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# Some Remarks on the Maxima of Inner Conformal Radius

**Abstract.** If $f(z)=z+a_2z^2+a_3z^3+\ldots$ is univalent in the unit disk $\mathbb{D}$ then $a_2=0$ and $|a_3|\le 1/3$ is necessary, whereas $a_2=0$ and $|a_3|<1/3$ is sufficient for the inner conformal radius $R(w,f(\mathbb{D}))$ to have a local maximum at $w=0$. The case $|a_3|=1/3$ is investigated. Moreover, a sufficient condition for $R(w,f(\mathbb{D}))$ to have a unique global maximum at $w=0$ is given.

**1. Preliminaries.** Suppose $G$ is a simply connected domain of hyperbolic type in the finite plane $\mathbb{C}$. If $w \in G$ and $\varphi$ maps $G$ conformally onto the disk $\{z : |z| < R\}$ so that $w$ and $z = 0$ correspond and $|\varphi'(w)| = 1$ then $R = R(w, G)$ is a well defined continuous real-valued function of $w \in G$ called *inner conformal radius* of $G$ at the point $w \in G$. The function $R(w, G)$ plays an important role in the geometric function theory, in particular $\rho(w) = 1/R(w, G)$ is the density of hyperbolic metric $\rho(w)|dw|$ in $G$.

Let $f$ be a conformal mapping of the unit disk $\mathbb{D}$ onto $G$. Then obviously

(1) $$R(w,G) = (1-|z|^2)|f'(z)|\,, \quad w = f(z)\,.$$

Hence

(2) $$u(z) := \log R(w,G) = \log(1-z\bar{z}) + \operatorname{Re}\log f'(z)\,.$$

Since

(3) $$\nabla_z u := u_{xx} + u_{yy} = 4\partial^2 u/\partial z\partial\bar{z} = -4(1-z\bar{z}))^{-2} < 0\,,$$

$u(z)$ is superharmonic as a function of $z \in \mathbb{D}$ and also of $w \in G$, in view of the equality $\Delta_w u = |dz/dw|^2 \Delta_z u$. This implies that any critical point of $u$, and also of $R$, is either a saddle point, or a local maximum.

The problem, how do the geometrical properties of $G$ affect the set of local maxima was investigated by many authors. Interesting results in this direction, as well as a fairly complete list of references can be found in [6].

Some properties of $R(w, G)$ can be immediately obtained in an elementary way:

(i) if $G_1 \subsetneq G_2$ then $R(w, G_1) < R(w, G_2)$;

(ii) if $\widetilde{G}$ is the image domain of $G$ under a conformal mapping $\varphi$ and $\widetilde{w} = \varphi(w)$ then $R(\widetilde{w}, \widetilde{G}) = |\varphi'(w)|R(w, G)$ which means conformal invariance of hyperbolic metric;

(iii) if $d(w) = \operatorname{dist}(w, \mathbf{C} \setminus G)$ then $d(w) \le R(w,G) \le 4d(w)$.

A non-elementary but very important property is the following:

(iv) If $G^*$ is obtained from $G$ by Steiner (or circular) symmetrization with respect to an axis passing through $w$ (or a ray emanating from $w$) then $R(w,G) \le R(w,G^*)$, cf. [2], [7]. The sign of equality occurs iff $G = G^*$ (Steiner symmetrization), or $G^* = aG$, $w = 0$, $|a| = 1$ (circular symmetrization), cf. [3].

The property (iii) immediately implies the following: $R(w,G)$ is bounded in $G$ if, and only if, $d(w)$ is bounded. Moreover, (ii) and (iv) imply that, for $G = \{w : |\operatorname{Im} w| < \pi/4\}$, any point $w$ on the real axis provides a local maximum of $R(w,G)$. If

$$(4) \qquad f(z) = \tfrac{1}{2}\log(1+z)/(1-z) = z + \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 + \dots ,$$

then $f(\mathbf{D}) = G$ and, by (1), we obtain $R(x,G) = 1$ for any $x \in \mathbf{R}$.

However, even for a bounded $R(w,G)$ the set of local maxima may be empty. To this end consider the function

$$(5) \qquad g(z) = -z + \log(1+z)/(1-z)\ , \quad z \in \mathbf{D}.$$

We have $\operatorname{Re} g'(z) = \operatorname{Re}(1+z)/(1-z) > 0$ which implies univalence of $g$ in $\mathbf{D}$. The domain $G = g(\mathbf{D})$ is symmetric w.r.t. the real axis, its boundary consisting of the curve $w(\theta) = \log\cot\theta/2 - \cos\theta + i(\pi/2 - \sin\theta)$, $0 < \theta < \pi$, and its reflection in the real axis. By (iv) $R(w,G)$ attains a maximal value at $w = u_0$ if $w = u_0 + iv \in G$ and $u_0$ is fixed. Then by (1) $R(u_0,G) = 1 + r^2$, where $r = g^{-1}(u_0)$, and consequently $R(w,G)$ increases strictly on the real axis as $|w|$ increases. Moreover, $R(w,G) < 1$ for all $w \in G$. On the imaginary axis $R(iv,G) = (1-y^2)^2/(1+y^2)$ strictly decreases to 0 as $|y| \to 1$. Therefore $w = 0$ is a saddle point. Using the characteristic equation (8) for critical points we arrive, after rejecting the case $r = 0$, at the equation $r^4\eta^4 + 2(1-r^2)\eta^2 - 1 = 0$, where $\eta = z/r$, $|\eta| = 1$. Hence $\eta^2$ must be real, i.e. $\eta^2 = \pm 1$ which shows to be impossible.

This means there exist no critical points apart from $w = 0$. Thus $R(w,G)$ being bounded has no local maximum and only one critical point.

The absence of local maxima is possible only if the area $|G| = +\infty$. This follows from the

**Proposition 1.** *If $G$ is a simply connected domain of finite area then there exists $w_0 \in G$ such that $R(w_0,G) \ge R(w,G)$ for all $w \in G$.*

**Proof.** Suppose $G = f(\mathbf{D})$, where $f$ is holomorphic in $\mathbf{D}$ and the area $|f(\mathbf{D})|$ is finite. Then, as it is well known, $\lim_{r\to 1}(1-r)M(r,f') = 0$, where $M(r,f') = \sup\{|f'(re^{i\theta})| : \theta \in \mathbf{R}\}$, cf. e.g. [4]. This implies, in view of (1), that $R(w,G) \to 0$, as $w \to \partial G$ in spherical metric. If $R(w_1,G) = d$ for some $w_1 \in G$, then $\{w \in G : R(w,G) \ge d\}$ is a non-empty compact subset of $G$ and $R(w,G)$, being continuous, attains its maximal value on this subset at some $w_0 \in G$, and this ends the proof.

In what follows we prove two lemmas which give necessary and sufficient conditions for a point $w \in G$ to be a local maximum of $R(w,G)$. Our approach is slightly different from that in [1] and [6], where analogous results appear.

Note first that critical points of $R(w,G)$ coincide with critical points of $u(z)$. We obtain from (2)

$$(6)\qquad r\frac{\partial}{\partial r}u(re^{i\theta}) = r\frac{\partial}{\partial r}[\log(1-r^2) + \operatorname{Re}\log f'(re^{i\theta})] = -2r^2/(1-r^2) + \operatorname{Re}\{zf''(z)/f'(z)\}\,, \quad z = re^{i\theta}\,,$$

on the radii $\theta$ =const.

On the other hand, we have on circles $|z| = r > 0$

$$(7)\qquad \frac{\partial}{\partial\theta}u(re^{i\theta}) = \frac{\partial}{\partial\theta}\operatorname{Re}\log f'(re^{i\theta}) = -\operatorname{Im}\{zf''(z)/f'(z)\}\,.$$

Hence we obtain

**Lemma 1.** *If $f$ is univalent in $\mathbf{D}$, and $G = f(\mathbf{D})$, then the point $w = f(re^{i\theta})$, $r > 0$, is critical for $R(w,G)$ if, and only if,*

$$(8)\qquad zf''(z)/f'(z) = 2r^2/(1-r^2)\,, \quad z = re^{i\theta}\,.$$

Due to the formula (1) and the property (ii) we may assume that the function $f$ mapping $\mathbf{D}$ onto $G$ belongs to the familiar class $S$, so that

$$(9)\qquad f(z) = z + a_2z^2 + a_3z^3 + \ldots\,, \quad z \in \mathbf{D}\,,$$

and $R(0, f(\mathbf{D})) = 1$.

We shall establish in terms of $a_2, a_3$ necessary and sufficient conditions for $R(0, f(\mathbf{D}))$ to be a local maximum.

**Lemma 2.** *If $R(w, f(\mathbf{D}))$ has a local maximum at $w = 0$, then $a_2 = 0$, $|a_3| \le 1/3$. Conversely, if $a_2 = 0$, $|a_3| < 1/3$, then $R(w, f(\mathbf{D}))$ has a strict local maximum at $w = 0$.*

**Proof.** Due to (2) we may preferably consider $u(z)$ instead of $R(w, f(\mathbf{D}))$. We have

$$\log f'(z) = \log[1 + (2a_2z + 3a_3z^2 + \ldots)] = 2a_2z + (3a_3 - 2a_2^2)z^2 + O(z^3)$$

and hence, using (2), we obtain for $z = re^{i\theta}$:

$$(10)\qquad u(re^{i\theta}) = (a_2e^{i\theta} + \overline{a}_2e^{-i\theta})r + \tfrac{1}{2}[(3a_3 - 2a_2^2)e^{2i\theta} + (3\overline{a}_3 - 2\overline{a}_2^2)e^{-2i\theta} - 2]r^2 + O(r^3)\,.$$

If $R(0, f(\mathbf{D})) = 1$ (or $u(0) = 0$) is a local maximum then obviously $a_2 = 0$ and $3a_3e^{2i\theta} - 3\overline{a}_3e^{-2i\theta} - 2 \le 0$ for all $\theta \in \mathbb{R}$ which means that $|a_3| \le 1/3$. Conversely, if $a_2 = 0$ and $|a_3| < 1/3$ then, as readily seen from (10), $R(w, f(\mathbf{D}))$ has a strict local maximum at $w = 0$.

**2. Some applications and remarks.** Lemma 2 leaves the case $a_2 = 0$, $|a_3| = 1/3$ open. We may obviously assume that $a_3 = 1/3$. We will give examples showing that all three possibilities can occur.

(I) *$w = 0$ is a strict local maximum.*

To this end consider $f(z) = z + \frac{1}{3} z^3$. We have $\operatorname{Re} f'(z) = \operatorname{Re}(1 + z^2) > 0$ in $\mathbb{D}$ and therefore $f \in S$. The domain $G$ is symmetric w.r.t. the real axis and hence, due to (iv), $R(u_0 + iv, G) < R(u_0, G)$ for any $u_0 + iv \in G$ ($u_0, v \in \mathbb{R}$, $v \neq 0$). This implies $R(iv, G) < R(0, G) = 1$; for $u_0 \neq 0$, $r = f^{-1}(u_0)$ we have $R(u_0 + iv, G) < R(u_0, G) = (1 - r^2)(1 + r^2) = 1 - r^4 < 1$ which proves (I). By means of (8) one verifies easily that $w = 0$ is the only critical point of $R(w, G)$ and consequently it attains its global maximum at $w = 0$.

(II) *$w = 0$ is a weak local maximum.*

This obviously occurs for $f$ as in formula (4).

(III) *$w = 0$ is a saddle point.*

Consider the function $f \in S$ satisfying $f'(z) = p(z) = (1+\omega(z))/(1+\omega(z))$, where $\omega(z) = z^2(1+2z)/(2+z) = z^2(2-\frac{3}{2}(1+\frac{1}{2}z)^{-1}) = \frac{1}{2}z^2 + \frac{3}{4}z^3 - \frac{3}{8}z^4 + O(z^5)$. Obviously $|\omega(z)| \leq |z|^2$ in $\mathbb{D}$ and hence $\operatorname{Re} f'(z) > 0$. We have $f'(z) = 1 + 2\omega(z) + 2(\omega(z))^2 + \cdots = 1 + z^2 + \frac{3}{2}z^3 - \frac{1}{4}z^4 + O(z^5)$. Since $f$ has real coefficients, $G$ is symmetric w.r.t. the real axis and by (iv) we have $R(iv, G) < R(0, G) = 1$. On the other hand, on the real axis

$$R(u, G) = (1 - x^2)(1 + x^2 + \tfrac{3}{2}x^3 + O(x^4)) = 1 + \tfrac{3}{2}x^3 + O(x^4)$$

which is $> 1$ for $x > 0$ sufficiently small and $< 1$ for small negative $x$ and this proves (III).

In [1] the author proved that, for convex domains, apart from the strip $\{w : |\operatorname{Im} w| < 1\}$ and its images under similarity, there exists at most one maximum of $R(w, G)$. A very simple proof of this result is given in [6], while in [5] a converse statement is disproved, i.e. a non-convex Jordan domain $G$ with exactly one maximum of $R(w, G)$ has been found. The domain $G$ in (I) has also the same properties, however, in both cases $\partial G$ is a piecewise analytic curve. The function $g$ in the formula (5) enables us to construct a non-convex Jordan domain with analytic boundary, one maximum and no other critical points of $R(w, G)$.

**Proposition 2.** *If $2\rho^2 = 1$ and $G = h(\mathbb{D})$, where*

$$h(z) = -z + \rho^{-1} \log(1 + \rho z)/(1 - \rho z) = z + \frac{2}{3}\rho^2 z^3 + \frac{2}{5}\rho^4 z^5 + \ldots ,$$

*then $R(w, G)$ has only one critical point $w = 0$ being a strict local maximum and $h(\partial\mathbb{D})$ is a non-convex analytic curve symmetric w.r.t. both coordinate axes.*

**Proof.** Obviously $h(z) = \rho^{-1} g(\rho z)$, with $g$ defined by (5), belongs to $S$. If $|z| = r$ then $R(w, G) \leq (1 - r^2)(1 + \rho^2 r^2)/(1 - \rho^2 r^2) \leq 1$ since $2\rho^2 r^2 = r^2 \leq r^2 + \rho^2 r^4$, with the sign of equality for $r = 0$ only. Thus $R(w, G)$ has a global maximum at $w = 0$. We have $\log h'(z) = \log(1 + \rho^2 z^2) - \log(1 - \rho^2 z^2)$, and hence

$$\operatorname{Im}\{zh''(z)/h'(z)\} = 4\rho^2(1 - \rho^4 r^4)|1 - \rho^4 z^4|^{-2} \operatorname{Im}(z^2) = 0$$

only for $z$ on coordinate axes. Thus, by (7), critical points may be situated on coordinate axes only. However, on both coordinate axes $R(w, G)$ tends monotonically to zero as $|w|$ increases. Hence no critical points $w \neq 0$ do exist. We have $zh''(z)/h'(z) = 4\rho^2 z^2(1-\rho^4 z^4)^{-1}$ and we now prove that $\varphi(z) = 1 + 4\rho^2 z^2(1-\rho^4 z^4)^{-1}$ is not in the familiar Carathéodory class. With $z = i$, $2\rho^2 = 1$ we obtain $\varphi(i) = -5/3 < 0$ and this proves that $h(\partial\mathbb{D})$ is a non-convex analytic Jordan curve.

We state now a simple sufficient condition for $R(w, G)$ to have only one local maximum.

**Theorem .** *If $f$ maps the unit disk $\mathbb{D}$ conformally onto $G$ and $R(w, G)$ has a strict local maximum at $w = 0$, then*

$$\text{(11)} \qquad \operatorname{Re} zf''(z)/f'(z) \le 2|z|^2(1-|z|^2)^{-1} \quad \textit{for all } z \in \mathbb{D}$$

*implies that $R(w, G)$ has only one local maximum $w = 0$.*

**Proof.** By (2) and (6) we have $\frac{\partial}{\partial r}\, u(re^{i\theta}) \le 0$ for $\theta$ =const with $\frac{\partial}{\partial r}\, u(re^{i\theta})$ being real–analytic in a neighbourhood of the ray $\theta$ =const. Therefore possible zeros of $\partial u/\partial r$ from a discrete set $\{r_k e^{i\theta}\}$ and so $\partial u/\partial r < 0$ in any interval $(r_k, r_{k+1})$. Hence $u$ is strictly decreasing for $\theta$ fixed and $r$ ranging over $(0,1)$. The same is true if $\partial u/\partial r$ has at most one zero in $(0,1)$. Hence $u(re^{i\theta})$ and also $R(w, G)$, $w = f(re^{i\theta})$, strictly decrease as $\theta$ is fixed and $r$ ranges over $(0,1)$. This ends the proof.

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