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Starlikeness and Convexity of Certain Integrals

O gwiaździstości i wypukłości pewnych całek

О звездообразности и выпуклости некоторых интегралов

Introduction: Let A denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular in the unit disc $E = \{z: |z| < 1\}$. We designate by S the subclass of univalent functions in A and by S^* and K the subclasses of S whose members are starlike and convex in E, respectively. Finally, we let R stand for the family of functions $f \in A$ which satisfy the condition $\text{Re}\, f'(z) > 0$, $z \in E$. It is known that R is a subclass of S. In 1952 Zmorovič [6] put the question whether R was a subclass of S^*. Later, Krzyż [3] gave an example of a function $f \in R$ such that $f \not\in S^*$. The problem of determining the radius of starlikeness of R is one of the open problems in the theory of univalent functions (see Goodman [1]). V. Singh and R. Singh [5] in 1977 showed that the radius of starlikeness of R was not less than 0.8534.

It is well known that if $f \in A$ and $|zf''(z)/f'(z)| < 1$, $z \in E$, then f is univalent and convex in E. In Theorem 1 of this paper we prove that one can replace the constant 1 by a larger one and still preserve the univalence (in fact, starlikeness) of f. In Theorem 2 we consider Zmorovič's problem for a subclass of R.

We shall need the following result due to Jack [2].

Lemma 1. *Let $w(z)$ be regular in the unit disc E, with $w(0) = 0$. If $|w(z)|$ attains its maximum value in the circle $|z| = r$ at a point z_0, then we can write*

$$z_0 w'(z_0) = k\, w(z_0),$$

where k is a real number ≥ 1.

Theorem 1. *If f belongs to A and satisfies $|zf''(z)/f'(z)| < 3/2$ in E, then f belongs to S^* and the function F, defined by*

$$F(z) = \frac{2}{z}\int_0^z f(t)\,dt \tag{1}$$

is in K.

Proof. Let us define a function w in E as follows:

$$\frac{zf'(z)}{f(z)} = 1 + w(z) \tag{2}$$

clearly $w(0) = 0$. To prove that $f \in S^*$ it suffices to show that $|w(z)| < 1$ in E.

From (2) we obtain

$$\frac{zf''(z)}{f'(z)} = w(z) + \frac{zw'(z)}{1 + w(z)}. \tag{3}$$

To prove that $|w(z)| < 1$ in E, assume that there exists a point z_0 in E such that $\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1$. Applying Lemma 1 to $w(z)$ at the point z_0 and letting $z_0 w'(z_0)/w(z_0) = k$, so that $k \geq 1$, we obtain from (3)

$$\left|\frac{z_0 f''(z_0)}{f' z_0}\right| = \left|e^{i\theta} + \frac{ke^{i\theta}}{1 + e^{i\theta}}\right| \geq \frac{3}{2}, w(z_0) = e^{i\theta},$$

which contradicts our hypothesis that $|zf''(z)/f'(z)| < 3/2$, $z \in E$. This contradiction establishes that $|w(z)| < 1$ in E and the assertion that $f \in S^*$ follows.

To show that the function F, defined by (1), is in K, we observe that our hypothesis: $|zf''(z)/f'(z)| < 3/2$, $z \in E$, implies that $\mathrm{Re}\,(1 + zf''(z)/f'(z)) > -1/2$, in E. The desired result now follows from [4], Cor, B. Theorem 1.

Theorem 2. *Let $f \in R$ and define g by*

$$g(z) = \int_0^z \frac{f(t)}{t}\,dt.$$

Then $g \in S^*$.

Proof. Since $f \in R$, we have $\mathrm{Re}\,(f(z)/z) > 0$, $z \in E$ and hence it follows that g belongs to R.

We are given that

$$\mathrm{Re}\,[g'(z) + zg''(z)] > 0, z \in E. \tag{4}$$

Define a function w in E as follows:

$$\frac{zg'(z)}{g(z)} = \frac{1+w(z)}{1-w(z)}. \tag{5}$$

clearly $w(0) = 0$, w is regular in E and of course $w(z) \neq 1$, $z \in E$. To prove that g belongs to R it clearly suffices to show that $|w(z)| < 1$ in E.

From (5) we obtain

$$g'(z) + zg''(z) = \frac{g(z)}{z}\left[\left(\frac{1+w(z)}{1-w(z)}\right)^2 + \frac{2zw'(z)}{(1-w(z))^2}\right] \tag{6}$$

Let us suppose that there exists a point z_0 in E such that $\max\limits_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1$.

Putting $z = z_0$ in (6) and applying Lemma 1 to $w(z)$ at the point z_0: letting $z_0 w'(z_0) = k\, w(z_0)$, so that $k \geqslant 1$, and $w(z_0) = e^{i\theta}$, $0 \leqslant \theta < 2\pi$, we obtain

$$\mathrm{Re}[g'(z_0) + z_0 g''(z_0)] = \mathrm{Re}\left\{\frac{g(z_0)}{z_0}\left[\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2 + \frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right]\right\} \tag{7}$$

It is readily seen that for all θ, $0 \leqslant \theta \leqslant 2\pi$, the expression within the square brackets is a negative real number. Also, since $g \in R$, we have $\mathrm{Re}\,(g(z_0)/z_0) > 0$. Thus (7) contradicts our hypothesis (4). This contradiction proves that $|w(z)| < 1$ in E and the assertion of our theorem follows.

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STRESZCZENIE

Niech

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

będzie funkcją regularną w kole jednostkowym E. W pracy otrzymano następujące wyniki:

(i) Jeśli

$$|zf''(z)/f'(z)| < \frac{3}{2}$$

w kole E, to f jest funkcją gwiaździstą, a całka

$$(2/z)\int_0^z f(t)\,dt$$

funkcją wypukłą w tym kole.

(ii) Jeśli $\operatorname{Re} f'(z) > 0$ w kole E, to całka

$$\int_0^z t^{-1} f(t)\,dt$$

jest funkcją gwiaździstą w tym kole.

РЕЗЮМЕ

Пусть

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n$$

голоморфная функция в одинтчном круге E. Получено следующе результаты:

(i) Если

$$|zf''(z)/f'(z)| < \frac{3}{2}$$

в E, тогда а звездноообразна а

$$(2/z)\int_0^z f(t)\,dt$$

выпукла в E.

(ii) Если $\operatorname{Re} f'(z) > 0$ в E, тогда

$$\int_0^z t^{-1} f(t)\,dt$$

звездноообразна в E.