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Some Results on the Commutative Neutrix Convolution Product of Distributions

Abstract. Let f, g be distributions in $\mathcal{D}'$ and let $f_n(x) = f(x)\tau_n(x), g_n(x) = g(x)\tau_n(x)$, where $\tau_n(x)$ is a certain function which converges to the identity function as n tends to infinity. Then the commutative neutrix convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit exists. The neutrix convolution product $\ln x_- \boxed{*} x_+^\mu$ is evaluated for $\mu = 0, \pm1, \pm2, \ldots$, from which other neutrix convolution products are deduced.

Keywords: distribution, neutrix, neutrix limit, commutative neutrix convolution product.

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In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The following definition for the convolution product of certain distributions f and g in $\mathcal{D}'$, was given by Gel'fand and Shilov [6].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ satisfying either of the following conditions:
(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side. Then the convolution product $f * g$ is defined by

$$\langle (f * g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x + y) \rangle \rangle$$

for arbitrary ϕ in $\mathcal{D}$.

It follows that if the convolution product $f * g$ exists by Definition 1, then

$$f * g = g * f , \tag{1}$$

$$(f * g)' = f * g' = f' * g . \tag{2}$$

Definition 1 is very restrictive and can only be used for a small class of distributions. In order to extend the convolution product to a larger class of distributions, Jones [7] gave the following definition.

Definition 2. Let f and g be distributions and let τ be an infinitely differentiable function satisfying the following conditions:
(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$ for $|x| \leq 1/2$,
(iv) $\tau(x) = 0$ for $|x| \geq 1$.

Let

$$f_n(x) = f(x)\tau(x/n), \quad g_n(x) = g(x)\tau(x/n)$$

for $n = 1, 2, \ldots$. Then the convolution product $f * g$ is defined as the limit of the sequence $\{f_n * g_n\}$, provided the limit h exists in the sense that

$$\lim_{n\to\infty} \langle f_n * g_n, \phi\rangle = \langle h, \phi\rangle$$

for all test functions ϕ in $\mathcal{D}$.

In this definition the convolution product $f_n * g_n$ exists by Definition 1 since f_n and g_n have bounded supports. It follows that if the limit of the sequence $\{f_n * g_n\}$ exists, so that the convolution product $f * g$ exists, then $g * f$ also exists and equation (1) holds. However equation (2) need not necessarily hold since Jones proved that

$$1 * \text{sgn } x = \text{sgn } x * 1 = x,$$

$$(1 * \text{sgn } x)' = 1, \quad 1' * \text{sgn } x = 0, \quad 1 * (\text{sgn } x)' = 2.$$

It can be proved that if a convolution product exists by Definition 1, then it exists by Definition 2 and defines the same distribution.

However, there were still many convolution products which did not exist by Definition 2 and in order that further convolution products could be defined the next definition was introduced in [3].

Definition 3. Let f and g be distributions and let

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$, where τ is defined as in Definition 3. Let $f_n(x) = f(x)\tau_n(x)$ and $g_n(x) = g(x)\tau_n(x)$ for $n = 1, 2, \ldots$. Then the commutative neutrix convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit h exists in the sense that

$$N\!-\!\lim_{n\to\infty} \langle f_n * g_n, \phi\rangle = \langle h, \phi\rangle$$

for all ϕ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n, \quad (\lambda > 0\,; r = 1, 2, \ldots)$$

and all functions $\epsilon(n)$ for which $\lim_{n\to\infty}\epsilon(n)=0$.

The convolution product $f_n * g_n$ in this definition is again in the sense of Definition 1, the support of f_n being contained in the interval $[-n-n^{-n}, n+n^{-n}]$. It was proved in [3] that if a convolution product exists by Definition 1, then the commutative neutrix convolution product exists and defines the same distribution.

The following theorems were proved in [3] and [4] respectively.

Theorem 1. *The neutrix convolution product $x_-^\lambda \boxed{*} x_+^\mu$ exists and*

$$x_-^\lambda \boxed{*} x_+^\mu = B(-\lambda-\mu-1,\mu+1)x_-^{\lambda+\mu+1} + B(-\lambda-\mu-1,\lambda+1)x_+^{\lambda+\mu+1},$$

for $\lambda,\mu,\lambda+\mu\neq 0,\pm1,\pm2,\dots$, where B denotes the Beta function.

Theorem 2. *The neutrix convolution product $x_-^\lambda \boxed{*} x_+^{r-\lambda}$ exists and*

$$x_-^\lambda \boxed{*} x_+^{r-\lambda} = B(-r-1,r+1-\lambda)x_-^{r+1} + B(-r-1,\lambda+1)x_+^{r+1} + \frac{(-1)^r(\lambda)_{r+1}}{(r+1)!}x^{r+1}\ln|x|,$$

for $\lambda\neq 0,\pm1,\pm2,\dots$ and $r=-1,0,1,2,\dots$.

In this theorem, B again denotes the Beta function but is defined as in [2] by

$$B(\lambda,\mu) = N\!-\!\lim_{n\to\infty}\int_{1/n}^{1-1/n} t^{\lambda-1}(1-t)^{\mu-1}\,dt\,.$$

In the following we are going to consider the commutative neutrix convolutions products $x_-^{-r} \boxed{*} x_+^\mu$ and $x_+^{-r} \boxed{*} x_-^\mu$, where x_+^{-r} is defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!}(\ln x_+)^{(r)}$$

and x_-^{-r} is defined by $x_-^{-r}=(-x)_+^{-r}$, but first of all we prove

Theorem 3. *The commutative neutrix convolution product $\ln x_- \boxed{*} x_+^\mu$ exists and*

$$\ln x_- \boxed{*} x_+^\mu = -\frac{x_+^{\mu+1}}{\mu+1}\ln x_+ + \frac{\gamma+\psi(-\mu-1)}{\mu+1}x_+^{\mu+1} \tag{3}$$

for $\mu\neq 0,\pm1,\pm2,\dots$, where γ denotes Euler's constant, $\psi=\Gamma'/\Gamma$ and Γ denotes the Gamma function.

Proof. We will first of all suppose that $\mu>-1$ and $\mu\neq 0,1,2,\dots$ so that x_+^μ is locally summable function. Put

$$(x_+^\mu)_n = x_+^\mu\tau_n(x)\,,\quad (\ln x_-)_n = \ln x_-\tau_n(x)\,.$$

Then the convolution product $(\ln x_-)_n * (x_+^\mu)_n$ exists by Definition 1 and

$$
\begin{aligned}
\langle(\ln x_-)_n * (x_+^\mu)_n &= \langle(\ln y_-)_n, ((x_+^\mu)_n, \phi(x+y))\rangle \\
&= \int_{-n-n^{-n}}^{0} \ln(-y)\tau_n(y) \int_a^b (x-y)_+^\mu \tau_n(x-y)\phi(x)\, dx\, dy \\
&= \int_a^b \phi(x) \int_{-n}^{0} \ln(-y)(x-y)_+^\mu \tau_n(x-y)\, dy\, dx \\
&\quad + \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)(x-y)_+^\mu \tau_n(x-y)\, dy\, dx
\end{aligned} \tag{4}
$$

for $n > -a$ and arbitrary ϕ in $\mathcal{D}$ with support of ϕ contained in the interval $[a,b]$.

When $x < 0$ and $-n \le y \le 0$, $\tau_n(x-y) = 1$ on the support of ϕ. Thus with $x < 0$ and $-n \le y \le 0$, we have on making the substitution $y = xu^{-1}$

$$
\begin{aligned}
\int_{-n}^{0} \ln(-y)(x-y)_+^\mu \tau_n(x-y)\, dy &= \int_{-n}^{x} \ln(-y)(x-y)^\mu\, dy \\
&= (-x)^{\mu+1}\ln(-x)\int_{-x/n}^{1} u^{-\mu-2}(1-u)^\mu\, du \\
&\quad - (-x)^{\mu+1}\int_{-x/n}^{1} u^{-\mu-2}\ln u(1-u)^\mu\, du \\
&= I_{1n} - I_{2n}\,.
\end{aligned}
$$

Choosing an integer $r > \mu + 1$ we have

$$
\begin{aligned}
\int_{-x/n}^{1} u^{-\mu-2}(1-u)^\mu\, ,du &= \int_{-x/n}^{1} u^{-\mu-2}\left[(1-u)^\mu - \sum_{i=0}^{r}\frac{(-1)^i(\mu)_i}{i!}u^i\right] du \\
&\quad + \sum_{i=0}^{r}\frac{(-1)^i(\mu)_i}{i!(i-\mu-1)}[1-(-x/n)^{i-\mu-1}]\,,
\end{aligned}
$$

where

$$
(\mu)_i = \begin{cases} 1, & i = 0, \\ \prod_{j=0}^{i-1}(\mu-j), & i \ge 1 \end{cases}
$$

and it follows that

$$
\underset{n\to\infty}{N-\lim}\, I_{1n} = B(-\mu-1, \mu+1)(-x)^{\mu+1}\ln(-x) = 0 \tag{5}
$$

see [6]. Further,

$$
\begin{aligned}
\int_{-x/n}^{1} u^{-\mu-2}\ln u(1-u)^\mu\, du &= \int_{-x/n}^{1} u^{-\mu-2}\ln u\left[(1-u)^\mu - \sum_{i=0}^{r}\frac{(-1)^i(\mu)_i}{i!}u^i\right] du \\
- \sum_{i=0}^{r}\frac{(-1)^i(\mu)_i}{i!(i-\mu-1)^2}&\left[(i-\mu-1)(-x/n)^{i-\mu-1}\ln(-x/n) + 1 - (-x/n)^{i-\mu-1}\right]
\end{aligned}
$$

and it follows that

$$N-\lim_{n\to\infty} I_{2n} = B_{10}(-\mu-1,\mu+1)(-x)^{\mu+1},$$

where

$$B_{10}(-\mu-1,\mu+1) = \frac{\partial}{\partial\lambda}B(\lambda,\mu+1)\Big]_{\lambda=-\mu-1} = 0,$$

see [2]. Thus

$$N-\lim_{n\to\infty} I_{2n} = 0 \tag{6}$$

and it follows from equations (5) and (6) that

$$N-\lim_{n\to\infty}\int_{-n}^{0}\ln(-y)(x-y)_+^{\mu}\tau_n(x-y)\,dy = 0\,. \tag{7}$$

When $x>0$ and $-n\le y\le 0$ we have

$$\int_{-n}^{0}\ln(-y)(x-y)_+^{\mu}\tau_n(x-y)\;dy = \int_{x-n}^{0}\ln(-y)(x-y)^{\mu}\;dy$$
$$+\int_{x-n-n^{-n}}^{x-n}\ln(-y)(x-y)^{\mu}\tau_n(x-y)\;dy.$$

Making the substitution $y = x(1-u^{-1})$, we have

$$\int_{x-n}^{0}\ln(-y)(x-y)^{\mu}\;dy = x^{\mu+1}\ln x\int_{x/n}^{1}u^{-\mu-2}\;du$$
$$+\,x^{\mu+1}\int_{x/n}^{1}u^{-\mu-2}\ln(1-u)\;du - x^{\mu+1}\int_{x/n}^{1}u^{-\mu-2}\ln u\;du$$
$$= I_{3n}+I_{4n}-I_{5n}\,.$$

We have

$$\int_{x/n}^{1}u^{-\mu-2}\;du = -\frac{1}{\mu+1}[1-(n/x)^{\mu+1}]$$

and it follows that

$$N-\lim_{n\to\infty} I_{3n} = -\frac{x^{\mu+1}\ln x}{\mu+1}\,. \tag{8}$$

Making the substitution $u = 1-v$, we have

$$\int_{x/n}^{1}u^{-\mu-2}\ln(1-u)\;du = \int_{0}^{1-x/n}\ln v(1-v)^{-\mu-2}\;dv$$
$$= \int_{0}^{1-x/n}\ln v\left[(1-v)^{-\mu-2}-\sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!}v^i\right]\;dv$$
$$+\sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!}\left[\frac{(1-x/n)^{i+1}\ln(1-x/n)}{i+1}-\frac{(1-x/n)^{i+1}}{(i+1)^2}\right],$$

where r is chosen greater than $\mu+1$. It follows that

$$N-\lim_{n\to\infty}\int_{x/n}^{1} u^{-\mu-2}\ln(1-u)\,du =$$

$$= \int_0^1 \ln v\left[(1-v)^{-\mu-2} - \sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!}v^i\right]dv - \sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!(i+1)^2}$$

$$= B_{10}(1,-\mu-1)\,.$$

Thus

$$N-\lim_{n\to\infty} I_{4n} = B(1,-\mu-1)x^{\mu+1}\,. \tag{9}$$

Next we have

$$\int_{x/n}^{1} u^{-\mu-2}\ln u\,du = \frac{(n/x)^{\mu+1}[\ln x - \ln n]}{\mu+1} - \frac{1}{(\mu+1)^2}[1-(n/x)^{\mu+1}]$$

and it follows that

$$N-\lim_{n\to\infty} I_{5n} = -\frac{x^{\mu+1}}{(\mu+1)^2}\,. \tag{10}$$

Now it is easily proved that

$$B_{10}(1,\mu) = \frac{-\gamma-\psi(1+\mu)}{\mu}, \quad \mu^{-1}+\psi(\mu) = \psi(\mu+1)$$

and so

$$B_{10}(1,-\mu-1) + (\mu+1)^{-2} = \frac{\gamma+\psi(-\mu-1)}{\mu+1}\,. \tag{11}$$

Thus, on using equations (8), (9), (10) and (11)

$$N-\lim_{n\to\infty}\int_{x-n}^{0}\ln(-y)(x-y)^{\mu}\,dy = -\frac{x^{\mu+1}\ln x}{\mu+1} + \left[\frac{\gamma+\psi(-\mu-1)}{\mu+1}\right]x^{\mu+1}\,. \tag{12}$$

Further, with $n > x > n^{-n}$

$$\left|\int_{x-n-n^{-n}}^{x-n}\ln(-y)(x-y)^{\mu}\tau_n(x-y)\,dy\right| \le \int_{n}^{n+n^{-n}} y^{\mu}\ln(y-x)\,dy$$

$$= O(n^{\mu-n}\ln n)\,,$$

and so

$$\lim_{n\to\infty}\int_{x-n-n^{-n}}^{x-n}\ln(-y)(x-y)^{\mu}\tau_n(x-y)\,dy = 0\,. \tag{13}$$

It now follows from equations (7), (12) and (13) that

$$\text{(14)} \quad \begin{aligned} N - \lim_{n\to\infty} \int_{-n}^{0} \ln(-y)(x-y)_+^{\mu}\tau_n(x-y)\,dy \\ = -\frac{x_+^{\mu+1}\ln x_+}{\mu+1} + \left[\frac{\gamma+\psi(-\mu-1)}{\mu+1}\right] x_+^{\mu+1} . \end{aligned}$$

Next, with $-\frac{1}{2}n < a \leq x \leq b < \frac{1}{2}n$, we have

$$\begin{aligned} \left| \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)(x-y)^{\mu}\tau_n(x-y)\,dy \right| \leq \int_{-n-n^{-n}}^{-n} \ln(-y)(x-y)^{\mu}\,dy \\ = O(n^{\mu-n}) \end{aligned}$$

and so

$$\text{(15)} \quad \lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} \ln(-y)(x-y)^{\mu}\tau_n(y)\tau_n(x-y)\,dy = 0 .$$

It now follows from equations (4), (7), (14) and (15) that

$$\begin{aligned} N - \lim_{n\to\infty} \langle (\ln x_-)_n * (x_+^{\mu})_n, \phi(x)\rangle \\ = (\mu+1)^{-1}\langle -x_+^{\mu+1}\ln x_+ + [\gamma+\psi(-\mu-1)]x_+^{\mu+1}, \phi(x)\rangle \end{aligned}$$

and equation (3) follows for $\mu > -1$ and $\mu \neq 0, 1, 2, \ldots$.

Now assume that equation (3) holds for $-k < \mu < -k+1$, where k is some positive integer. This is certainly true when $k = 1$. The convolution product $(\ln x_-)_n * (x_+^{\mu})_n$ exists by Definition 1 and so equations (1) and (2) hold. Thus if ϕ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$, where we may suppose that $a < 0 < b$,

$$\begin{aligned} \langle [(\ln x_-)_n * (x_+^{\mu})_n]', \phi(x)\rangle &= -\langle (\ln x_-)_n * (x_+^{\mu})_n, \phi'(x)\rangle \\ &= \mu\langle (\ln x_-)_n * (x_+^{\mu-1})_n, \phi(x)\rangle \\ &\quad + \langle (\ln x_-)_n * [x_+^{\mu}\tau_n'(x)], \phi(x)\rangle \end{aligned}$$

and so

$$\text{(16)} \quad \begin{aligned} \mu\langle (\ln x_-)_n * (x_+^{\mu-1})_n, \phi(x)\rangle &= -\langle (\ln x_-)_n * (x_+^{\mu})_n, \phi'(x)\rangle \\ &\quad - \langle (\ln x_-)_n * [x_+^{\mu}\tau_n'(x)], \phi(x)\rangle . \end{aligned}$$

The support of $x_+^{\mu}\tau_n'(x)$ is contained in the interval $[n, n+n^{-n}]$ and so with $n > b > n^{-n}$, it follows as above that

$$\langle (\ln x_-)_n * [x_+^{\mu}\tau_n'(x)], \phi(x)\rangle = \int_a^b \phi(x) \int_n^{n+n^{-n}} y^{\mu}\tau_n'(y)\ln(y-x)\tau_n(x-y)\,dy\,dx$$

where on domain of integration y^μ and $\ln(y-x)$ are locally summable functions. It is easily seen that

$$\left|\int_a^b \phi(x)\int_n^{n+n^{-n}} y^\mu \tau_n'(y)\ln(y-x)\tau_n(x-y)\,dy\,dx\right| = O(n^\mu \ln n)$$

and so

$$\lim_{n\to\infty}\langle(\ln x_-)_n * [x_+^\mu \tau_n'(x)], \phi(x)\rangle = 0\,, \tag{17}$$

since $\mu<0$.

It now follows from equations (16) and (17) that

$$\begin{aligned}N-\lim_{n\to\infty}\mu\langle(\ln x_-)_n*(x_+^{\mu-1})_n,\phi(x)\rangle &= -N-\lim_{n\to\infty}\langle(\ln x_-)_n*(x_+^\mu)_n,\phi'(x)\rangle\\ &= -\langle \ln x_- \boxed{*}\, x_+^\mu, \phi'(x)\rangle\end{aligned}$$

by our assumption. This proves that the neutrix convolution product $\ln x_- \boxed{*}\, x_+^{\mu-1}$ exists and

$$\begin{aligned}\ln x_- \boxed{*}\, x_+^{\mu-1} &= \mu^{-1}[\ln x_- \boxed{*}\, x_+^\mu]'\\ &= \mu^{-1}\{-x_+^\mu \ln x_+ - (\mu+1)^{-1}x_+^\mu + [\gamma+\psi(-\mu-1)]x_+^\mu\}\\ &= \mu^{-1}\{-x_+^\mu \ln x_+ + [\gamma+\psi(-\mu-1)]x_+^\mu\}\end{aligned}$$

since $\psi(-\mu-1)-(\mu+1)^{-1}=\psi(-\mu)$.

Equation (3) now follows by induction for $\mu\neq 0,\pm1,\pm2,\ldots$. This completes the proof of the theorem.

Corollary . *The neutrix convolution products* $\ln|x| \boxed{*}\, x_+^\mu$, $\ln|x| \boxed{*}\, x_-^\mu$ *and* $\ln|x| \boxed{*}\, |x|^\mu$ *exist and*

$$\ln|x| \boxed{*}\, x_+^\mu = \frac{\pi\cot\mu\pi}{\mu+1}x_+^{\mu+1} \tag{18}$$

$$\ln|x| \boxed{*}\, x_-^\mu = \frac{\pi\cot\mu\pi}{\mu+1}x_-^{\mu+1} \tag{19}$$

$$\ln|x| \boxed{*}\, |x|^\mu = \frac{\pi\cot\mu\pi}{\mu+1}|x|^{\mu+1} \tag{20}$$

for $\mu\neq 0,\pm1,\pm2,\ldots$.

Proof. The convolution product $\ln x_+ * x_+^\mu$ exists by Gel'fand and Shilov's definition and it is easily proved that

$$\begin{aligned}\ln x_+ * x_+^\mu &= (\mu+1)^{-1}x_+^{\mu+1}\ln x_+ + B_{10}(1,\mu+1)x_+^{\mu+1}\\ &= (\mu+1)^{-1}x_+^{\mu+1}\ln x_+ - \left[\frac{\gamma+\psi(\mu+2)}{\mu+1}\right]x_+^{\mu+1}.\end{aligned}$$

Since the neutrix convolution product is clearly distributive with respect to addition, it follows that

$$\ln x_- \boxed{*} x_+^\mu + \ln x_+ * x_+^\mu = \frac{\psi(-\mu-1)-\psi(\mu+1)}{\mu+1} x_+^{\mu+1}$$
$$= \frac{\pi \cot \mu\pi}{\mu+2} x_+^{\mu+1}$$

since it can be easily proved that

$$\psi(-\mu-1)-\psi(\mu+2) = \pi \cot \mu\pi \ .$$

This proves equation (18).

Replacing x by $-x$ in equation (18) gives equation (19) and equation (20) follows on noting that $|x|^\mu = x_+^\mu + x_-^\mu$.

Theorem 4. *The neutrix convolution product* $x_-^{-r} \boxed{*} x_+^\mu$ *exists and*

$$x_-^{-r} \boxed{*} x_+^\mu = \frac{(\mu)_{r-1}}{(r-1)!}\{x_+^{\mu-r+1}\ln x_+ - [\gamma + \psi(-\mu+r+1)]x_+^{\mu-r+1}\} \tag{21}$$

for $\mu \neq 0, \pm1, \pm2, \ldots$ *and* $r = 1, 2, \ldots$.

Proof. Let ϕ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$, where we may suppose that $a < 0 < b$. Then

$$\langle[(\ln x_-)_n * (x_+^\mu)_n]', \phi(x)\rangle = -\langle(\ln x_-)_n * (x_+^\mu)_n, \phi'(x)\rangle$$
$$= -\langle(x_-^{-1})_n * (x_+^\mu)_n, \phi(x)\rangle + \langle[\ln x_- \tau_n'(x)] * (x_+^\mu)_n, \phi(x)\rangle$$

and so

$$\langle(x_-^{-1})_n * (x_+^\mu)_n, \phi(x)\rangle = \langle(\ln x_-)_n * (x_+^\mu)_n, \phi'(x)\rangle + \langle[\ln x_- \tau_n'(x)] * (x_+^\mu)_n, \phi(x)\rangle \ . \tag{22}$$

The support of $\ln x_- \tau_n'(x)$ is contained in the interval $[-n-n^{-n}, -n]$ and so with $n > -a > n^{-n}$, it follows as above that

$$\langle[\ln x_- \tau_n'(x)] * (x_+^\mu)_n, \phi(x)\rangle$$
$$= \int_a^b \phi(x) \int_{-n-n^{-n}}^{x-n} \ln(-y)\tau_n'(y)(x-y)^\mu \tau_n(x-y)\, dy\, dx$$
$$= \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{x-n} \ln(-y)\tau_n'(y)(x-y)^\mu \tau_n(x-y)\, dy\, dx \tag{23}$$
$$+ \int_a^0 \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^\mu\, dy\, dx$$
$$- \int_{-n^{-n}}^0 \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^\mu\, dy\, dx \ ,$$

where on the domain of integration $\ln(-y)$ and $(x-y)^\mu$ are locally summable functions.

It is easily seen that

$$\left|\int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^\mu \tau_n(x-y)\, dy\, dx\right|$$
$$= \left|\int_{-n^{-n}}^{0} \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^\mu\, dy\, dx\right|$$
$$= O(n^{\mu-n}\ln n)$$

and it follows that

$$\lim_{n\to\infty} \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^\mu \tau_n(x-y)\, dy\, dx$$
$$= \lim_{n\to\infty} \int_{-n^{-n}}^{0} \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^\mu\, dy\, dx = 0\,. \tag{24}$$

Integrating by parts, it follows that

$$\int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^\mu\, dy = (x+n)^\mu \ln n$$
$$+ \int_{-n-n^{-n}}^{-n} [y^{-1}(x-y)^\mu + \mu\ln(-y)(x-y)^{\mu-1}]\tau_n(y)\, dy\,. \tag{25}$$

Choosing an integer $r > \mu$, we have

$$(x+n)^\mu \ln n = \sum_{i=0}^{r-1} \frac{(\mu)_i x^i}{i!} n^{\mu-i}\ln n + \sum_{i=r}^{\infty} \frac{(\mu)_i x^i}{i!} n^{\mu-i}\ln n$$

and it follows that

$$N-\lim_{n\to\infty}(x+n)^\mu \ln n = 0\,. \tag{26}$$

Further,

$$\left|\int_{-n-n^{-n}}^{-n} [y^{-1}(x-y)^\mu + \mu\ln(-y)(x-y)^{\mu-1}]\tau_n(y)\, dy\right| = O(n^{\mu-n-1}\ln n) \tag{27}$$

and it follows from equations (25), (26) and (27) that

$$N-\lim_{n\to\infty} \int_a^0 \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^\mu\, dy\, dx = 0 \tag{28}$$

and then from equations (23), (24) and (28) that

$$\langle [\ln x_- \tau_n'(x)] * (x_+^\mu)_n, \phi(x)\rangle = 0\,. \tag{29}$$

It now follows from equations (22) and (29) that

$$\begin{aligned} N-\lim_{n\to\infty}\langle (x_-^{-1})_n * (x_+^{\mu})_n, \phi(x)\rangle &= N-\lim_{n\to\infty}\langle (\ln x_-)_n * (x_+^{\mu})_n, \phi'(x)\rangle \\ &= \langle \ln x_- \boxed{*} x_+^{\mu}, \phi'(x)\rangle. \end{aligned}$$

This proves that the neutrix convolution product $x_-^{-1} \boxed{*} x_+^{\mu}$ exists and

$$x_-^{-1} \boxed{*} x_+^{\mu} = -[\ln x_- \boxed{*} x_+^{\mu}]' = x_+^{\mu}\ln x_+ - [\gamma + \psi(-\mu-1)]x_+^{\mu}$$

as above for $\mu \neq 0, \pm 1, \pm 2, \ldots$. Equation (21) is therefore proved for case $r = 1$.

Now assume that equation (21) holds for some positive integer r. Then it follows as above that

$$\begin{aligned} \langle [(x_-^{-r})_n * (x_+^{\mu})_n]', \phi(x)\rangle &= r\langle (x_-^{-r-1})_n * (x_+^{\mu})_n, \phi(x)\rangle \\ &\quad + \langle [x_-^{-r}\tau_n'(x)] * (x_+^{\mu})_n, \phi(x)\rangle \ . \end{aligned} \tag{30}$$

It follows as above that

$$N-\lim_{n\to\infty}\langle [x_-^{-r}\tau_n'(x)] * (x_+^{\mu})_n, \phi(x)\rangle = 0$$

and so

$$\begin{aligned} N-\lim_{n\to\infty} r\langle (x_-^{-r-1})_n * (x_+^{\mu})_n, \phi(x)\rangle &= -N-\lim_{n\to\infty}\langle (x_-^{-r})_n * (x_+^{\mu})_n, \phi'(x)\rangle \\ &= -\langle x_-^{-r} \boxed{*} x_+^{\mu}, \phi'(x)\rangle \end{aligned}$$

by our assumption. Thus $x_-^{-r-1} \boxed{*} x_+^{\mu}$ exists and

$$\begin{aligned} x_-^{-r-1} \boxed{*} x_+^{\mu} &= r^{-1}[x_-^{-r} \boxed{*} x_+^{\mu}]' \\ &= \frac{(\mu)_{r-1}}{r!}\{(\mu-r+1)x_+^{\mu-r}\ln x_+ - (\mu-r+1)[\gamma+\psi(-\mu+r+1)]x_+^{\mu-r}\} \\ &= \frac{(\mu)_r}{r!}\{x_+^{\mu-r}\ln x_+ - [\gamma+\psi(-\mu+r)]x_+^{\mu-r}\} \ . \end{aligned}$$

Equation (21) now follows by induction for $\mu \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.

Corollary . *The neutrix convolution products $x^{-r} \boxed{*} x_+^{\mu}$ and $x^{-r} \boxed{*} |x|^{\mu}$ exist and*

$$x^{-r} \boxed{*} x_+^{\mu} = \frac{(-1)^{r-1}(\mu)_{r-1}\pi\cot\mu\pi}{(r-1)!}x_+^{\mu-r+1} \tag{31}$$

$$x^{-r} \boxed{*} |x|^{\mu} = \begin{cases} -\frac{(\mu)_{r-1}\pi\cot\mu\pi}{(r-1)!}|x|^{\mu-r+1}, & \textit{even } r\,, \\ \frac{(\mu)_{r-1}\pi\cot\mu\pi}{(r-1)!}\operatorname{sgn} x\cdot|x|^{\mu-r+1}, & \textit{odd } r \end{cases} \tag{32}$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.

Proof. The convolution product $x_+^{-r} * x_+^{\mu}$ exists by Gel'fand and Shilov's definition and it is easily proved that

$$x_+^{-r} * x_+^{\mu} = \frac{(-1)^{r-1}(\mu)_{r-1}}{(r-1)!}\{x_+^{\mu-r+1}\ln x_+ - [\gamma+\psi(\mu-r+2)]x_+^{\mu-r+1}\}$$
$$= x_+^{-r} \boxed{*} x_+^{\mu}.$$

Since $x^{-r} = x_+^{-r} + (-1)^r x_-^{-r}$, we have

$$x^{-r} \boxed{*} x_+^{\mu} = x_+^{-r} \boxed{*} x_+^{\mu} + (-1)^r x_-^{-r} \boxed{*} x_+^{\mu}$$
$$= \frac{(-1)^r(\mu)_{r-1}}{(r-1)!}[\psi(\mu-r+2) - \psi(-\mu+r-1)]x_+^{\mu-r+1}$$
$$= \frac{(-1)^{r-1}(\mu)_{r-1}\cot\mu\pi}{(r-1)!}x_+^{\mu-r+1}$$

since $\psi(\mu-r+2) - \psi(-\mu+r-1) = -\cot(\mu-r)\pi = -\cot\mu\pi$. This proves equation (31).

Equation (32) follows from equation (31) on noting that $|x|^{\mu} = x_+^{\mu} + x_-^{\mu}$ and $\operatorname{sgn} x.|x|^{\mu} = x_+^{\mu} - x_-^{\mu}$.

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