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A Note on the Blasis's Method of an Approximation to an upper semicontinuous multifunction

Abstract. It is proved that a product measurable multifunction $F : T \times X \to \mathcal{U}(Z)$, where T, X and $\mathcal{U}(Z)$ denote a measurable space, metric space and a family of all nonempty closed convex and bounded subsets of real normed space Z, respectively, is an upper semi-Caratheodory multifunction if and only if it is a limit of a decreasing sequence $\{G_n\}$ of Caratheodory multifunctions $G_n : T \times X \to \mathcal{U}(Z)$.

1. Introduction. It is well known that an upper semicontinuous closed convex valued multifunction can be approximated from the exterior by continuous multifunctions (see [1], [2], [4], [7]) ; from them we especially recommend results given in the Blasi's paper [2] in which the author obtained several theorems characterizing certain classes of semicontinuous multifunctions by continuous approximations. The purpose of the present note is to show that de Blasi's method can be used in proving analogous results to those of [2] for semicontinuous multifunctions which depend measurably on the parameter.

2. Preliminaries. We assume the reader is familiar with notions concerning multifunctions (the necessary information can be found in [3, 5, 7]). To show the close relation between our considerations and those of [2], and for the reader's convenience, we mostly use notations given in [2] :

T	—	a complete measurable space with $\mathcal{A}$ as a σ-field of measurable subsets,
X	—	a Polish space,
Z	—	a real separable normed space,
$2^X (2^Z)$	—	a family of all nonempty subsets of X (Z),
$\mathcal{U}(Z)$	—	a family of all nonempty closed convex and bounded subsets of Z,
$\mathcal{B}(X)$	—	a σ-field of Borel subsets of X,
$\mathcal{A} \times \mathcal{B}(X)$	—	a product σ-field on $T \times X$,
$S(a,r)$	—	an open ball centered at $x \in X$ ($a \in Z$) of radius r,
$S = S(\theta,1)$	—	where θ denotes the zero of Z,
$S(A,r) = \bigcup_{a\in A} S(a,r)$	—	for $A \in 2^X$ ($A \in 2^Z$),
$\overline{A}$	—	the closure of $A \in 2^X$ ($A \in 2^Z$),
$\overline{\text{co}}\, A$	—	a closed convex hull of $A \in 2^Z$,

$d(z, A)$	—	distance of a point $z \in X$ ($z \in Z$) to a set $A \in 2^X$ ($A \in 2^Z$),
$h(A, B) = \inf\{t > 0 : A \subset S(B, t), B \subset S(A, t)\}$		for bounded $A, B \in 2^X$ (bounded $A, B \in 2^Z$),
$A + B = \{a + b; a \in A, b \in B\}$		for $A, B \in 2^Z$,
$rA = \{ra; a \in A\}$		for $A \in 2^Z$, $r \in \mathbf{R}$ (real numbers)

Let $Y \subset X$ be a nonempty set. A multifunction $F : T \times Y \to \mathcal{U}(Z)$ is said to be a H–Carathéodory multifunction (resp. H upper semi–Carathéodory multifunction) if $F(\cdot, x)$ is weakly $\mathcal{A}$–measurable for each $x \in Y$ and $F(t, \cdot)$ is Hausdorff continous (resp. Hausdorff upper semicontinous) for each $t \in T$. A multifunction $F : T \times Y \to \mathcal{U}(Z)$ is bounded if there is a constant $M \geq 0$ such that $h(F(t, x), \{\theta\}) \leq M$ for all $(t, x) \in T \times Y$. Further, we say that a multifunction $F : T \times Y \to \mathcal{U}(Z)$ is product measurable if it is weakly Σ–measurable, where Σ is the trace σ–field $\{(T \times Y) \cap C : C \in \mathcal{A} \times \mathcal{B}(X)\}$; it is obvious that $\Sigma \subset \mathcal{A} \times \mathcal{B}(X)$, provided $Y \in \mathcal{B}(X)$.

3. Auxillary results. We start with the following four lemmas. The first three of them are obvious and we give them without proofs.

Lemma 1. *If $h(A_n, A) \to 0$ as $n \to \infty$, where A_n, A, $n \in \mathbf{N}$ (positive integers) are nonempty bounded subsets of X (Z), then for each $z \in X$ ($z \in Z$) $d(z, A_n) \to d(z, A)$ as $n \to \infty$.*

Lemma 2. *Let $A, B, C \in 2^Z$. Then*

i) $\overline{co}\, A = \overline{co}\, \overline{A}$,

ii) $\overline{\overline{A + B} + C} = \overline{A + B + C}$.

Lemma 3. *Let a multifunction $G : T \to 2^Z$ be weakly $\mathcal{A}$-measurable. Then for each $r \in \mathbf{R}$ the multifunction $rG : T \to 2^Z$, defined by $(rG)(t) = rG(t)$, is also weakly $\mathcal{A}$-measurable.*

Lemma 4 will play the crucial role in our further considerations.

Lemma 4. *Let $A \subset X$ and let $F : T \times Y \to 2^Z$ be a product measurable multifunction. If A and Y belong to $\mathcal{B}(X)$, then a multifunction $G : T \to 2^Z$, defined by $G(t) = \overline{co}\, F(t, A)$, where $F(t, A) = \bigcup_{a \in A} F(t, a)$, is weakly $\mathcal{A}$-measurable.*

Proof. First we shall show that a multifunction $G^* : T \to 2^Z$, defined by $G^*(t) = F(t, A)$, is weakly $\mathcal{A}$–measurable. Let us take an arbitrary open subset U of Z. By the product measurability of F, the inverse image of U under F, i.e. the set $F^-(U) = \{(t, x) \in T \times Y : F(t, x) \cap U \neq \emptyset\}$ belongs to $\Sigma = \{(T \times Y) \cap C : C \in \mathcal{A} \times \mathcal{B}(X)\} \subset \mathcal{A} \times \mathcal{B}(X)$. Hence using the projection theorem [9, Theorem 4] we obtain

$$G^{*-}(U) = \{t \in T : G^*(t) \cap U \neq \emptyset\} = \{t \in T : F(t,A) \cap U \neq \emptyset\} =$$
$$= \mathrm{proj}_T \{F^-(U) \cap (T \times A)\} \in \mathcal{A}\ .$$

Therefore the multifunction G^* is weakly $\mathcal{A}$-measurable. Then by [6, Proposition 2.6] its closure $\overline{G^*}$, i.e. the closed valued multifunction defined by $\overline{G^*}(t) = \overline{G^*(t)}$, is also weakly $\mathcal{A}$-measurable. Applying lemma 2i) we get $G(t) = \overline{\mathrm{co}}\ F(t,A) = \overline{\mathrm{co}}\, G^*(t) = \overline{\mathrm{co}}\, \overline{G^*(t)}$ and now weak $\mathcal{A}$-measurability of the multifunction G follows from [5, Corollaire 3.2.3] (cf. also [6, Theorem 9.1]).

Proposition 1. *Let $Y \subset X$ be a nonempty open set, $\mathcal{V} = \{V_j\}_{j\in J}$ be a covering of Y by nonempty open sets $V_j \subset Y$ and $Q = \{q_j\}_{j\in J}$ be a partition of unity subordinated to $\mathcal{V}$. Suppose that to each $j \in J$ there corresponds a bounded H-Carathéodory multifunction $G_j : T \times W_j \to \mathcal{Z}$, where $V_j \subset W_j \subset Y$. Then for $G(t,x) = \overline{G^*(t,x)}$, where $G^*(t,x) = \sum_{j\in J} q_j(x) G_j(t,x)$, $(t,x) \in T \times Y$, we have $G(t,x) \in \mathcal{Z}$ and the multifunction $G : T \times Y \to \mathcal{Z}$ is H-Carathéodory multifunction.*

Proof. The proof of Hausdorff continuity of the multifunction $G(t,\cdot)$ for each fixed $t \in T$ is analogous to the proof of Proposition 2.6 in [2] and we omit it.

Now let us fix $x \in Y$ and let $I_x = \{j_1, j_2, \dots, j_{n_x}\} \subset J$ be such a finite set of indexes that $\sum_{j\in I_x} \varphi_j(x) = 1$. Then $G(\cdot,x) = \overline{\sum_{j\in I_x} \varphi_j G_j^*(\cdot,x)}$.

Hence, by Lemma 2ii), Lemma 3 and [5, Corollaire 3.2.3], $G(\cdot,x)$ is weakly $\mathcal{A}$-measurable and so $G : T \times Y \to \mathcal{U}(Z)$ is a H-Carathéodory multifunction.

Proposition 2. *Let $F : T \times X \to \mathcal{U}(Z)$ be a H-upper semi-Carathéodory multifunction. Let $\mathcal{V} = \{V_j\}_{j\in J}$ be a covering of X by nonempty open sets $V_j \subset X$ and $Q = \{q_j\}_{j\in J}$ be a partition of unity subordinated to $\mathcal{V}$. Suppose that to each $j \in J$ there corresponds a sequence $\{G_n^j\}$ of bounded H-Carathéodory multifunctions $G_n^j : T \times W_j \to \mathcal{U}(Z)$, where $V_j \subset W_j \subset X$, satisfying for each $(t,x) \in T \times W_j$ the properties*

(a_1) $G_n^j(t,x) \subset \Omega_j = \overline{\mathrm{co}}\ F(t,W_j)$, $n \in \mathbf{N}$,
(a_2) $G_n^j(t,x) \supset F(t,x)$, $n \in \mathbf{N}$,
(a_3) $G_1^j(t,x) \supset G_2^j(t,x) \supset \cdots$, and
(a_4) $h(G_n^j(t,x), F(t,x)) \to 0$ as $n \to \infty$.

For any fixed $n \in \mathbf{N}$, set $G_n(t,x) = \overline{G_n^*(t,x)}$, where $G_n^*(t,x) = \sum_{j\in J} q_j G_n^j(t,x)$, $(t,x) \in T \times X$. Then $G_n(t,x) \in \mathcal{U}(Z)$ and for each $n \in \mathbf{N}$ the multifunction $G_n : T \times X \to \mathcal{U}(Z)$ is a H-Carathéodory multifunction and satisfies, for each $(t,x) \in T \times X$, the properties

(b_1) $G_n(t,x) \subset \Omega = \overline{co}\ F(t,x)$, $n \in \mathbf{N}$,
(b_2) $G_n(t,x) \supset F(t,x)$, $n \in \mathbf{N}$,
(b_3) $G_1(t,x) \supset G_2(t,x) \supset \cdots$, and
(b_4) $h(G_n(t,x), F(t,x)) \to 0$ as $n \to \infty$.

Proof. By Proposition 1, for each $n \in \mathbf{N}$ G_n is well-defined a H-Carathéodory multifunction from $T \times X$ to $\mathcal{U}(Z)$. To prove that $\{G_n\}$ satisfies properties $(b_1)-(b_n)$ it is enough to consider the same as in [2, Proposition 4.4].

Proposition 3. *Let $F : T \times Y \to \mathcal{U}(Z)$, where $Y \subset X$ is a nonempty open set, be a product measurable and bounded H-upper semi-Carathéodory multifunction. Then there is a sequence $\{G_n\}$ of H- Carathéodory multifunctions $G_n : T \times Y \to \mathcal{U}(Z)$ satisfying, for each $(t,x) \in T \times Y$, the properties :*
(a_1) $G_n(t,x) \subset \Omega = \overline{co}\, F(t,Y)$, $n \in \mathbf{N}$,
(a_2) $G_n(t,x) \supset F(t,x)$, $n \in \mathbf{N}$,
(a_3) $G_1(t,x) \supset G_2(t,x) \supset \cdots$, and
(a_4) $h(G_n(t,x), F(t,x)) \to 0$ as $n \to \infty$.

Proof. For each $(t,u) \in T \times Y$ and $n \in \mathbf{N}$ put $F^n_{(t,u)} = \overline{co}\, F(t, S(u, \frac{2}{3^n}))$. Then $F^n_{(t,u)}$ is bounded because F is bounded and, clearly, $F^n_{(t,u)} \in \mathcal{U}(Z)$. Moreover each multifunction $G^n_u : T \times S(u, \frac{2}{3^n}) \to \mathcal{U}(Z)$, defined by $G^n_u(t,x) = F^n_{(t,u)}$, is H - Carathéodory multifunction. Indeed, $G^n_u(t,\cdot) : S(u, \frac{2}{3^n}) \to \mathcal{U}(Z)$ is H -continuous as a constant multifunction $F^n_{(t,u)}$ and $G^n_u(\cdot,x) : T \to \mathcal{U}(Z)$ is weakly $\mathcal{A}$-measurable by Lemma 4. Put $S^n_u = S(u, \frac{1}{3^n})$, $u \in Y$, $n \in \mathbf{N}$. For each (fixed) $n \in \mathbf{N}$, $\mathcal{S}_n = \{S^n_u\}_{u \in Y}$ is an open covering of Y and hence there is a partition $Q = \{q^n_u\}_{u \in Y}$ of unity subordinated to $\mathcal{S}_n$ (see [8, Lemma 8.12]). Let us put
$G_n(t,x) = \overline{G^*_n(t,x)}$, where

$$G^*_n(t,x) = \sum_{u \in Y} q^n_u(x) \cdot G^n_u(t,x) \; . \tag{1}$$

By Proposition 1, $G_n(t,x) \in \mathcal{U}(Z)$ and the multifunction $G : T \times Y \to \mathcal{U}(Z)$, defined by (1), is a H-Carathéodory multifunction. Now to prove that $\{G_n\}$ satisfies properties $(a_1)-(a_4)$ it is enough, for fixed $(t_0, x_0) \in T \times Y$, to argue identically as in the corresponding part of the proof of Proposition 4.1 in [2].

4. Main Theorem.

Theorem . *Let a weakly $\mathcal{A} \times \mathcal{B}(X)$-measurable multifunction $F : T \times X \to \mathcal{U}(Z)$ be given. Then the following two statements are equivalent :*
(a) F is H-upper semi-Carathéodory,
(b) There exists a sequence $\{G_n\}$ of H-Carathéodory multifunctions $G_n : T \times X \to \mathcal{U}(Z)$, satisfying, for each $(t,x) \in T \times X$, the properties :
(b_1) $G_n(t,x) \subset \Omega = \overline{co}\, F(t,X)$, $n \in \mathbf{N}$,
(b_2) $G_n(t,x) \supset F(t,x)$, $n \in \mathbf{N}$,
(b_3) $G_1(t,x) \supset G_2(t,x) \supset \cdots$, and
(b_4) $h(G_n(t,x), F(t,x)) \to 0$ as $n \to \infty$.

Proof.
$(b) \Rightarrow (a)$ The proof of H-upper semicontinuity of the multifunction $F(t,\cdot)$, $t \in T$, is the same as the proof of Theorem 4.5 part $(b) \Rightarrow (a)$ in [2] and therefore we omit it. The weak $\mathcal{A}$-measurability of $F(\cdot,x)$, $x \in X$, follows from [6, Theorem

3.5(i)] because for each $z \in \mathcal{U}(Z)$ the real function $t \to d(z, F(t,x))$ is measurable as a limit, by Lemma 1, of measurable functions $t \to d(z, G_n(t,x))$. Thus $F : T \times X \to \mathcal{U}(Z)$ is a H-upper semi-Carathéodory multifunction.

$(a) \Rightarrow (b)$ First of all let us observe that weak $\mathcal{A} \times \mathcal{B}(X)$-measurability of F implies the product measurability of each restriction $F|_{T\times Y}$, $Y \subset X$. On the other hand, each such restriction is a H-upper semi-Carathéodory multifunction since F is so. Further, since $F(t,\cdot)$ is H-upper semicontinuous, for each $u \in X$ there is $\delta(u) > 0$ such that $F(t,x) \subset F(t,u) + S$, whenever $x \in S(u,\delta(u))$. Put $S_u = S(u,\delta(u))$ and denote by F^u the restriction of F to $T \times S(u,\delta(u))$. For each $u \in X$, $F^u : T \times S_u \to \mathcal{U}(Z)$ is a bounded product measurable and H-upper semi-Carathéodory multifunction. Thus by Proposition 3 (with S_u and F^u in the place of Y and F, respectively) there is a sequence $\{G_n^u\}$ of bounded H-Carathéodory multifunctions $G_n^u : T \times S_u \to \mathcal{U}(Z)$ satisfying properties $(a_1) - (a_4)$ of Proposition 3. Let $Q = \{q_u\}_{u\in X}$ be a partition of unity subordinated to the open covering $\mathcal{S} = \{S_u\}_{u\in X}$ of X. For any $n \in \mathbf{N}$, set $G_n(t,x) = \overline{G_n^*(t,x)}$, where

$$G_n^*(t,x) = \sum_{u\in X} q_u(x) G_n^u(t,x)\,, \quad (t,x) \in T \times X\,. \tag{2}$$

From Proposition 2 (with $\mathcal{S}$ and G_n^u in the place of $\mathcal{V}$ and G_n^j, respectively) it follows that, for each $n \in \mathbf{N}$, we have $G_n(t,x) \in \mathcal{U}(Z)$ and the multifunction $G_n : T \times X \to \mathcal{U}(Z)$ defined by (2) is H-Carathéodory. Moreover, the sequence $\{G_n\}$ satisfies the properties $(b_1) - (b_4)$ of the statement. The proof is completed.

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