ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLVI, 4 SECTIO A 1992

Lech GRUSZECKI (Lublin)

A Note on a Metric on $D_E[0,1]$ Space

Abstract. The aim of this note is to give a metric on $D_E[0,1]$ space modeling a metric for $D_E[0,\infty)$ of [2]. We show that in order to obtain the Skorohod topology in this case we should change the formula given by Stone.

Introduction. Let (E,r) be a metric space. Denote by $D_E[0,1]$ the space of all E-valued functions on $[0,1]$ which are right-side continuous on $[0,1)$, left-side continuous at 1 and have left-side limits everywhere on $(0,1]$.

The distance between elements x and y of $D_E[0,1]$ can be defined as

$$d(x,y)=\inf_{\lambda\in\Lambda}\sup_{0\le t\le 1}|t-\lambda(t)|\vee r(x(t),y(\lambda(t)))\ ,$$

where Λ is the set of all continuous, strictly increasing real functions λ on $[0,1]$ such that $\lambda(0)=0$ and $\lambda(1)=1$.

Another, a more useful distance in $D_E[0,1]$ can be defined as

$$d_0(x,y)=\inf_{\lambda\in\Lambda_0}\operatorname{ess}\sup_{0\le t\le 1}|\log\lambda'(t)|\vee r(x(t),y(\lambda(t)))\ ,$$

where Λ_0 is the subset of Λ formed by Lipschitz functions with Lipschitz inverse.

Topology of $(D_E[0,1],d)$ coincides with topology of $(D_E[0,1],d_0)$ and it is called Skorohod's topology (cf. [1]).

A direct application of the metrization of $D_E[0,\infty)$ given in [2] suggests the following metric for $D_E[0,1]$:

$$\rho(x,y)=\inf_{\lambda\in\Lambda_0}\left(\gamma(\lambda)\vee\int_0^1\omega(x,y,\lambda,u)du\right)\ , \tag{1}$$

where

$$\omega(x,y,\lambda,u)=\sup_{0\le t\le 1}q(x(t\wedge u),y(\lambda(t)\wedge u))\ ,$$

$q=r\wedge 1$ and

$$\gamma(\lambda)=\operatorname{ess}\sup_{0\le t\le 1}|\log\lambda'(t)|=\sup_{0\le t<s\le 1}\left|\log\frac{\lambda(s)-\lambda(t)}{s-t}\right|$$

However, the metric (1) does not induce the Skorohod topology on $D_E[0,1]$ as it shows the following example.

Example. Let e and e' be distinct elements of E. Define

$$x_n(t)=\begin{cases} e & , \text{ for } t\in[0,1-\frac{1}{n}) \\ e' & , \text{ for } t\in[1-\frac{1}{n},1] \end{cases}$$

and

$$x(t)=e \quad \text{for } t\in[0,1] .$$

Note that

$$\rho(x_n,x)\le q(e,e')/n\ , \quad n\in\mathbf{N}\ ,$$

implies $\lim_{n\to\infty}\rho(x_n,x)=0$.
But the sequence $\{x_n\}$ does not converge in the Skorohod topology as $d(x_n,x)\ge r(e,e')$, $n\in\mathbf{N}$.

In Section 2 we introduce a metric on $D_E[0,1]$ which has no such drawback. However, before giving the main result we analyse properties of Stone's type metric on $D_E[0,1]$ (Section 1).

1. Properties of the metric ρ. Following the argument of [2] we can get the following useful fact on ρ given by (1).

Lemma 1. *If $\{x_n\}$, $\{y_n\}\subset D_E[0,1]$ then $\lim_{n\to\infty}\rho(x_n,y_n)=0$ iff there exists $\{\lambda_n\}\subset\Lambda_0$ such that*

(2) $$\lim_{n\to\infty}\gamma(\lambda_n)=0$$

and for every $\varepsilon>0$ and $a\in(0,1]$

$$\lim_{n\to\infty} m\{u\in[0,a]:w(x_n,y_n,\lambda_n,u)\ge\varepsilon\}=0\ ,$$

where m is the Lebesgue measure.

Proposition 1. *The function ρ given by (1) is a metric on $D_E[0,1]$.*

Proposition 2. *Let $\{x_n\}\subset D_E[0,1]$ and $x\in D_E[0,1]$. Then $\lim_{n\to\infty}\rho(x_n,x)=0$ iff (2) holds and*

(3) $$\lim_{n\to\infty} w(x_n,x,\lambda_n,u)=0$$

at every continuity point u of x, $u\in(0,1)$.

Corollary. *If $\lim_{n\to\infty}\rho(x_n,x)=0$ and u is a continuity point of x, then*

$$\lim_{n\to\infty}x_n(u)=\lim_{n\to\infty}x_n(u-)=x(u)\ .$$

Theorem 1. *Let $\{x_n\} \subset D_E[0,1]$ and $x \in D_E[0,1]$. Then $\lim_{n\to\infty} \rho(x_n, x) = 0$ iff there exists $\{\lambda_n\} \subset \Lambda_0$ such that (2) holds and for $T \in (0,1)$*

$$\lim_{n\to\infty} \sup_{0\le t\le T} r(x_n(t), x(\lambda_n(t))) = 0\,. \tag{4}$$

Proof. Assume that $\lim_{n\to\infty} \rho(x_n, x) = 0$. Then there exist $\{\lambda_n\} \subset \Lambda_0$ and $\{u_n\} \subset [0,1]$ such that (2) holds and

$$w(x_n, x, \lambda_n, u_n) \to 0 \quad \text{with } u_n \to 1,\ n \to \infty\,.$$

Thus

$$\lim_{n\to\infty} \sup_{0\le t\le 1} r(x_n(t\wedge u_n), x(\lambda_n(t)\wedge u_n)) = 0\,.$$

If $T \in (0,1)$ then for all sufficiently large n we have $u_n \ge T \vee \lambda_n(T)$. Therefore (4) is satisfied.

Conversely, let $\{\lambda_n\} \subset \Lambda_0$ be such that (2) holds and assume that (4) is satisfied. Then for $u \in (0,1)$ and $\{u_n\} \subset (u,1]$ we see, after using the triangle inequality and properties of functions λ_n, that

$$\begin{aligned} \sup_{0\le t\le 1} r(x_n(t\wedge u), x(\lambda_n(t)\wedge u)) &\le \sup_{0\le t\le u} r(x_n(t), x(\lambda_n(t)\wedge u_n)) \\ &+ \sup_{u\le s\le(\lambda_n(u)\wedge u_n)\vee u} r(x(u), x(s)) \vee \sup_{\lambda_n(u)\wedge u\le s\le u} r(x(\lambda_n(u)\wedge u_n), x(s))\,. \end{aligned} \tag{5}$$

Let now u be a continuity point of x and let us choose $\{u_n\}$ such that $u_n > \lambda_n(u) \vee u$, $n \in \mathbb{N}$. Then by (4) and (5) we see that (3) holds. Hence the assumption (2) and Proposition 2 complete the proof.

Theorem 2. *Let $\{x_n\} \subset D_E[0,1]$ and $x \in D_E[0,1]$. If $\lim_{n\to\infty} d(x_n, x) = 0$ then $\lim_{n\to\infty} \rho(x_n, x) = 0$.*

Proof. It is known that $\{x_n\} \subset D_E[0,1]$ converges to x in the Skorohod topology induced by d iff there exists $\{\lambda_n\} \subset \Lambda_0$ such that (2) holds and

$$\lim_{n\to\infty} \sup_{0\le t\le 1} r(x_n(t), x(\lambda_n(t))) = 0\,, \quad \text{(cf. [1])}\,.$$

Hence we conclude by Theorem 1 that the implication of Theorem 2 is true.

2. The main result. We give a new metric δ on $D_E[0,1]$ which determines the Skorohod topology.

Definition. For $x, y \in D_E[0,1]$ we define

$$\delta(x,y) = \inf_{\lambda\in\Lambda_0} \left(\gamma(\lambda) \vee \int_0^1 (w(x,y,\lambda,u) + w_1(x,y,\lambda,u))du\right), \tag{6}$$

where

$$w_1(x,y,\lambda,u) = \sup_{0\le t\le 1} q(x(t\vee u), y(\lambda(t)\vee u))$$

and $\Lambda_0, \gamma(\cdot), w(\cdot,\cdot,\cdot,\cdot)$ are the quantities defined in Introduction.

Following the argument of Section 1 we have

Proposition 3. *The function δ given by* (6) *is a metric on $D_E[0,1]$.*

Now we show that the topology of $(D_E[0,1],\delta)$ coincides with the Skorohod topology of $(D_E[0,1],d)$.

Theorem 3. *The metric δ determines the Skorohod topology on $D_E[0,1]$.*

Proof. Let $\{x_n\} \subset D_E[0,1]$ and $x \in D_E[0,1]$. Assume that there exists $\{\lambda_n\} \subset \Lambda_0$ and $\{u_n\},\{v_n\} \subset [0,1]$ such that (2) holds, $u_n \to 1, v_n \to 0$ and

$$\lim_{n\to\infty} w(x_n, x, \lambda_n, u_n) = \lim_{n\to\infty} w_1(x_n, x, \lambda_n, v_n) = 0 .$$

In particular we have

$$\lim_{n\to\infty} \sup_{0\le t\le 1} r(x_n(t\wedge u_n), x(\lambda_n(t)\wedge u_n)) = 0$$

and at the same time

$$\lim_{n\to\infty} \sup_{0\le t\le 1} r(x_n(t\vee v_n), x(\lambda_n(t)\vee v_n)) = 0 .$$

Let $T \in (0,1)$. Then by the assumptions for all sufficiently large n

$$u_n \ge T \vee \lambda_n(T)$$

and

$$v_n \le T \wedge \lambda_n(T) .$$

Hence

$$\lim_{n\to\infty} \sup_{0\le t\le T} r(x_n(t), x(\lambda_n(t))) = 0 , \quad T \in (0,1) ,$$

and

$$\lim_{n\to\infty} \sup_{T\le t\le 1} r(x_n(t), x(\lambda_n(t))) = 0 , \quad T \in (0,1) .$$

Thus we get

$$\lim_{n\to\infty} \sup_{0\le t\le 1} r(x_n(t), x(\lambda_n(t))) = 0 ,$$

which together with (2) give the Skorohod convergence of $\{x_n\}$ to x.

Now let $\lim_{n\to\infty} d(x_n,x) = 0$, $\{x_n\} \subset D_E[0,1]$, $x \in D_E[0,1]$. Then there exists $\{\lambda_n\} \subset \Lambda_0$ such that (2) holds and

$$\lim_{n\to\infty} \sup_{0\le t\le 1} r(x_n(t), x(\lambda_n(t))) = 0 .$$

Therefore, by (5) with $u_n > \lambda_n(u) \vee u$ for every continuity point u of x, $u \in (0,1)$, we get

$$\lim_{n\to\infty} \sup_{0\le t\le 1} q(x_n(t\wedge u), x(\lambda_n(t)\wedge u)) = 0\,,$$

which implies that

(7) $$\lim_{n\to\infty} w(x_n, x, \lambda_n, u) = 0$$

for every continuity point u of x.

Similary, we see that for every continuity point u of x, $u \in (0,1)$, and $\{u_n\} \subset [0,u)$ we have

$$\begin{aligned} w_1(x_n, x, \lambda_n, u) &= \sup_{0\le t\le 1} q(x_n(t\vee u), x(\lambda_n(t)\vee u)) \\ &\le \sup_{0\le t\le 1} q(x_n(t\vee u), x(\lambda_n(t\vee u)\vee u_n)) \\ &+ \sup_{0\le t\le 1} q(x(\lambda_n(t\vee u)\vee u_n), x(\lambda_n(t)\vee u)) \\ &\le \sup_{u\le t\le 1} q(x_n(t), x(\lambda_n(t)\vee u_n)) \\ &+ \sup_{u\le s\le \lambda_n(u)\vee u} q(x(\lambda_n(u)\vee u_n), x(s)) \vee q \sup_{u\wedge(\lambda_n(u)\vee u)\le s\le u} (x(u), x(s))\,. \end{aligned} \tag{8}$$

Letting now $u_n < u \wedge \lambda_n(u)$, $n \in \mathbf{N}$, we state that the first term of the last inequality (8) tends to zero. Therefore for every continuity point u of x, $u \in (0,1)$, we have

$$\lim_{n\to\infty} w_1(x_n, x, \lambda_n, u) = 0\,.$$

Thus by (2), (7) and (8)

$$\lim_{n\to\infty} \delta(x_n, x) = 0$$

which completes the proof of Theorem 3.

Now we note similarly as for $D_E[0,\infty)$ that the metric space $(D_E[0,1], \delta)$ is complete and separable whenever (E,r) is complete and separable.

Theorem 4. *If (E,r) is complete and separable, then $(D_E[0,1], \delta)$ is complete and separable.*

Proof. Assume that $\{x_k\} \subset D_E[0,1]$ is a Cauchy sequence. Then there exists subsequence $\{y_n\} = \{x_{k_n}\}$ of $\{x_n\}$ such that

$$\delta(y_n, y_{n+1}) \le 2^{-n}\,, \quad n \in \mathbf{N}\,.$$

Therefore, we can choose $\{\lambda_n\} \subset \Lambda_0, \{u_n\}, \{v_n\} \subset [0,1]$ such that $\lim_{n\to\infty} u_n = 1$, $\lim_{n\to\infty} v_n = 0$ and

(9) $$\gamma(\lambda_n) \vee w(y_n, y_{n+1}, \lambda_n, u_n) \vee w_1(y_n, y_{n+1}, \lambda_n, v_n) < 2^{-n}\,, \quad n \in \mathbf{N}\,.$$

Note that there exists uniformly on $[0,1]$ the limit

$$\mu_n(t) = \lim_{k\to\infty} (\lambda_{n+k} \circ \cdots \circ \lambda_{n+1} \circ \lambda_n)(t) \quad \text{(cf. [1])}$$

Hence using (9) we get

$$\gamma(\mu_n) \le \sum_{i=n}^{\infty} \gamma(\lambda_i) \le 2^{-n+1} .$$

Thus $\mu_n \in \Lambda_0, n \in \mathbf{N}$. Taking into account that

$$\sup_{0\le t\le 1} q(y_n(\mu_n^{-1}(t) \wedge u_n), y_{n+1}(\mu_{n+1}^{-1}(t) \wedge u_n)) \le 2^{-n}$$

and

$$\sup_{0\le t\le 1} q(y_n(\mu_n^{-1}(t) \vee v_n), y_{n+1}(\mu_{n+1}^{-1}(t) \vee v_n)) \le 2^{-n} , \quad n \in \mathbf{N} \quad \text{(cf. [2], p 121)} ,$$

and (9) we see that $y_n \circ \mu_n^{-1}$ converges uniformly on $[0,1]$ to a function $y \in D_E[0,1]$ as (E,r) is a complete space. But $\lim_{n\to\infty} \gamma(\mu_n^{-1}) = 0$ and

$$\lim_{n\to\infty} \sup_{0\le t\le 1} r(y_n(\mu_n^{-1}(t)), y(t)) = 0 ,$$

imply that $\lim_{n\to\infty} \delta(y_n, y) = 0$.

Finally, we note that $(D_E[0,1], \delta)$ is a separable space as the set of the functions given by

$$x(t) = \begin{cases} a_{i_n}, & t \in [t_{n-1}, t_n),\ n = 1,2,\ldots,k-1, \\ a_{i_k}, & t \in [t_{k-1}, 1] \end{cases}$$

with $\{a_i\}$ being a countable dense subset of E, where $0 = t_0 < t_1 < \cdots < t_k = 1$ are rationals, $i_1, \ldots, i_n \in \mathbf{N}, n \in \mathbf{N}$, is a dense subset of $(D_E[0,1], \delta)$.

3. The conditions for compactness and relative compactness. We present here the conditions for compactness of sets in $D_E[0,1]$ using our metric δ (6).

In the proof of that result we need the following proposition (cf. Proposition 6.5 [2], p. 125 and [3]).

Proposition 4. *Let (E,r) be a metric space. Suppose that $\{x_n\} \subset D_E[0,1], x \in D_E[0,1], t \in [0,1], \{t_n\} \subset [0,1]$, and $\lim_{n\to\infty} t_n = t$. Then $\lim_{n\to\infty} \delta(x_n, x) = 0$ iff*

(i) $\lim_{n\to\infty} r(x_n(t_n), x(t)) \wedge r(x_n(t_n), x(t-)) = 0$.

(ii) *If* $\lim_{n\to\infty} r(x_n(t_n), x(t)) = 0$, $s_n \ge t_n$ *for each* n, *and* $\lim_{n\to\infty} s_n = t$, *then* $\lim_{n\to\infty} r(x_n(s_n), x(t)) = 0$.

(iii) *If* $\lim_{n\to\infty} r(x_n(t_n), x(t-)) = 0, 0 \le s_n \le t_n$ *for each* n, *and* $\lim_{n\to\infty} s_n = t$, *then* $\lim_{n\to\infty} r(x_n(s_n), x(t-)) = 0$.

and iff (a) *holds and*

(iv) *If* $s_n \le t_n \le v_n$ *for each* $n \in \mathbf{N}$, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} v_n = t$, *and* $\lim_{n\to\infty} r(x_n(s_n), g) = \lim_{n\to\infty} r(x_n(v_n), g) = 0$ *for an element* g *of* E, *then* $\lim_{n\to\infty} r(x_n(t_n), g) = 0$.

The proof of Proposition 6.5 of [2] and Corollary [3] needs only small changes.

Let $x \in D_E[0,1]$ be a step function. Write

$$S_0(x) = 0 ,$$
$$S_k(x) = \inf\{t \in (S_{k-1}(x), 1] : x(t) \neq x(t-) \vee t = 1\}$$

whenever $S_{k-1}(x) < 1$, and

$$S_k(x) = 1 \quad \text{if } S_{k-1}(x) = 1, \; k = 1, 2, \ldots .$$

Lemma 2. *Let $\Gamma \subset E$ be a compact set and let η be a positive number. If $A(\Gamma, \eta)$ is the set of step functions $x \in D_E[0,1]$ such that $x(t) \in \Gamma$ for all $t \in [0,1]$ and $S_k(x) - S_{k-1}(x) > \eta, k \geq 1$, whenever $S_{k-1}(x) < 1$, then the closure of $A(\Gamma, \eta)$ is compact.*

Proof of Lemma 2 can be given by the argument used in the proof of Lemma 6.1 of [2], p. 122.

Now for $x \in D_E[0,1]$ and $\eta > 0$ we define the modulus of continuity $\omega'(x, \eta)$ as follows

$$\omega'(x, \eta) = \inf_{\{t_i\}} \max_i \sup_{s,t \in [t_{i-1}, t_i)} r(x(s), x(t)) ,$$

where $\{t_i\}$ ranges over all partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ with $\min_{1 \leq i \leq n}(t_i - t_{i-1}) > \eta, n \in \mathbf{N}$.

The following theorem contains the conditions for compactness in $D_E[0,1]$.

Theorem 5. *Let (E, r) be a complete space. Then the closure of $A \subset D_E[0,1]$ is compact iff:*

(a) *For every rational $t \in [0,1]$ there exists a compact set $\Gamma_t \subset E$ such that $x(t) \in \Gamma_t$ for all $x \in A$,*

(b) $\lim_{\eta \to \infty} \sup_{x \in A} \omega'(x, \eta) = 0$.

Proof. Suppose that A satisfies (a) and (b). For $l \in \mathbf{N}$ choose $\eta_l \in (0,1)$ such that

$$\sup_{x \in A} \omega'(x, \eta_l) \leq \frac{1}{l} ,$$

and $m_l \in \mathbf{N} - \{1\}$ such that $\frac{1}{m_l} < \eta_l$. Write

$$\Gamma^{(l)} = \bigcup_{i=0}^{m_l} \Gamma_{i/m_l} ,$$

and put $A_l = A(\Gamma^{(l)}, \eta_l)$, where $A(\Gamma^{(l)}, \eta_l)$ is defined as in Lemma 2.

Then for every $x \in A$ there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ with $\min_{1 \leq i \leq n}(t_i - t_{i-1}) > \eta_l$ such that

$$\max_{1 \leq i \leq n} \sup_{s,t \in [t_{i-1}, t_i)} r(x(s), x(t)) \leq \frac{2}{l} .$$

Define

$$x_l(t) = \begin{cases} x(([m_l t_i]+1)/m_l) & , t_i \le t < t_{i+1},\ i = 0,1,\dots,n-1, \\ x(([m_l t_{n-1}]+1)/m_l) & , t = 1. \end{cases}$$

Then

$$\sup_{t\in[0,1]} r(x_l(t), x(t)) \le \frac{2}{l}$$

and

$$\delta(x_l, x) \le 2\int_0^1 \sup_{t\in[0,1]} q(x_l(t), x(t))du \le \frac{4}{l}\ .$$

Hence $A \subset A_l^{4/l}$. By Lemma 2 the sets A_l, $l \in \mathrm{N}$, are compact. Taking into account that $A \subset \bigcap_{l\in\mathrm{N}} A_l^{4/l}$ we see that A is totally bounded and hence has a compact closure.

Suppose now that A has a compact closure. Then the standard analysis with using Proposition 4 gives (a).

Now we show that (b) holds. Let $\beta > 0$ and $\{x_n\} \subset A$ be such that

$$\text{(10)} \qquad \omega'(x_n, \frac{1}{n}) \ge \beta, \quad n \in \mathrm{N}\ .$$

By compactness $\bar{A}$ there exists $x \in D_E[0,1]$ such that $\lim_{n\to\infty} \delta(x_n, x) = 0$ which is equivalent to $\lim_{n\to\infty} d(x_n, x) = 0$. Therefore, there exists a sequence $\{\lambda_n\} \subset \Lambda$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,1]} |\lambda_n(t) - t| = 0$$

and

$$\text{(11)} \qquad \lim_{n\to\infty} \sup_{t\in[0,1]} r(x_n(t), x(\lambda_n(t))) = 0\ .$$

Let $\eta > 0$. For each $n \in \mathrm{N}$ put

$$y_n(t) = x(\lambda_n(t))\ , \quad t \in [0,1]$$

and

$$\eta_n = \sup_{0\le t\le 1-\eta} [\lambda_n(t+\eta) - \lambda_n(t)]\ .$$

Taking into account the inequalities

$$\omega'(x,\eta) \le \omega'(y,\eta) + 2\sup_{s\in[0,1]} r(x(s), y(s))\ ,$$

$$\omega'(y,\eta) \le \omega'(x,\eta) + 2\sup_{s\in[0,1]} r(x(s), y(s))\ ,$$

and the fact that the function $\eta \to \omega'(x,\eta)$ is right continuous (cf. [2], p. 123), we get

$$\begin{aligned} \limsup_{n\to\infty} \omega'(x_n,\eta) &= \limsup_{n\to\infty} \omega'(y_n,\eta) \\ &\le \limsup_{m\to\infty} \omega'(x,\eta_n) \\ \text{(12)} \qquad &\le \lim_{n\to\infty} \omega'(x, \eta_n \vee \eta) = \omega'(x,\eta)\ . \end{aligned}$$

Letting $\eta \to 0$ we see that the right side of (12) tends to zero which contradicts to (10).

Now, we give conditions for relative compactness of a family of stochastic processes with sample path in $D_E[0,1]$.

Theorem 6. *Let (E,r) be the Polish space, and let $\{X_\alpha\}$ be a family of processes taking values in $D_E[0,1]$. Then $\{X_\alpha\}$ is relatively compact if and only if the two following conditions hold:*

(a) *For every $\eta > 0$ and rational $t \in [0,1]$, there exists a compact set $\Gamma_{\eta,t} \subset E$ such that*

(13) $$\inf_\alpha P(X_\alpha(t) \in \Gamma^\eta_{\eta,t}) \geq 1 - \eta \,.$$

(b) *For every $\eta > 0$ there exists $\delta > 0$ such that*

(14) $$\sup_\alpha P(\omega'(X_\alpha, \delta) \geq \eta) \leq \eta \,.$$

Proof. If $\{X_\alpha\}$ is relatively compact then by Theorem 5 and the Prohorov's theorem ([1], p. 58), we immediately obtain (a) and (b).

Conversely, let $\varepsilon > 0$ and choose $\delta > 0$ such that (14) holds with $\eta = \varepsilon/4$. Let $m \in \mathbf{N} \cap (1/\delta, \infty)$.

Write

$$\Gamma = \bigcup_{i=0}^{m} \Gamma_{\varepsilon 2^{-i-2}, i/m} \,.$$

Note that

$$\sup_\alpha P\left(\bigcup_{i=0}^{m} \{X_\alpha(i/m) \notin \Gamma^{\varepsilon/4}\right) \leq \frac{\varepsilon}{2} \,.$$

Hence

(15) $$\inf_\alpha P(X_\alpha(i/m) \in \Gamma^{\varepsilon/4}, i = 0, 1, \ldots, m) \geq 1 - \frac{\varepsilon}{2} \,.$$

Put $A = A(\Gamma, \delta)$ (cf. Lemma 2). Let $x \in D_E[0,1]$ be such that $\omega'(x,\delta) < \frac{\varepsilon}{4}$ and $x(i/m) \in \Gamma^{\varepsilon/4}$ for $i = 0, 1, \ldots, m$, and choose a portition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, $n \in \mathbf{N}$, such that $min_{1 \leq i \leq n} > \delta$ and

(16) $$\max_{1 \leq i \leq n} \sup_{s,t \in [t_{i-1}, t_i)} r(x(s), x(t)) < \frac{\varepsilon}{4} \,.$$

Now, select $\{y_i\} \subset \Gamma$ with $r(x(i/m), y_i) < \frac{\varepsilon}{4}$, $i = 0, 1, \ldots, m$. If we define $x' \in A$ by

$$x'(t) = \begin{cases} y_{[mt_{i-1}]} + 1 & , t_{i-1} \leq t < t_i, \ i = 1, \ldots, n \\ y_{[mt_{n-1}]} + 1 & , t = 1 \end{cases}$$

then we have

$$\sup_{0 \leq t \leq 1} r(x(t), x'(t)) < \frac{\varepsilon}{2} \,.$$

Therefore

$$\delta(x, x') < \varepsilon$$

which implies $x \in A^\varepsilon$. Consequently, $\inf_\alpha P(X_\alpha \in A^\varepsilon) \geq 1 - \varepsilon$, so the relative compactness follows from Theorem 4 and the Prohorov's theorem ([1], p. 58).

REFERENCES

[1] Billingsley, P., *Convergence of Probability Measures*, John Wiley & Sons, Inc., 1968.

[2] Ethier, S. N., Kurtz, T. G., *Markov Processes. Characterization and Convergence*. John Wiley & Sons, Inc., 1986.

[3] Kisyński, J., *Metrization of* $D_E[0,1]$ *by Hausdorff distance between graphs*, Ann. Pol. Math., Vol. LI, 1988.

Katedra Matematyki
Politechnika Lubelska
ul. Nadbystrzycka 38
20-618 Lublin, Poland

(received February 10, 1993)