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On the *-Weak Law of Large Numbers in the Incomplete Tensor Product of W^*-algebras

Abstract. The sequence of operators $\widetilde{A}_n$ of the form $\widetilde{A}_n=\frac{1}{n}\sum_{i=1}^{n} 1\otimes\ldots\otimes 1\otimes A_i\otimes 1\otimes\ldots$ is *-weak convergent if and only if the sequence of values $\phi(\widetilde{A}_n)$ converges for some normal normed state ϕ.

Let $\mathcal{A}_i$, for each positive integer i, be a W^*-algebra with the normal normed state α_i. Denote by $\mathcal{A}$ the incomplete tensor product $\bigotimes_{i=1}^{\infty}(\mathcal{A}_i,\alpha_i)$ [1]. Assume that, for $i \in \mathrm{N}$, A_i is a self-adjoint element of $\mathcal{A}_i$ and consider the sequence of elements of $\mathcal{A}$ of the form

$$\overline{A}_i = \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \ldots \otimes A_i \otimes \mathbf{1}_{i+1} \otimes \ldots \tag{1}$$

(where $\mathbf{1}_j$ denotes the identity in $\mathcal{A}_j$) and the sequence of the corresponding mean-values

$$\widetilde{A}_n = \frac{1}{n}\sum_{i=1}^{n} \overline{A}_i\,. \tag{2}$$

Assume that the sequence of norms of A_i is bounded, i.e. there exists M such that $\|A_i\| < M$ for any positive integer i. We say that, for the sequence A_i (or $\overline{A}_i$), the *-weak law of large numbers holds if the sequence $\psi(\widetilde{A}_n)$ converges for any normal normed state ψ on $\mathcal{A}$.

The aim of the paper is to show that, for the sequence A_i, the *-weak law of large numbers holds if and only if the sequence $\phi(\widetilde{A}_n)$ converges for some normal normed state on $\mathcal{A}$. This fact can easily be deduced from the following

Theorem . *Let ϕ,ψ be two normal normed states on $\mathcal{A}$. Then*

$$\lim_{n\to\infty} |\phi(\widetilde{A}_n) - \psi(\widetilde{A}_n)| = 0\,. \tag{3}$$

Proof. Consider first the case when ϕ and ψ are product states in $\mathcal{A}$, i.e. there exists a sequence of states ϕ_i and ψ_i on $\mathcal{A}_i$, respectively, such that $\phi = \bigotimes_{i=1}^{\infty}\phi_i$, $\psi =$

$\bigotimes_{i=1}^{\infty} \psi_i$. It is well known that every $\mathcal{A}_i$ can be represented as the operator algebra acting in some Hilbert space H_i in such a way that there exist in each H_i unit vectors x_i and y_i such that ϕ_i and ψ_i can be represented as the pure states given by x_i and y_i, respectively. The fact that there exist products of ϕ_i and ψ_i on the same incomplete tensor product of $\mathcal{A}_i$ means that $\sum_{i=1}^{\infty} |1-(x_i,y_i)| < \infty$, [1], and, by [2], we have that

(4) $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |1-(x_i,y_i)| = 0 \,.$$

On the other hand, for any unit vectors ξ,η in some Hilbert space $\mathcal{K}$ and for $x \in \mathcal{B}(\mathcal{K})$, we have

(5) $$2|1-(\xi,\eta)| \geq 2-2\mathrm{Re}(\xi,\eta) = \|\xi-\eta\|^2 \,,$$

and hence,

(6) $$\begin{aligned} |(\xi,x\xi)-(\eta,x\eta)| &= |(\xi,x\xi)-(\eta,x\xi)+(\eta,x\xi)-(\eta,x\eta)| \\ &= |(\xi-\eta,x\xi)+(\eta,x(\xi-\eta))| \leq 2\|x\|\,\|\xi-\eta\| \\ &\leq 2\sqrt{2}\|x\|\,|1-(\xi,\eta)|^{1/2} \,. \end{aligned}$$

Using the inequality and putting $x = \bigotimes_{i=1}^{\infty} x_i$, $y = \bigotimes_{i=1}^{\infty} y_i$, we have

(7) $$\begin{aligned} |\phi(\widetilde{A}_n)-\psi(\widetilde{A}_n)| &= |(x,\widetilde{A}_n x)-(y,\widetilde{A}_n y)| \\ &\leq \frac{1}{n}\sum_{i=1}^{n} |(x_i,A_i x_i)-(y_i,A_i y_i)| \\ &\leq 2\sqrt{2}M\frac{1}{n}\sum_{i=1}^{n} |1-(x_i,y_i)|^{1/2} \,. \end{aligned}$$

So, in the case considered, we have proved that

$$|\phi(\widetilde{A}_n)-\psi(\widetilde{A}_n)|$$

tends to zero.

Assume now that ϕ is a pure state generated by a vector x which is a linear combination of pairwise orthogonal product vectors in $\mathcal{A}$, i.e. that ϕ is of the form

(8) $$x = \sum_{j=1}^{m} a^j \xi^j$$

where all ξ^j are product vectors lying in $\mathcal{A}$.

Now,

(9) $$\begin{aligned} \phi(\widetilde{A}_n)-\psi(\widetilde{A}_n) &= \sum_{j=1}^{m} |a^j|^2 |(x^j,\widetilde{A}_n x_j)-(y,\widetilde{A}_n y)| \\ &\quad + \sum_{r\neq j} \overline{a}^r a^j (x^r,\widetilde{A}_n x^j) \,. \end{aligned}$$

The first sum in (9) tends to zero by the first part of the proof. The convergence of the second sum to zero can be obtained by Lemma 2.2. in [3].

Now, we consider the case when ϕ is an arbitrary pure state on $\mathcal{A}$, i.e. there exists a unit vector x in $\mathcal{K}$ such that $\phi(A) = (Ax, x)$. Evidently, x can be written as

$$x = \sum_{j=1}^{\infty} a_j \xi^j \tag{10}$$

where all ξ^j are as in (8), and

$$\sum_{j=1}^{\infty} |a_j|^2 = 1 . \tag{11}$$

Decompose now x into two sums

$$x = \sum_{j=1}^{k} a_j \xi^j + \sum_{j=k+1}^{\infty} a_j \xi^j \tag{12}$$

and consider x as a linear combination of two vectors with norm one, say

$$\sum_{j=1}^{k} a_j \xi^j = b_k \beta_k \tag{13}$$

and

$$\sum_{j=k+1}^{\infty} a_j \xi^j = c_k \gamma_k ,$$

where

$$b_k = \sqrt{\sum_{j=1}^{k} |a_j|^2} , \quad c_k = \sqrt{1 - b_k^2} . \tag{14}$$

Then we can calculate

$$\begin{aligned} |\phi(\widetilde{A}_n) - \psi(\widetilde{A}_n)| &= |(x, \widetilde{A}_n x) - (y, \widetilde{A}_n y)| \\ &\le b_k^2 |(\beta_k, \widetilde{A}_n \beta_k) - (y, \widetilde{A}_n y)| + |(b_k \beta_k, \widetilde{A}_n c_k \gamma_k)| \\ &\quad + |(c_k \gamma_k, \widetilde{A}_n b_k \gamma_k)| + |(c_k \gamma_k, \widetilde{A}_n c_k \gamma_k)| \\ &\le b_k^2 |(\beta_k, \widetilde{A}_n \beta_k) - (y, \widetilde{A}_n y)| + 4 M c_k . \end{aligned} \tag{15}$$

Since, for a sufficiently large k, c_k is so small as we want, the proof in the case considered is finished.

Assume now that ϕ is quite arbitrary, i.e. ϕ is a convex combination of pure states, say,

$$\phi = \sum_{i=1}^{\infty} m_i \phi_i , \quad \sum_{i=1}^{\infty} m_i = 1 . \tag{16}$$

We have

$$
\begin{aligned}
(17)\qquad |\phi(\widetilde{A}_n)-\psi(\widetilde{A}_n)| &= \left|\sum_{i=1}^{\infty} m_i(\phi_i(\widetilde{A}_n)-\psi(\widetilde{A}_n))\right| \\
&\le \sum_{i=1}^{\infty} m_i|\phi(\widetilde{A}_n)-\psi(\widetilde{A}_n)| = \sum_{i=1}^{l} m_i|\phi_i(\widetilde{A}_n)-\psi(\widetilde{A}_n)| \\
&+ \sum_{i=l+1}^{\infty} m_i|\phi_i(\widetilde{A}_n)-\psi(\widetilde{A}_n)| .
\end{aligned}
$$

The first sum can be arbitrarily small for large n, the second – for large l.

Repeating the same considerations for the state ψ, we obtain that $|\phi(\widetilde{A}_n)-\psi(\widetilde{A}_n)|$ tends to zero for any states ϕ, ψ. So, the sequence $\phi(\widetilde{A}_n)$ is convergent if and only if any sequence $\psi(\widetilde{A}_n)$ is convergent. This ends the proof.

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