}}$ we set $K(\gamma) = \inf\{K : \gamma \in \mathbb{Q}_{\mathbf{T}}(K)\}$. Due to Krzyż characterization, cf. [9], the class $\mathbb{Q}_{\mathbf{T}}$ coincides with the class of all quasisymmetric (abbreviated: qs.) automorphisms of the unit circle $\mathbf{T}$, i.e. all sense-preserving homeomorphisms $\gamma : \mathbf{T} \to \mathbf{T}$ satisfying

$$k^{-1} \leq |\gamma(I_1)|/|\gamma(I_2)| \leq k \tag{0.3}$$

for each pair of adjacent closed arcs $I_1, I_2 \subset \mathbf{T}$ of equal length: $0 < |I_1| = |I_2| \leq \pi$ where the constant k depends on γ only. We denote by $L^p_{\mathbf{T}}$, $1 \leq p < \infty$, the space of all functions $f : \mathbf{T} \to \mathbf{R}$, p-integrable on $\mathbf{T}$, i.e. $\|f\|_p = (\int_{\mathbf{T}} |f(z)|^p |dz|)^{1/p} < \infty$ and we set $L^\infty_{\mathbf{T}} = \{f \in L^1_{\mathbf{T}} : \|f\|_\infty = \sup\text{ess}_{z\in\mathbf{T}} |f(z)| < \infty\}$. The space $L^2_{\mathbf{T}}$ is a real Hilbert space with the usual inner product $(f,g) = \int_{\mathbf{T}} f(z)g(z)|dz|$, $f,g \in L^2_{\mathbf{T}}$. To any function $f \in L^1_{\mathbf{T}}$ we can assign an analytic function $f_\Delta : \Delta \to \mathbf{C}$ given by the formula

$$f_\Delta(z) = \frac{1}{2\pi}\int_{\mathbf{T}} f(u)\frac{u+z}{u-z}|du| = \frac{1}{2\pi}\int_{\mathbf{T}} f(u)|du| + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big(\int_{\mathbf{T}} f(u)\overline{u}^n|du|\Big)z^n\,, \quad z \in \Delta\,.$$

The space $\mathbf{H} = \{f \in L^1_{\mathbf{T}} : \int_\Delta |f'_\Delta|^2 dS < \infty \text{ and } f_\Delta(0) = 0\}$, where $f'_\Delta = (f_\Delta)'$, equipped with the inner product $(f,g)_{\mathbf{H}} = \operatorname{Re}\int_\Delta f'_\Delta \overline{g'_\Delta}\, dS$, $f,g \in \mathbf{H}$, is a real Hilbert space isometric to the space $\tilde{L}^2_{\mathbf{T}} = \{f \in L^2_{\mathbf{T}} : f_\Delta(0) = 0\}$, cf. [19, Theorem 1.2]. If $L : \mathbf{H} \to \mathbf{H}$ is a bounded linear operator then we denote by $\Lambda(L)$ its spectrum, i.e. the set of all $\lambda \in \mathbf{R}$ for which the operator $\lambda I - L$ is not a homeomorphism of $\mathbf{H}$ onto itself. The spectrum $\Lambda(L)$ contains the discrete spectrum $\Lambda^*(L)$ of L consisting of all its eigenvalues i.e. all $\lambda \in \mathbb{R}$ for which there exists $f \in \mathbf{H}$, $\|f\|_{\mathbf{H}} = 1$ such that $L(f) = \lambda f$.

In the paper [19, Theorem 3.1] the linear bounded symmetric operator $R_\gamma : \mathbf{H} \to \mathbf{H}$ was associated with every qs. automorphism $\gamma \in \mathbf{Q_T}$. The purely real spectrum $\Lambda(R_\gamma)$ of R_γ was applied to define a spectral value of a qs. automorphism of the unit circle $\mathbf{T}$, cf. [19, Definition 4.1]. The implicit construction of the operator R_γ, based on functional analysis methods, enabled us to prove the basic properties of the spectrum $\Lambda(R_\gamma)$ and, consequently, spectral values of a qs. automorphism γ, cf. [19, Theorems 3.1, 4.2]. On the other hand, in the paper [20, (1.7)], the linear homeomorphism A_γ of the Hilbert space $\mathbf{H}$ onto itself, was assigned explicitly to any qs. automorphism $\gamma \in \mathbf{Q_T}$ by the equality

$$A_\gamma = B_\gamma A B_{\gamma^{-1}}. \tag{0.4}$$

The operator B_γ is a linear homeomorphism of the space $(\mathbf{H}, \|\cdot\|_{\mathbf{H}})$ onto itself such that

$$B_\gamma(f) = f \circ \gamma - (f \circ \gamma)_\Delta(0) \tag{0.5}$$

for every continuous function $f \in \mathbf{H}$ and A is the harmonic conjugation operator, defined by the singular integral

$$A(f)(z) = \frac{1}{\pi}\mathrm{Re}\ \mathrm{P.V.}\int_{\mathbf{T}} \frac{f(u)}{z-u}\,du = \lim_{r \to 1^-} \mathrm{Im} f_\Delta(rz) \tag{0.6}$$

for every function $f \in L^2_{\mathbf{T}} \supset \mathbf{H}$ and for a.e. $z \in \mathbf{T}$, cf. [5] or [32]. The basic properties of the operator A_γ justify calling it the generalized harmonic conjugation operator, cf. [20, Theorems 1.3, 1.4, Corollary 2.3]. For more detailes about A_γ we refer to [20]. The relation between operators R_γ and A_γ is given by the following deep equality, cf. [20, (2.4)]

$$R_\gamma = I + AA_\gamma \tag{0.7}$$

where I is the identity operator. This essential equality can be regarded as a decomposition of the operator R_γ and it enables us to apply the results concerning the operator A_γ obtained in [20] to study the spectrum $\Lambda(R_\gamma)$ of the operator R_γ. In particular, we can characterize more naturally and conveniently spectral values Λ_γ and eigenvalues Λ^*_γ of a qs. automorphism $\gamma \in \mathbf{Q_T}$ by means of the generalized hormonic conjugation operator A_γ. Thus we state in the first section Definitions 1.1 and 1.2 which show to be equivalent to those in [19], [12] defining Λ_γ, Λ^*_γ by means of $\Lambda(R_\gamma)$ (see Lemma 1.3 and the equalities (1.6)). Their advantage depends on the fact that they enable to apply directly the properties of the operator A_γ established in [20] which evidently simplify considerations in the present paper. For example, Theorem 1.4 presenting the basic properties of spectral values and eigenvalues of a qs. automorphism $\gamma \in \mathbf{Q_T}$, is a direct conclusion from [20, Theorem 2.2]. As another consequence of [20, Theorem 2.2] we establish at the end of the first section the inclusion (1.7) and the estimate (1.8) which improve Theorem 3.1 (ii), (iii) in [19], presenting basic properties of the operator R_γ. The second section is devoted to an eigenvalue existence problem. We study the cases $\Lambda^*_\gamma \neq \emptyset$ and $\Lambda^*_\gamma = \Lambda_\gamma$. In the third section we define in Definition 3.1 spectral values and eigenvalues of a quasicircle as spectral values and eigenvalues of a qs. automorphism of the unit circle being its welding homeomorphism. This correspondence between spectral values and eigenvalues of a quasicircle on the one hand and qs. automorphism of the unit circle on the other hand, enables in a natural way to apply results established in the previous sections. This way Theorems 3.2, 3.3, 3.4, giving basic properties of spectral values and eigenvalues of a quasicircle, are consequences of Theorems 1.4, 2.4 and 2.5, respectively. In the last section we show that if Γ is a sufficiently regular Jordan curve in the complex plane $\mathbb{C}$ then spectral values and eigenvalues defined in Definition 3.1 coincide with classical Fredholm eigenvalues of Γ studied by Bergman and Schiffer in [2] and [24].

1. Spectral values and eigenvalues of a qs. automorphism of the unit circle. In the papers [19] and [12], the notions of a spectral value and an eigenvalue of a qs. automorphism $\gamma \in \mathbf{Q_T}$ of the unit circle were introduced by means of the spectrum $\Lambda(R_\gamma)$ of the operator R_γ. In what follows, we define them equivalently but in a more convenient way applying the generalized harmonic conjugation operator A_γ.

Definition 1.1. A real number λ is said to be an eigenvalue of a qs. automorphism $\gamma \in \mathbf{Q_T}$, if there exists a function $f \in \mathbf{H}$ with the norm $\|f\|_{\mathbf{H}} = 1$ such that

$$(\lambda + 1)A(f) = (\lambda - 1)A_\gamma(f) \,. \tag{1.1}$$

The function f is said to be an eigenfunction of γ associated with the eigenvalue λ.

The set of all eigenvalues of $\gamma \in \mathbf{Q_T}$ is denoted by Λ^*_γ.

Definition 1.2. A real number λ is said to be a spectral value or an approximate eigenvalue of a qs. automorphism $\gamma \in \mathbf{Q_T}$, if there exist functions $f_n \in \mathbf{H}$, $\|f_n\|_{\mathbf{H}} = 1$, $n = 1, 2, \ldots$ such that

$$\|(\lambda + 1)A(f_n) - (\lambda - 1)A_\gamma(f_n)\|_{\mathbf{H}} \to 0 \quad \text{as } n \to \infty \,. \tag{1.2}$$

The set of all spectral values of $\gamma \in \mathbf{Q_T}$ is denoted by Λ_γ.

The following lemma points out a strict relation between the sets Λ^*_γ, Λ_γ and $\Lambda^*(AA_\gamma)$, $\Lambda(AA_\gamma)$.

Lemma 1.3. *For every qs. automorphism $\gamma \in \mathbf{Q_T}$*

$$\lambda \in \Lambda^*_\gamma \quad \textit{iff} \quad \frac{1+\lambda}{1-\lambda} \in \Lambda^*(AA_\gamma) \setminus \{-1\} \tag{1.3}$$

$$\lambda \in \Lambda_\gamma \quad \textit{iff} \quad \frac{1+\lambda}{1-\lambda} \in \Lambda(AA_\gamma) \setminus \{-1\} \tag{1.4}$$

Proof. Assume that $\lambda \in \Lambda^*_\gamma$. Then the equality (1.1) holds for some $f \in \mathbf{H}$, $\|f\|_{\mathbf{H}} = 1$. Hence $\lambda \neq 1$ and letting the operator A act on both sides in the equality

(1.1) we get

$$\frac{1+\lambda}{1-\lambda} f = AA_\gamma(f) \tag{1.5}$$

so $(1+\lambda)/(1-\lambda) \in \Lambda^*(AA_\gamma) \setminus \{-1\}$. Conversely, if $\lambda' \in \Lambda^*(AA_\gamma) \setminus \{-1\}$ then there exists $f \in \mathbf{H}$, $\|f\|_{\mathbf{H}} = 1$, satisfying (1.5) with $\lambda = (\lambda'-1)/(\lambda'+1)$. Letting once again the operator A act on both sides in the equality (1.5) we obtain the equality (1.1). This means that $\lambda \in \Lambda^*_\gamma$ which proves (1.3).

Suppose now that $\lambda \in \Lambda_\gamma$. It can not be equal 1 because of (1.2) and [20, Theorem 1.3]. Then setting $\lambda' = (1+\lambda)/(1-\lambda)$ we derive from (1.2) and [20, Theorem 1.3] that

$$\|\lambda' f_n - AA_\gamma(f_n)\|_{\mathbf{H}} = \frac{1}{|1-\lambda|}\|A\|\|(\lambda+1)A(f_n) - (\lambda-1)A_\gamma(f_n)\|_{\mathbf{H}} \to 0 \quad \text{as } n \to \infty\,.$$

This means that the operator $\lambda' I - AA_\gamma$ is not a homeomorphism of the space $\mathbf{H}$ onto itself so $\lambda' \in \Lambda(AA_\gamma) \setminus \{-1\}$. Conversely, let now $\lambda' \in \Lambda(AA_\gamma) \setminus \{-1\}$ be arbitrary. Since AA_γ is a symmetric operator, cf. [20, Theorem 2.2 (i)], so λ' is an approximate eigenvalue of AA_γ, i.e. there exist functions $f_n \in \mathbf{H}$, $\|f_n\|_{\mathbf{H}} = 1$, $n = 1, 2, \ldots$ such that

$$\|\lambda' f_n - AA_\gamma(f_n)\|_{\mathbf{H}} \to 0 \quad \text{as } n \to \infty\,.$$

Hence, setting $\lambda = (\lambda'-1)/(\lambda'+1)$, we get by (1.2) and [20, Theorem 1.3] that

$$\|(\lambda+1)A(f_n) - (\lambda-1)A_\gamma(f_n)\|_{\mathbf{H}} = |1-\lambda|\|A\|\|\lambda' f_n - AA_\gamma(f_n)\|_{\mathbf{H}} \to 0 \quad \text{as } n \to \infty\,.$$

Thus $\lambda \in \Lambda_\gamma$ which proves (1.4).

Applying the above lemma we easily derive from [20, Theorem 2.2] the following basic properties of the spectral values and eigenvalues of a qs. automorphism $\gamma \in \mathbf{Q_T}$.

Theorem 1.4. [11, (3.6)] *For any qs. automorphism $\gamma \in \mathbf{Q_T}$ the following properties hold:*
(i) $\Lambda_\gamma = \emptyset$ *iff* $\gamma \in \mathbf{Q_T}(1)$;
(ii) $\Lambda^*_\gamma \subset \Lambda_\gamma$;
(iii) *if* $\lambda \in \Lambda_\gamma$ *then* $|\lambda| \geq \frac{K(\gamma)+1}{K(\gamma)-1}$;
(iv) *for every* $\nu, \eta \in \mathbf{Q_T}(1)$ $\Lambda_\gamma = \Lambda_{\nu\circ\gamma\circ\eta}$ *and* $\Lambda^*_\gamma = \Lambda^*_{\nu\circ\gamma\circ\eta}$;
(v) $\Lambda_\gamma = \Lambda_{\gamma^{-1}} = \Lambda_{\bar{i}_{\mathbf{T}}\circ\gamma\circ i_{\mathbf{T}}}$ *and* $\Lambda^*_\gamma = \Lambda^*_{\gamma^{-1}} = \Lambda^*_{\bar{i}_{\mathbf{T}}\circ\gamma\circ i_{\mathbf{T}}}$;
(vi) *if* $\lambda \in \Lambda_\gamma$ *then* $-\lambda \in \Lambda_\gamma$ *and if* $\lambda \in \Lambda^*_\gamma$ *then* $-\lambda \in \Lambda^*_\gamma$
where $i_{\mathbf{T}}(z) = z$, $z \in \mathbf{T}$.

Proof. The properties (i)–(vi) are direct consequences of the respective properties (ii)–(viii) in [20, Theorem 2.2] and the equivalences (1.3) and (1.4).

Remark. The equality (0.7) yields for every qs. automorphism $\gamma \in \mathbf{Q_T}$

$$\Lambda(R_\gamma) = \{\lambda+1 : \lambda \in \Lambda(AA_\gamma)\} \quad \text{and} \quad \Lambda^*(R_\gamma) = \{\lambda+1 : \lambda \in \Lambda^*(AA_\gamma)\}\,. \tag{1.6}$$

The equalities (1.6), (1.3) and (1.4) mean that Definitions 1.1, 1.2 and the definitions of a spectral value and an eigenvalue of a qs. automorphism $\gamma \in \mathbf{Q_T}$ inserted in [19] and [12] are equivalent. This way Theorem 1.4 completes and improves [19, Theorem 4.2]. Moreover, by the equalities (1.6) and [20, Theorem 2.2 (viii)] we get the inclusion[1)]

$$\Lambda(R_\gamma) \subset [1 - K(\gamma), 1 - \tfrac{1}{K(\gamma)}] \tag{1.7}$$

which leads in view of the symmetry of the operator R_γ, cf. [19, Theorem 3.1 (i)], to the following estimate of its norm[2)]

$$\|R_\gamma\| = \sup\{|\lambda| : \lambda \in \Lambda(R_\gamma)\} \leq K(\gamma) - 1 . \tag{1.8}$$

The properties (1.7) and (1.8) improve [19, Theorem 3.1 (ii), (iii)].

2. Eigenvalues of a qs. automorphism of the unit circle. In this section we restrict our considerations to eigenvalues of a qs. automorphism of the unit circle $\mathbf{T}$. In particular we study the eigenvalue existence problem. More precisely, we try to characterize qs. automorphisms $\gamma \in \mathbf{Q_T}$ for which $\Lambda^*_\gamma \neq \emptyset$, or $\Lambda^*_\gamma = \Lambda_\gamma$.

Let τ be the Teichmüller pseudometric in the class $\mathbf{Q_T}$ i.e. $\tau(\gamma,\sigma) = \log K(\gamma \circ \sigma^{-1})$. The closure of a subset $X \subset \mathbf{Q_T}$ with respect to τ will be denoted by $\mathrm{cl}_\tau(X)$. We denote by $\mathbf{A_T}$ the class of all analytic automorphisms of $\mathbf{T}$, i.e. homeomorphic self-mapping of $\mathbf{T}$ which have a conformal extension on some annulus containing $\mathbf{T}$.

Theorem 2.1. *If a homeomorphism $\gamma \in \mathrm{cl}_\tau(\mathbf{A_T})$ then $\Lambda^*_\gamma = \Lambda_\gamma$.*

Proof. Let $\gamma \in \mathrm{cl}_\tau(\mathbf{A_T})$ be arbitrary. Then there exist analytic automorphisms $\gamma_n \in \mathbf{A_T}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \tau(\gamma_n, \gamma) = 0$. Hence, in view of the equality (0.7) and [20, Theorem 1.3, Theorem 3.1 (iv)], we get

$$\begin{aligned} &\|R_\gamma - R_{\gamma_n}\| = \|AA_\gamma - AA_{\gamma_n}\| = \|A\|\|A_\gamma - A_{\gamma_n}\| \leq \\ &(K(\gamma_n \circ \gamma^{-1}) - 1)\min\{K(\gamma), K(\gamma_n)\} = (e^{\tau(\gamma_n,\gamma)} - 1)K(\gamma) \to 0 \quad \text{as } n \to \infty . \end{aligned} \tag{2.1}$$

But in view of [19, Theorem 3.1 (vi)] every operator R_{γ_n}, $n \in \mathbb{N}$, is compact. Thus it follows from (2.1) that R_γ is a compact operator, too. Moreover, R_γ is a symmetric operator, cf. [19, Theorem 3.1 (i)], so $\Lambda^*(R_\gamma) = \Lambda(R_\gamma)$. Hence, by the equalities (1.6) and Lemma 1.3 we get $\Lambda^*_\gamma = \Lambda_\gamma$ which ends the proof.

From this theorem and Theorem 1.4 (i) we immediately obtain

Corollary 2.2. *If a qs. automorphism $\gamma \in \mathrm{cl}_\tau(\mathbf{A_T}) \setminus \mathbf{Q_T}(1)$ then $\Lambda^*_\gamma = \Lambda_\gamma \neq \emptyset$.*

Remark. It is quite easy to check that $\mathrm{cl}_\tau(\mathbf{A_T}) \neq \mathbf{Q_T}$. So a natural problem appears to find the widest subclass X of $\mathbf{Q_T}$ for which $\Lambda^*_\gamma \neq \emptyset$ as $\gamma \in X \setminus \mathbf{Q_T}(1)$. It is quite possible that $X = \mathbf{Q_T}$.

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1) cf. [12, (3.5), (i)]

2) cf. [12, (3.5), (ii)]

Another characterization of the class $\mathrm{cl}_\tau(\mathbf{A_T})$ without the metric τ will be considered in a separate paper. Now, we shall prove by a modified the convolution technique used in [18, Theorem] the following

Proposition 2.3. *If a sense-preserving homeomorphism $\gamma : \mathbf{T} \to \mathbf{T}$ has a nonvanishing derivative at every point $\zeta \in \mathbf{T}$, i.e. $\gamma'(\zeta) = \lim_{z\to\zeta} \frac{\gamma(z)-\gamma(\zeta)}{z-\zeta} \neq 0$, $z, \zeta \in \mathbf{T}$, and $\log|\gamma'|$ is continuous on $\mathbf{T}$ then $\gamma \in \mathrm{cl}_\tau(\mathbf{A_T})$.*

Proof. Let $P_n(z) = \frac{1}{\pi}\frac{n}{n^2z^2+1}$, $n \in \mathbb{N}$. Setting for every $z \in \Omega_n = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < 1/n\}$ and $n \in \mathbb{N}$

$$\omega_n(z) = c_n \int_0^z \exp\Big(\int_{-\infty}^{\infty} P_n(w-t)\log|\gamma'(e^{it})|\, dt\Big)\, dw$$

where integrating runs over the line segment $[0, z]$ and

$$2\pi c_n^{-1} = \int_0^{2\pi} \exp\Big(\int_{-\infty}^{\infty} P_n(x-t)\log|\gamma'(e^{it})|\, dt\Big)\, dx \tag{2.2}$$

we see that ω_n is an analytic function in the strip Ω_n. Moreover, for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$

$$\omega_n'(x) = c_n \exp\Big(\int_{-\infty}^{\infty} P_n(x-t)\log|\gamma'(e^{it})|\, dt\Big) > 0 \tag{2.3}$$

and in view of (2.2)

(2.4)

$$\omega_n(z+2\pi) = c_n \int_0^{2\pi} \exp\Big(\int_{-\infty}^{\infty} P_n(w-t)\log|\gamma'(e^{it})| dt\Big)\, dw +$$
$$c_n \int_{2\pi}^{z+2\pi} \exp\Big(\int_{-\infty}^{\infty} P_n(w-t)\log|\gamma'(e^{it})|\, dt\Big)\, dw =$$
$$= 2\pi + c_n \int_0^z \exp\Big(\int_{-\infty}^{\infty} P_n(w+2\pi-t)\log|\gamma'(e^{it})|\, dt\Big)\, dw = 2\pi + \omega_n(z)$$

as $z \in \Omega_n$. Hence and by (2.3) we conclude that ω_n is a conformal mapping in some strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \varepsilon_n\}$, $n \in \mathbb{N}$. Thus in view of (2.4) the mapping $z \mapsto \exp(i\omega_n(-i\log z))$ is conformal in the annulus $\{z \in \mathbb{C} : \big|\log|z|\big| < \varepsilon_n\}$ for every $n \in \mathbb{N}$. This way setting $\gamma_n(e^{it}) = e^{i\omega_n(t)}$, $t \in \mathbb{R}$, we see that all $\gamma_n \in \mathbf{A_T}$, $n \in \mathbb{N}$. Moreover, by the properties of Poisson integral and by (2.4) we get

$$\|\log|\gamma_n'| - \log|\gamma'|\|_\infty \leq \max_{x\in\mathbb{R}} \Big| \int_{-\infty}^{\infty} P_n(x-t)\log|\gamma'(e^{it})| dt - \log|\gamma'(e^{ix})| \Big| \to 0$$

as $n \to \infty$. Hence, setting $\log k_n = 2\|\log|(\gamma_n \circ \gamma^{-1})'|\,\|_\infty$, we have

$$\lim_{n\to\infty} \log k_n = 2 \lim_{n\to\infty} \Big\|\log \frac{|\gamma_n' \circ \gamma^{-1}|}{|\gamma' \circ \gamma^{-1}|}\Big\|_\infty = 2 \lim_{n\to\infty} \big\|\log|\gamma_n'| - \log|\gamma'|\,\big\|_\infty = 0\,. \tag{2.5}$$

Obviously, every homeomorphism $\gamma_n \circ \gamma^{-1}$, $n \in \mathbb{N}$, satisfies the condition (0.3) with the constant $k := k_n$ so it is a k_n-qs. automorphism. Thus applying the results from [9] and [16] we obtain that $\gamma_n \circ \gamma^{-1} \in \mathbf{Q_T}(k_n^2)$, $n \in \mathbb{N}$. This together with (2.5) leads to

$$\tau(\gamma_n, \gamma) = \log K(\gamma_n \circ \gamma^{-1}) \leq \log k_n^2 \to 0 \quad \text{as } n \to \infty \,,$$

which proves that $\gamma \in \mathrm{cl}_\tau(\mathbf{A_T})$.

The following theorems provide criteria of another sort which guarantee the existence of an eigenvalue of a qs. automorphism $\gamma \in \mathbf{Q_T}$. We remind that a quasiconformal selfmapping φ of the unit disc Δ is said to be a regular Teichmüller mapping if there exists an analytic function $\psi : \Delta \to \mathbf{C}$ and a constant k, $0 \leq k < 1$, such that the complex dilatation of φ has the form

$$\frac{\overline{\partial}\varphi}{\partial\varphi} = k\frac{\overline{\psi}}{|\psi|} \,.$$

Theorem 2.4. *Suppose $\gamma \in \mathbf{Q_T}$ and there exist non-constant continuous functions $G, F : \overline{\Delta} \to \mathbf{C}$, analytic in Δ satisfying on $\mathbf{T}$ the following equalities*

$$\mathrm{Im}\, G = \mathrm{Im}\, F \circ \overline{\gamma} \quad , \quad (1-\lambda)\mathrm{Re}\, G = (1+\lambda)\mathrm{Re}\, F \circ \overline{\gamma} \tag{2.6}$$

with some real constant λ and at least one of Dirichlet integrals $\int_\Delta |G'|^2 dS$, $\int_\Delta |F'|^2 dS$ is finite. Then $\lambda \in \Lambda_\gamma^*$. Moreover, if the functions G, F are locally univalent in $\overline{\Delta}$ then $|\lambda|$ is the smallest positive eigenvalue of γ and the qs. automorphism γ has a K-quasiconformal regular Teichmüller extension φ on Δ with $K = \frac{|\lambda|+1}{|\lambda|-1}$. The complex dilatation of φ is represented by the formula

$$\frac{\overline{\partial}\varphi}{\partial\varphi} = -\frac{1}{\lambda}\frac{\overline{(G'(z))^2}}{|(G'(z))^2|} \,, \quad z \in \Delta \tag{2.7}$$

and φ satisfies on Δ the equality

$$G = l \circ F \circ \overline{\varphi} \tag{2.8}$$

where $l(z) = \frac{z+\lambda\overline{z}}{1-\lambda}$, $z \in \mathbf{C}$, is an affine mapping. The mapping φ is the unique extremal quasiconformal extension of γ i.e. the extension with the smallest maximal dilatation.

Proof. Assume $\gamma \in \mathbf{Q_T}$ is arbitrary. Setting $E(z) = \overline{F(\overline{z})}$, $z \in \Delta$, we get by (2.6) that

$$\mathrm{Im}\, G = -\mathrm{Im}\, E \circ \gamma \quad , \quad (1-\lambda)\mathrm{Re}\, G = (1+\lambda)\mathrm{Re}\, E \circ \gamma \tag{2.9}$$

on $\mathbf{T}$. Hence, in view of the equality (0.5) and [20, (1.2) and Lemma 1.1] both integrals $\int_\Delta |G'|^2 dS$, $\int_\Delta |F'|^2 dS = \int_\Delta |E'|^2 dS$ are finite if one of them is finite. Thus there exist functions $f, g \in \mathbf{H}$ such that

$$E(z) = f_\Delta(z) + E(0) \quad \text{and} \quad G(z) = g_\Delta(z) + G(0) \,, \quad z \in \Delta \,.$$

Hence and by the equality (0.6) we get on $\mathbf{T}$

$$\operatorname{Im} E = A(f) + \operatorname{Im} E(0) \quad \text{and} \quad \operatorname{Im} G = A(g) + \operatorname{Im} G(0)$$

so, in view of the equality (0.5), the equalities (2.9) take the form

$$A(g) = -B_\gamma A(f) \quad \text{and} \quad (1-\lambda)g = (1+\lambda)B_\gamma(f)\,.$$

Then, applying the equality (0.4) and [20, Lemma 1.1 and Theorem 1.3], we obtain that $(1-\lambda)g = (1+\lambda)B_\gamma A B_{\gamma^{-1}} A(g) = (1+\lambda)A_\gamma A(g)$ from which $(\lambda+1)A(g) = (\lambda-1)A_\gamma(g)$. But G is a non-constant function so $\|g\|_{\mathbb{H}} \neq 0$ and the same $\lambda \in \Lambda_\gamma^*$. Assume now that G, F are locally univalent functions in $\overline{\Delta}$. Following the proof of Theorem in [8] and using the argument principle and the monodromy principle applied to the function F^{-1} we see that γ has the homeomorphic extension φ on Δ satisfying the equality (2.8) because of the equalities (2.6). By (2.8) we have the local representation $\varphi = \overline{F^{-1}} \circ l^{-1} \circ G = \overline{F^{-1}}\left(\frac{G-\lambda\overline{G}}{1-\lambda}\right)$ from which

$$(1-\lambda)\partial\varphi = \overline{(F^{-1})'(l^{-1}\circ G)}\,(-\lambda G') \quad \text{and} \quad (1-\lambda)\overline{\partial}\varphi = \overline{(F^{-1})'(l^{-1}\circ G)}\,\overline{G'}\,.$$

Thus φ is a regular quasiconformal Teichmüller extension of the qs. automorphism γ to Δ with the complex dilatation (2.7). It follows from [21, Theorem 2.2] that $|\lambda|$ is the smallest positive eigenvalue of γ and φ is an extremal quasiconformal extension of γ with the maximal dilatation $K = \frac{|\lambda|+1}{|\lambda|-1}$. Moreover, $\int_\Delta |G'|^2 dS < \infty$, and in view of Strebel theorem, cf. [28], [29] and also [17], the mapping φ is the unique extremal quasiconformal extension of γ.

Remark. The equalities (2.6) are equivalent to the following one

$$G - F\circ\overline{\gamma} = \lambda(G + \overline{F}\circ\overline{\gamma})\,.$$

which used to appear in [19] instead of (2.6).

If γ is a more regular qs. automorphism of $\mathbf{T}$ then the first part of Theorem 2.4 may be improved as follows

Theorem 2.5. *If $\gamma, \gamma^{-1} \in \mathrm{Q}_{\mathbf{T}}$ are absolutely continuous on $\mathbf{T}$ then a real number $\lambda \in \Lambda_\gamma^*$ iff there exist non-constant analytic functions $G, F : \Delta \to \mathbb{C}$ whose radial limits*

$$G(z) = \lim_{r\to 1^-} G(rz) \quad , \quad F(z) = \lim_{r\to 1^-} F(rz) \quad \textit{for a.e. } z \in \mathbf{T}$$

satisfy a.e. on $\mathbf{T}$ the equalities (2.6) and at least one of Dirichlet integrals $\int_\Delta |G'|^2 dS$, $\int_\Delta |F'|^2 dS$ is finite.

Proof. If the radial limits of the functions G, F satisfy a.e. on $\mathbf{T}$ the equalities (2.6) then we prove, in a similar way as in the proof of Theorem 2.4, that $\lambda \in \Lambda_\gamma^*$. This time we apply [20, Lemma 1.2] instead of the equality (0.5).

Conversely, assume that $\lambda \in \Lambda_\gamma^*$ is an arbitrary eigenvalue of a qs. automorphism γ. Then there exists a function $f \in \mathbf{H}$, $\|f\|_{\mathbf{H}} = 1$ satisfying the equality (1.1) and by [20, Lemma 1.1, Theorem 1.3] the functions $AA_\gamma(f), B_{\gamma^{-1}}(f) \in \mathbf{H}$. Hence both non-constant analytic functions $E = (B_{\gamma^{-1}}(f))_\Delta + E(0)$ and $G = (AA_\gamma(f))_\Delta$ have finite Dirichlet integrals and in view of (0.6) their radial limits satisfy a.e. on $\mathbf{T}$ the following equalities

$$\operatorname{Re} E = B_{\gamma^{-1}}(f) + \operatorname{Re} E(0) \quad , \quad \operatorname{Im} E = AB_{\gamma^{-1}}(f) + \operatorname{Im} E(0)$$
$$\operatorname{Re} G = AA_\gamma(f) \quad , \quad \operatorname{Im} G = AAA_\gamma(f) = -A_\gamma(f) .$$

Then, setting $\operatorname{Re} E(0) = -(B_{\gamma^{-1}}(f) \circ \gamma)_\Delta(0)$, $\operatorname{Im} E(0) = -(AB_{\gamma^{-1}}(f) \circ \gamma)_\Delta(0)$ and applying the equality (0.4), as well as [20, Lemma 1.1, Lemma 1.2 and Theorem 1.3], we get a.e. on $\mathbf{T}$

$$\operatorname{Im} E \circ \gamma = AB_{\gamma^{-1}}(f) \circ \gamma + \operatorname{Im} E(0) = B_\gamma AB_{\gamma^{-1}}(f) = A_\gamma(f) = -\operatorname{Im} G$$

and using additionally the equality (1.1)

$$(\lambda+1)\operatorname{Re} E \circ \gamma = (\lambda+1)B_\gamma B_{\gamma^{-1}}(f) = (\lambda+1)f = -A((\lambda+1)A(f))$$
$$= -A((\lambda-1)A_\gamma(f)) = (1-\lambda)AA_\gamma(f) = (1-\lambda)\operatorname{Re} G .$$

Hence, substituting $F(z) = \overline{E(\overline{z})}$ we easily derive the equalities (2.6) which ends the proof.

3. Spectral values and eigenvalues of a quasicircle. Let Γ be a Jordan curve in the finite plane $\mathbb{C}$ and let Ω and $\Omega_* \ni \infty$ be its complementary domains in the extended plane $\hat{\mathbb{C}}$. Due to the Riemann and Taylor-Osgood-Carathéodory theorems there exist homeomorphisms $\Phi : \overline{\Delta} \to \overline{\Omega}$ and $\Phi_* : \overline{\Delta}_* \to \overline{\Omega}_*$, conformal in Δ and $\Delta_* = \{z \in \hat{\mathbb{C}} : |z| > 1\}$, respectively. The homeomorphisms Φ and Φ_* generate a homeomorphism $\gamma = \Phi_*^{-1} \circ \Phi$ of the unit circle $\mathbf{T}$ onto itself which is said to be *the welding homeomorphism or the conformal parametrization of a Jordan curve* Γ, cf. [19, Definition 5.1]. The set of all welding homeomorphisms of Γ will be denoted by $\Gamma_{\mathbf{T}}$. It is easy to show that any two welding homeomorphisms γ, σ belong to $\Gamma_{\mathbf{T}}$ iff there exist Möbius transformations $\nu, \eta \in \mathbf{Q}_{\mathbf{T}}(1)$ such that $\sigma = \nu \circ \gamma \circ \eta$. Moreover, $\Gamma_{\mathbf{T}} \subset \mathbf{Q}_{\mathbf{T}}(K)$ iff Γ is a K-quasicircle, $K \geq 1$, i.e. Γ admits a K-quasiconformal reflection, cf. [19, (5.1) and (5.2)]. These properties and Theorem 1.4 (iv) enable us to define, in a natural way, eigenvalues and spectral values of a quasicircle by means of eigenvalues and spectral values of its arbitrary welding homeomorphism. In fact, we shall show in the next section that the so defined eigenvalues and spectral values of a quasicircle Γ coincide with classical Fredholm eigenvalues $\Lambda_k(\Gamma)$ defined by the Neumann–Poincaré kernel k, cf. (0.1) and (0.2), if Γ is sufficiently regular. This way the following definition extends classical Fredholm eigenvalues $\Lambda_k(\Gamma)$ to the case of Γ being a quasicircle.

Definition 3.1. A real number λ is said to be an eigenvalue or a spectral value of a quasicircle $\Gamma \subset \mathbb{C}$ if $\lambda \in \Lambda_\gamma^*$ or $\lambda \in \Lambda_\gamma$, respectively, for $\gamma \in \Gamma_{\mathbf{T}}$ being any welding homeomorphism of Γ.

The set of all eigenvalues and spectral values of a quasicircle $\Gamma \subset \mathbb{C}$ is denoted by $\Lambda^*(\Gamma)$ and $\Lambda(\Gamma)$, respectively. In view of Lemma 1.3 and the equalities (1.6) the above definition is equivalent to those in [19, Definition 5.2] and [12]. Thus the following theorem dealing with the basic properties of spectral values and eigenvalues of a quasicircle $\Gamma \subset \mathbb{C}$ and Corollary 4.3 improve [19, Theorem 5.3].

Theorem 3.2. [12, (3.7) (i)–(v)] *If a Jordan curve $\Gamma \subset \mathbb{C}$ admits a K-quasiconformal reflection, $1 \leq K < \infty$ then the following properties hold:*
(i) $\Lambda(\Gamma) = \emptyset$ *iff* Γ *is a circle;*
(ii) $\Lambda^*(\Gamma) \subset \Lambda(\Gamma)$;
(iii) *if* $\lambda \in \Lambda(\Gamma)$ *then* $|\lambda| \geq \frac{K+1}{K-1}$;
(iv) *if* η *is a homography such that* $\infty \notin \eta(\Gamma)$ *then* $\Lambda(\Gamma) = \Lambda(\eta(\Gamma))$ *and* $\Lambda^*(\Gamma) = \Lambda^*(\eta(\Gamma))$;
(v) *if* $\lambda \in \Lambda(\Gamma)$ *then* $-\lambda \in \Lambda(\Gamma)$ *and if* $\lambda \in \Lambda^*(\Gamma)$ *then* $-\lambda \in \Lambda^*(\Gamma)$.

Proof. Assume that a Jordan curve $\Gamma \subset \mathbb{C}$ admits a K-quasiconformal reflection, $1 \leq K < \infty$. Then in view of [19, (5.2)], $\Gamma_{\mathbf{T}} \subset Q_{\mathbf{T}}(K)$ and the properties (i), (ii), (iii) and (v) immediately follow from the properties (i), (ii), (iii), (vi) of Theorem 1.4. Let γ be an arbitrary welding homeomorphism of Γ. If a homography η, $\infty \notin \eta(\Gamma)$, maps the interior of Γ onto the interior of $\eta \circ \Gamma$ then also $\gamma \in (\eta \circ \Gamma)_{\mathbf{T}}$ and the equalities in (iv) are obvious. But if η maps the interior of Γ onto the exterior of $\eta \circ \Gamma$ then $i_{\mathbf{T}} \circ \gamma^{-1} \circ i_{\mathbf{T}} \in (\eta \circ \Gamma)_{\mathbf{T}}$. This together with Theorem 1.4 (v) leads to (iv) which ends the proof.

As a corrolary from Theorem 2.4 we obtain the following counterpart of Krzyż theorem, cf. [8].

Theorem 3.3. [3] *Let $\Gamma \subset \mathbb{C}$ be a quasicircle and let $\Omega, \Omega_* \ni \infty$ be its complementary domains in the extended plane $\hat{\mathbb{C}}$. Suppose there exist non-constant continuous functions $G : \overline{\Omega} \to \mathbb{C}$, $F : \overline{\Omega}_* \to \mathbb{C}$, analytic in Ω and Ω_*, respectively, satisfying on Γ the following equalities*

$$\text{(3.1)} \qquad \operatorname{Im} G = \operatorname{Im} F \quad , \quad (1-\lambda)\operatorname{Re} G = (1+\lambda)\operatorname{Re} F$$

with some real constant λ and at least one of Dirichlet integrals $\int_\Omega |G'|^2 dS$, $\int_{\Omega_*} |F'|^2 dS$ is finite. Then $\lambda \in \Lambda^*(\Gamma)$. Moreover, if the functions G, F are locally univalent in $\overline{\Omega}$ and $\overline{\Omega}_*$, respectively, then $|\lambda|$ is the smallest positive eigenvalue of Γ and, as shown in [8], *Γ admits the unique extremal K-quasiconformal reflection Ψ with $K = \frac{|\lambda|+1}{|\lambda|-1}$ which satisfies on Ω the equality $G = l \circ F \circ \Psi$ where l is an affine mapping described in* Theorem 2.4 .

Proof. Assume $\Phi : \overline{\Delta} \to \overline{\Omega}$ and $\Phi_* : \overline{\Delta}_* \to \overline{\Omega}_*$ are homeomorphisms conformal in Δ and Δ_*, respectively. Then $\gamma = \Phi_*^{-1} \circ \Phi$ is a welding homeomorphism of the quasicircle Γ. Setting $\tilde{G}(z) = G \circ \Phi(z)$ and $\tilde{F}(z) = (F \circ \Phi_*(z^{-1}))$, $z \in \Delta$, we infer from (3.1) that

$$\text{(3.2)} \qquad \operatorname{Im} \tilde{G} = \operatorname{Im} \tilde{F} \circ \overline{\gamma} \quad , \quad (1-\lambda)\operatorname{Re} \tilde{G} = (1+\lambda)\operatorname{Re} \tilde{F} \circ \overline{\gamma}$$

[3] This is an improved version of [12, Theorem 3.1]

on $\mathbf{T}$ and at least one of the integrals $\int_\Delta |\tilde{G}'|^2 dS$, $\int_\Delta |\tilde{F}'|^2 dS$ is finite because of the conformal invariance of the Dirichlet integral. Hence and by Theorem 2.4 $\lambda \in \Lambda^*_\gamma = \Lambda^*(\Gamma)$. Moreover, if G, F are locally univalent on $\overline{\Omega}$ and $\overline{\Omega}_*$, respectively, then the functions $\tilde{G}, \tilde{F}$ are locally univalent on $\overline{\Delta}$, too. Then applying Theorem 2.4 once again we get that $|\lambda|$ is the smallest positive eigenvalue of Γ. The mapping $\Psi : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ defined as follows

$$\Psi(z) = \Phi_*\Big(\frac{1}{\overline{\varphi \circ \Phi^{-1}(z)}}\Big)\,,\ z \in \Omega \quad \text{and} \quad \Psi(z) = \Psi^{-1}(z)\,,\ z \in \Omega_*\,,$$

where φ is the K-quasiconformal ($K = \frac{|\lambda|+1}{|\lambda|-1}$) regular Teichmüller extension of the qs. automorphism γ to Δ described in Theorem 2.4, is an extremal K-quasiconformal reflection in Γ. Since φ is the unique extremal quasiconformal extension of γ so Ψ is the unique extremal quasiconformal reflection as well and it satisfies on Ω the equality $G = l \circ \bar{F} \circ \Psi$ because $\tilde{G} = l \circ \tilde{F} \circ \overline{\varphi}$ on Δ. This proves the theorem.

Remark. It is easy to show that the equalities (3.1) are equivalent to the following

$$G - F = \lambda(G + \overline{F})$$

on Γ so the above theorem is a slightly improved version of [12, Theorem 3.1] being in fact a counterpart of J. G. Krzyż theorem, cf. [8], in the case when both functions F and G are locally univalent. Then $|\lambda|$ coincides with the Schober constant $\lambda(\Gamma)$, cf. [26] and also [27].

If Γ is a rectifiable quasicircle then the first part of Theorem 3.3 may be improved as follows

Theorem 3.4. *If Ω and $\Omega_* \ni \infty$ are complementary domains of a rectifiable quasicircle $\Gamma \subset \mathbf{C}$ then a real number $\lambda \in \Lambda^*(\Gamma)$ iff there exist non-constant analytic functions $G : \Omega \to \mathbf{C}$, $F : \Omega_* \to \mathbf{C}$ whose angular limits $F(z)$, $G(z)$, existing for a.e. $z \in \Gamma$, satisfy a.e. on Γ the equalities (3.1) and at least one of the Dirichlet integrals $\int_\Omega |G'|^2 dS$, $\int_{\Omega_*} |F'|^2 dS$ is finite.*

Proof. Let Ω, Ω_*, Φ, Φ_*, $\tilde{F}$, $\tilde{G}$ and γ be the same as in the proof of the previous theorem. It is a well known classical fact that, if Γ is a rectifiable Jordan curve then for every subset $E \subset \mathbf{T}$ the set $\Phi(E)$ ($\Phi_*(E)$) has the arc-length measure zero on Γ iff E is of the arc-length measure zero on $\mathbf{T}$, cf. e.g. [23]. Hence $\gamma, \gamma^{-1} \in Q_\mathbf{T}$, $\gamma = \Phi_*^{-1} \circ \Phi$, are absolutely continuous qs. automorphisms of $\mathbf{T}$. Moreover, there exists a tangent at a.e. point $\zeta \in \Gamma$ and for a.e. $z \in \mathbf{T}$ the curves $\Phi(rz)$, $\Phi_*(rz)$, $0 \le r \le 1$, are orthogonal to Γ. Thus the radial (or equivalently angular) limits of the functions $\tilde{F}$ and $\tilde{G}$ satisfy the equalities (3.2) a.e. on $\mathbf{T}$ iff the angular limits of the functions F and G satisfy the equalities (3.1) a.e. on Γ. In addition $\int_\Omega |G'|^2 dS = \int_\Delta |\tilde{G}'|^2 dS$ and $\int_{\Omega_*} |F'|^2 dS = \int_\Delta |\tilde{F}'|^2 dS$, because of the conformal invariance of the Dirichlet integral. This way the theorem follows as a direct consequence of Theorem 2.5.

4. Classical Fredholm eigenvalues of a smooth Jordan curve. In this section we prove Corollary 4.3 which says that $\Lambda_k(\Gamma) = \Lambda^*(\Gamma) = \Lambda(\Gamma)$ as $\Gamma \in C^3$, i.e.

Γ is a rectifiable Jordan curve in the compiex plane $\mathbb{C}$ with a three times differentiable arc length parametrization. This justifies to consider spectral values and eigenvalues defined in Definition 3.1 as a generalization of classical Fredholm eigenvalues of Γ studied by Bergman and Schiffer in [2] for Γ being an analytic Jordan curve and later on by Schiffer in the case when $\Gamma \in C^3$, cf. [24]. We shall derive Corollary 4.3 from the inclusions $\Lambda_k(\Gamma) \subset \Lambda^*(\Gamma)$ and $\Lambda(\Gamma) \subset \Lambda_k(\Gamma)$, $\Gamma \in C^3$, proved in an extended form in Theorems 4.1 and 4.2, respectively.

Theorem 4.1. *If a rectifiable Jordan curve $\Gamma \subset \mathbb{C}$ is of the class C^3 and $\mu : \Gamma \to \mathbb{R}$ is a non-constant measurable function such that*

$$(4.1) \qquad \int_\Gamma |k(\zeta,t)\mu(\zeta)||d\zeta| < +\infty \quad \textit{for a.e. } t \in \Gamma$$

and satisfies a.e on Γ the equation (0.2) with a real constant λ then μ is an integrable function on Γ and $\lambda \in \Lambda(\Gamma)$. In particular $\Lambda_k(\Gamma) \subset \Lambda^*(\Gamma)$.

Proof. Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ of length d and an arc length parametrization $\zeta : [0,d] \to \mathbb{C}$ determining the positive orientation with respect to its inside Ω. We say that $\Gamma \in C^n$, $n = 1,2,\ldots$ if ζ is n-times continuously differentiable on $\mathbb{R}$ after its periodic extension. If $\Gamma \in C^1$ then

$$(4.2) \qquad k(\zeta,t) = \frac{1}{\pi}\mathrm{Im}\frac{\zeta'(s)}{\zeta - t} = \frac{1}{\pi}\frac{1}{|\zeta - t|^2}\mathrm{Im}(\zeta'(s)\overline{\zeta - t}) \ , \quad \zeta, t \in \Gamma,\ \zeta \neq t.$$

Assume now that $\Gamma \in C^3$. Then by Taylor expansion

$$(4.3) \qquad 2\pi k(\zeta(s),\zeta(s+h)) = \mathrm{Im}(\overline{\zeta'(s)}\zeta''(s)) + \frac{1}{3}h\mathrm{Im}(\overline{\zeta'(s)}\zeta'''(s)) + o_1(s,h)$$

for $s,h \in \mathbb{R}$, $0 < |h| < d$, where $o_1(s,h)/h \to 0$ as $h \to 0$ uniformly w.r.t. $s \in \mathbb{R}$. It is easy to show that the term $\mathrm{Im}(\overline{\zeta'(s)}\zeta''(s))$ in (4.3) is the curvature of Γ at the point $\zeta(s) \in \Gamma$. Let

$$(4.4) \qquad K(s,t) = \begin{cases} k(\zeta(s),\zeta(t)) & \text{as } \zeta(s) \neq \zeta(t) \\ \dfrac{1}{2\pi}\mathrm{Im}(\overline{\zeta'(s)}\zeta''(s)) & \text{as } \zeta(s) = \zeta(t) \end{cases}$$

Hence and by (4.2) and (4.3) we get

$$(4.5) \qquad K'_{|2}(s,t) =: \frac{\partial}{\partial t}K(s,t) = \begin{cases} \dfrac{1}{\pi}\mathrm{Im}\dfrac{\zeta'(s)\zeta'(t)}{(\zeta(s)-\zeta(t))^2} & \text{as } \zeta(s) \neq \zeta(t) \\ \dfrac{1}{6\pi}\mathrm{Im}(\overline{\zeta'(s)}\zeta'''(s)) & \text{as } \zeta(s) = \zeta(t) \end{cases}$$

By this and the equality $\mathrm{Im}(\zeta'(s)\overline{\zeta''(s)})^2 = 0$ we obtain that

$$K'_{|2}(s,s+h) = \frac{1}{\pi}\mathrm{Im}\frac{\zeta'(s)\zeta'(s+h)}{(\zeta(s)-\zeta(s+h))^2} = \frac{h^4}{|\zeta(s+h)-\zeta(s)|^4}\times$$

$$\frac{1}{\pi h^4}\mathrm{Im}(\zeta'(s+h)\zeta'(s)\overline{(\zeta(s+h)-\zeta(s))^2}) \to K'_{|2}(s,s) \quad \text{as } h \to 0 \ ,$$

uniformly w.r.t. $s \in \mathbb{R}$. Thus $K'_{|2}(s+h, s+h') = K'_{|2}(s+h, s+h+(h'-h)) - K'_{|2}(s+h, s+h) + K'_{|2}(s+h, s+h) \to K'_{|2}(s,s)$ as $h, h' \to 0$ so $K'_{|2}$ is a continuous function on $\mathbb{R}^2$. Hence there exists a constant q such that for all $s, t', t'' \in \mathbb{R}$

$$(4.6) \qquad |K(s,t') - K(s,t'')| \le q|t' - t''| .$$

This and (4.3), (4.4) yield the continuity of the function K on $\mathbb{R}^2$. This way the condition (4.1) reduces to the following

$$(4.7) \qquad \int_0^d |K(s,t)\nu(s)|ds < +\infty \quad \text{for a.e. } t \in \mathbb{R} ,$$

where $\nu = \mu \circ \zeta : \mathbb{R} \to \mathbb{R}$ is a measurable non-constant function, and ν satisfies the following linear Fredholm integral equation with a continuous kernel K and a real constant $\lambda \in \Lambda_k(\Gamma)$

$$(4.8) \qquad \nu(t) = \lambda \int_0^d K(s,t)\nu(s)ds \quad \text{for a.e. } t \in \mathbb{R}$$

because μ is a solution of the equation (0.2). We shall show that ν is an integrable function on the interval $[0,d]$. Suppose, to the contrary, that $\int_0^d |\nu(s)|ds = +\infty$. Then there exists $x \in [0,d]$ such that for every $\delta > 0$ $\int_{[x-\delta, x+\delta]} |\nu(s)|ds = +\infty$. The function $\mathbb{R} \ni s \mapsto K(x,s) \in \mathbb{R}$ is a non-vanishing function on $\mathbb{R}$ because Γ is not a straight line. Thus there exists $y \in [0,d]$ such that $|K(x,y)| = 2m > 0$. Hence and by the continuity of K $|K(s,t)| \ge m > 0$ for all $(s,t) \in [x-\delta, x+\delta] \times [y-\delta, y+\delta]$ where δ is some positive constant, $2\delta \le d$. Then for every $t \in [y-\delta, y+\delta]$ we have

$$\int_0^d |K(s,t)\nu(s)|ds \ge \int_{[x-\delta,x+\delta]} |K(s,t)||\nu(s)|ds \ge m \int_{[x-\delta,x+\delta]} |\nu(s)|ds = +\infty$$

which contradicts (4.7). Therefore $\int_0^d |\nu(s)|ds < +\infty$. Let for every $s, t \in \mathbb{R}$, $\nu(s,h) = 1/h(K(s,t+h) - K(s,t))\nu(s)$. By the inequality (4.6) $|\nu(s,h)| \le q|\nu(s)|$ for all $s, h \in \mathbb{R}$, $h \ne 0$ and $q|\nu|$ is an integrable function on $[0,d]$. Then by the Lebesgue bounded convergence theorem and the equality (4.8)

$$(4.9) \qquad \nu'(t) = \lambda \lim_{h \to 0} \int_0^d \nu(s,h)ds = \lambda \int_0^d K'_{|2}(s,t)\nu(s)ds , \quad t \in \mathbb{R} .$$

Hence

$$|\nu'(t+h) - \nu'(t)| \le |\lambda| \sup_{s \in \mathbb{R}} |K'_{|2}(s,t+h) - K'_{|2}(s,t)| \int_0^d |\nu(s)|ds \to 0 \quad \text{as } h \to 0$$

because of the continuity of the function $K'_{|2}$. Thus ν is continuously differentiable on $\mathbb{R}$, see also [7], and the function μ satisfies the Lipschitz property $|\mu(\zeta_1) - \mu(\zeta_2)| \le L|\zeta_1 - \zeta_2|$, $\zeta_1, \zeta_2 \in \Gamma$ where $L > 0$ is a constant. Then, cf. [23], the functions

$G : \Omega \to \mathbf{C}$, $F : \Omega_* \to \mathbf{C}$ defined in complementary domains Ω, $\Omega_* \ni \infty$ of a Jordan curve Γ by the Cauchy integrals

$$G(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mu(\zeta)}{\zeta - z}d\zeta\,,\ z \in \Omega \quad\text{and}\quad F(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mu(\zeta)}{\zeta - z}d\zeta\,,\ z \in \Omega_*$$

have continuous extensions to $\overline{\Omega}$ and $\overline{\Omega}_*$, respectively, and their boundary values satisfy Plemelj's formulas

(4.10)
$$G(z) = \frac{1}{2}\mu(z) + \text{P.V.}\ \frac{1}{2\pi i}\int_\Gamma \frac{\mu(\zeta)}{\zeta - z}d\zeta\,,\quad F(z) = -\frac{1}{2}\mu(z) + \text{P.V.}\ \frac{1}{2\pi i}\int_\Gamma \frac{\mu(\zeta)}{\zeta - z}d\zeta\,,$$

$z \in \Gamma$. On the other hand, by the regularity of the kernel function k and the equalities (4.2), as well as (0.2)

$$\text{(4.11)}\qquad \lambda\,\text{Re}\,\text{P.V.}\ \frac{1}{2\pi i}\int_\Gamma \frac{\mu(\zeta)}{\zeta - z}d\zeta = \frac{\lambda}{2\pi}\text{Im}\,\text{P.V.}\int_\Gamma \frac{\mu(\zeta)}{\zeta - z}d\zeta =$$
$$\frac{1}{2}\lambda\int_\Gamma k(\zeta, z)\mu(\zeta)|d\zeta| = \frac{1}{2}\mu(z)\,,\quad z \in \Gamma\,.$$

Hence and by (4.10) the boundary values of G and F satisfy on Γ the equalities (3.1). Moreover, it follows from the regularity of the function μ and by the Green formula that the functions G, F have finite Dirichlet integrals $\int_\Omega |G'|^2 dS$, $\int_{\Omega_*} |F'|^2 dS$, cf. [24]. Then, in view of Theorem 3.3, $\lambda \in \Lambda^*(\Gamma)$ which proves the inclusion $\Lambda_k(\Gamma) \subset \Lambda^*(\Gamma)$ as $\Gamma \in C^3$.

It follows from Corollary 2.2 that $\Lambda^*(\Gamma) = \Lambda(\Gamma)$ if $\Gamma_{\mathbf{T}} \subset \text{cl}_r(A_{\mathbf{T}})$. It seems to be a not trivial problem to characterize all quasicircles $\Gamma \subset \mathbf{C}$ whose welding homeomorphisms belong to the class $\text{cl}_r(A_{\mathbf{T}})$, in terms of a standard parametrization of Γ. Now, we restrict considerations to Dini smooth Jordan curves. We remind that a rectifiable Jordan curve Γ with a natural parametrization $\zeta(s)$ is Dini smooth if the argument of the tangent vector at the point $\zeta(s)$ is a Dini continuous function of the natural parameter s, cf. [22].

Theorem 4.2.[4)] *If Ω and $\Omega_* \ni \infty$ are complementary domains of a rectifiable Dini smooth Jordan curve Γ then $\Gamma_{\mathbf{T}} \subset \text{cl}_r(A_{\mathbf{T}})$ and $\Lambda^*(\Gamma) = \Lambda(\Gamma)$. Moreover, if Γ does not coincide with a circle then $\Lambda^*(\Gamma) \neq \emptyset$ and for every eigenvalue $\lambda \in \Lambda^*(\Gamma)$ there exist analytic functions $G : \Omega \to \mathbf{C}$, $F : \Omega_* \to \mathbf{C}$, $F(\infty) = 0$, and a square integrable real valued function $\mu : \Gamma \to \mathbf{R}$ such that*

(i) *$\int_\Omega |G'|^2 dS < \infty$ and $\int_{\Omega_*} |F'|^2 dS < \infty$;*

(ii) *Angular limits of F and G satisfy the equalities (3.1) a.e. on Γ as well as $\mu = G - F$ a.e. on Γ;*

(iii)
$$\lambda\,\text{Re}\,P.V.\ \frac{1}{\pi i}\int_\Gamma \frac{\mu(u)}{u - z}du = \mu(z) \quad \textit{for a.e. } z \in \Gamma\,.$$

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4) This is an improved version of [12, Theorem 3.2]

In particular $\Lambda^*(\Gamma) \subset \Lambda_k(\Gamma)$ *as* $\Gamma \in C^3$.

Proof. Let $\Phi : \overline{\Delta} \to \overline{\Omega}$, $\Phi_* : \overline{\Delta}_* \to \overline{\Omega}_*$ be homeomorphisms conformal in Δ and Δ_*, respectively. Then $\gamma = \Phi_*^{-1} \circ \Phi : \mathbf{T} \to \mathbf{T}$ is a welding homeomorphism of Γ. It follows from Warschawski theorem, cf. [30] or [22], that the derivatives Φ' and Φ'_* have continuous extensions to the unit circle $\mathbf{T}$ and

$$\min\{|\Phi'(z)|\,,\,|\Phi'_*(z)|\} > 0 \tag{4.12}$$

for every $z \in \mathbf{T}$. Differentiating both sides of the equality $\Phi_*(\gamma(e^{it})) = \Phi(e^{it})$, $t \in \mathbb{R}$, we get $\Phi'_*(\gamma(z))\gamma'(z)iz = \Phi'(z)iz$ for every $z \in \mathbf{T}$. Hence $\log|\gamma'| = \log|\Phi'| - \log|\Phi'_* \circ \gamma|$ is a continuous function on $\mathbf{T}$ so in view of Proposition 2.3 and Corollary 2.2 $\gamma \in \mathrm{cl}_\tau(\mathbf{A_T})$ and $\Lambda^*(\Gamma) = \Lambda^*_\gamma = \Lambda_\gamma = \Lambda(\Gamma)$. If Γ does not coincide with a circle then $\Lambda^*(\Gamma) \neq \emptyset$ because of Theorem 3.2 (i). Assume $\lambda \in \Lambda^*(\Gamma)$ is an arbitrary eigenvalue of Γ. It follows from Theorem 3.4 that there exist non-constant analytic functions $G : \Omega \to \mathbb{C}$, $F : \Omega_* \to \mathbb{C}$ with finite Dirichlet integrals (i) (both integrals (i) are finite if one of them is finite, cf. the proof of Theorem 2.4) whose angular limits satisfy the equalities (3.1) a.e. on Γ. Without loss of generality we may assume that $F(\infty) = 0$. It can be always achieved after adding to the functions F, G suitable constants. The analytic functions $\tilde{F}$, $\tilde{G}$ assigned to the functions F, G as in the proof of Theorem 3.3 have also finite Dirichlet integrals in the unit disc Δ so their radial limits $\tilde{F}(z) = \lim_{r\to 1^-} \tilde{F}(rz)$, $\tilde{G}(z) = \lim_{r\to 1^-} \tilde{G}(rz)$, for a.e. $z \in \mathbf{T}$ form square integrable functions on $\mathbf{T}$. Hence and by the regularity of Φ', Φ'_* on $\mathbf{T}$ angular limits of F, G are square integrable functions on Γ as well and so is the function $\mu(\zeta) = G(\zeta) - F(\zeta)$, for a.e. $\zeta \in \Gamma$. By Cauchy integral theorem

$$G(z) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{G(u)}{u-z}du & , z \in \Omega \\ 0 & , z \in \Omega_* \end{cases} \quad \text{and} \quad F(z) = \begin{cases} -\dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{F(u)}{u-z}du & , z \in \Omega_* \\ 0 & , z \in \Omega \end{cases}$$

because of $F(\infty) = 0$. Hence

$$G(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mu(u)}{u-z}du\ ,\ z \in \Omega \quad \text{and} \quad F(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mu(u)}{u-z}du\ ,\ z \in \Omega_*$$

and by Privalov's theorem, cf. [23], there exists a singular integral P.V. $\frac{1}{\pi i}\int_\Gamma \frac{\mu(u)}{u-z}du$ at every point $z \in \Gamma$ for which the functions F, G have angular limits. Then the equalities (4.10) hold for a.e. point $z \in \Gamma$ and so by the equalities (3.1) μ is real valued a.e. on Γ which consequently leads to (iii). If $\Gamma \in C^3$ then by (iii) and (4.2) we obtain the equality (0.2) so $\lambda \in \Lambda_k(\Gamma)$. Thus $\Lambda^*(\Gamma) \subset \Lambda_k(\Gamma)$ as $\Gamma \in C^3$ which ends the proof.

Corollary 4.3. *If a Jordan curve* $\Gamma \in C^3$ *then* $\Lambda_k(\Gamma) = \Lambda^*(\Gamma) = \Lambda(\Gamma)$.

Remark. Due to David results, cf. [3], Krzyż was able to generalize the Neumann–Poincaré operator for any AD–regular Jordan curves by means of the singular integral Cauchy operator. This way he obtained another generalization of classical

Fredholm eigenvalues for AD-regular Jordan curves, cf. [10], [11]. Thus a natural problem appears whether the generalization of classical Fredholm eigenvalues considered by him coincides with that considered here where Γ is an AD-regular quasicircle, i.e. a chord-arc curve, cf. [31].

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