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Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

Dmitri V. PROKHOROV, Jan SZYNAL

Inverse Coefficients for (α, β)-convex Functions

Współczynniki funkcji odwrotnych do funkcji (α, β)-wypukłych

Коэффициенты обратных функций к функциям (α, β)-выпуклым

1. In this note we deal with some classes of holomorphic functions f in the unit disc $D = \{z : |z| < 1\}$ which have the form

$$f(z) = z + a_2 z^2 + \dots, \quad z \in D. \tag{1}$$

By $M(\alpha, \beta)$, $\alpha \geq 0$, $0 \leq \beta < 1$, we denote the set of functions f of the form (1) which satisfy the conditions $z^{-1} f(z) f'(z) \neq 0$, $z \in D$, and

$$\operatorname{Re} \left\{ (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \beta, \ z \in D. \tag{2}$$

The class $M(\alpha, \beta)$ is known as the class of α-convex functions of order β in the sense of Mocanu [6], [9].

We remark the obvious relations $M(0, \beta) = S^*_\beta$, $M(1, \beta) = K_\beta$, where S^*_β, K_β denote the familiar classes of starlike and convex functions of the order β respectively.

The important role within considered classes plays the so called "Koebe-type" function

$$m_{(\alpha, \beta)}(z) = z \left[\frac{1}{\alpha} \int_0^1 t^{1/\alpha - 1} (1 - tz)^{-2(1-\beta)/\alpha} dt \right]^\alpha \in M(\alpha, \beta). \tag{3}$$

*) This work was done while the first author visited the Institute of Mathematics of M. Curie-Skłodowska University, Lublin, Poland.

In this note we are concerned with the estimates for the coefficients of the inverses of functions in the class $M(\alpha,\beta)$.

We denote by

$$\hat{M}(\alpha,\beta)=\left\{F: F=f^{-1}, f\in M(\alpha,\beta)\right\}$$

where F is defined by restricting f to a sufficiently small neighbourhood of the origin.

We have $F(w)=w+A_2w^2+\ldots$ and notice the relations

$$A_2=-a_2,\ A_3=-a_3+2a_2^2,\ A_4=-a_4+5a_2a_3-5a_2^3\ . \tag{4}$$

So far there are known the estimates for the coefficients of the inverses of functions in the class S of holomorphic and univalent functions obtained by Loewner [4] and very nice and surprising result for the class Σ_0 done by Netanyahu in [7].

In both cases there exists only one extremal function namely the inverse to the classical Koebe function.

In [2] Kirwan and Schober found the exact bounds for $|A_2|$, $|A_3|$ $(k\geqslant 2)$ and for $|A_4|$ $(k\geqslant 2\frac{1}{3})$ if function f belongs to the class V_k of functions of bounded boundary rotation ($V_2=K_0$). Moreover they remarked the interesting fact for the class K_0 : $\max|A_{10}|>1$ and is not attained for the "Koebe-type" function $F(w)=w(1-w)^{-1}$.

In [3] Krzyż, Libera and Złotkiewicz determined the exact bounds for $|A_2|$, $|A_3|$ as well as the order of magnitude for $|A_n|$ if $f\in S^*_\beta$.

For further references concerning the problem of inverse coefficients we send the reader to [1, pp. 183–188].

In this note we find the precise bound for the functional

$$J_4(f)=|a_4+sa_2a_3+ua_2^3| \tag{5}$$

for arbitrary real numbers s and u within the class $M(\alpha,\beta)$.

As an application we obtain the exact estimate for $|A_2|$, $|A_3|$, $|A_4|$ if $F\in\hat{M}(\alpha,\beta)$ as well as some other results.

The main key which we use is the lemma (it has also an independent interest) concerning the sharp estimate of the functional

$$\Psi(\omega)=|c_3+\mu c_1c_2+\nu c_1^3|,\ \mu,\nu \text{ are real,} \tag{6}$$

within the class Ω of all holomorphic functions ω of the form

$$\omega(z)=c_1z+c_2z^2+c_3z^3+\ldots,\ z\in D, \tag{7}$$

and satisfying the condition $|\omega(z)|<1,\ z\in D$.

By the way it is worthwile to mention that the estimate of the functional of the fourth order (5) within the class of bounded univalent functions in D has been found by Ławrynowicz and Tammi [5].

2. We will need in the sequel the following

Lemma 1. [9] *If $f \in M(\alpha, \beta)$ then for complex number σ the following sharp estimate*

$$|a_3 - \sigma a_2^2| \leqslant \frac{1-\beta}{1+2\alpha} \max\left(1, \frac{|4\sigma(1-\beta)(1+2\alpha) - 2(1+3\alpha)(1-\beta) - (1+\alpha)^2|}{(1+\alpha)^2}\right) \tag{8}$$

holds.

Now in order to formulate the next lemma we should write down the following denotations, where μ and ν are real numbers:

$$D_1 = \left\{ (\mu, \nu):\ |\mu| \leqslant \frac{1}{2},\quad -1 \leqslant \nu \leqslant 1 \right\}$$

$$D_2 = \left\{ (\mu, \nu):\ \frac{1}{2} \leqslant |\mu| \leqslant 2,\quad \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leqslant \nu \leqslant 1 \right\}$$

$$D_3 = \left\{ (\mu, \nu):\ |\mu| \leqslant \frac{1}{2},\quad \nu \leqslant -1 \right\}$$

$$D_4 = \left\{ (\mu, \nu):\ |\mu| \geqslant \frac{1}{2},\quad \nu \leqslant -\frac{2}{3}(|\mu|+1) \right\}$$

$$D_5 = \left\{ (\mu, \nu):\ |\mu| \leqslant 2,\quad \nu \geqslant 1 \right\}$$

$$D_6 = \left\{ (\mu, \nu):\ 2 \leqslant |\mu| \leqslant 4,\quad \nu \geqslant \frac{1}{12}(\mu^2 + 8) \right\}$$

$$D_7 = \left\{ (\mu, \nu):\ |\mu| \geqslant 4,\quad \nu \geqslant \frac{2}{3}(|\mu|-1) \right\} \tag{9}$$

$$D_8 = \left\{ (\mu, \nu):\ \frac{1}{2} \leqslant |\mu| \leqslant 2,\quad -\frac{2}{3}(|\mu|+1) \leqslant \nu \leqslant \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \right\}$$

$$D_9 = \left\{ (\mu, \nu):\ |\mu| \geqslant 2,\quad -\frac{2}{3}(|\mu|+1) \leqslant \nu \leqslant \frac{2|\mu|(|\mu|+1)}{\mu^2 + 2|\mu| + 4} \right\}$$

$$D_{10} = \left\{ (\mu, \nu):\ 2 \leqslant |\mu| \leqslant 4,\quad \frac{2|\mu|(|\mu|+1)}{\mu^2 + 2|\mu| + 4} \leqslant \nu \leqslant \frac{1}{12}(\mu^2 + 2) \right\}$$

$$D_{11} = \left\{ (\mu, \nu):\ |\mu| \geqslant 4,\quad \frac{2|\mu|(|\mu|+1)}{\mu^2 + 2|\mu| + 4} \leqslant \nu \leqslant \frac{2|\mu|(|\mu|-1)}{\mu^2 - 2|\mu| + 4} \right\}$$

$$D_{12} = \left\{ (\mu; \nu):\ |\mu| \geqslant 4,\quad \frac{2|\mu|(|\mu|-1)}{\mu^2 - 2|\mu| + 4} \leqslant \nu \leqslant \frac{2}{3}(|\mu|-1) \right\}$$

Now we may state the following

Lemma 2. *If $\omega \in \Omega$, then for any real numbers μ and ν the following sharp estimate: $\Psi(\omega) \leqslant \Phi(\mu, \nu)$ holds, where*

$$\Phi(\mu,\nu)=\begin{cases} 1 & \text{if } (\mu,\nu)\in D_1\cup D_2\cup\{(2,1)\} \\ |\nu| & \text{if } (\mu,\nu)\in \bigcup_{k=3}^{7} D_k \\ \frac{2}{3}(|\mu|+1)\left(\frac{|\mu|+1}{3(|\mu|+1+\nu)}\right)^{1/2} & \text{if } (\mu,\nu)\in D_8\cup D_9 \\ \frac{1}{3}\nu\left(\frac{\mu^2-4}{\mu^2-4\nu}\right)\left(\frac{\mu^2-4}{3(\nu-1)}\right)^{1/2} & \text{if } (\mu,\nu)\in D_{10}\cup D_{11}-\{(2,1)\} \\ \frac{2}{3}(|\mu|-1)\left(\frac{|\mu|-1}{3(|\mu|-1-\nu)}\right)^{1/2} & \text{if } (\mu,\nu)\in D_{12}\,. \end{cases} \tag{10}$$

Proof. If $\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots \in \Omega$ then the function

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + 2b_1 z + 2b_2 z^2 + 2b_3 z^3 + \ldots \tag{11}$$

has a positive real part in D.

For the coefficients of p the following Carathéodory–Toeplitz inequalities

$$\begin{vmatrix} 1 & b_1 \\ \bar{b}_1 & 1 \end{vmatrix} \geqslant 0, \quad \begin{vmatrix} 1 & b_1 & b_2 \\ \bar{b}_1 & 1 & b_1 \\ \bar{b}_2 & \bar{b}_1 & 1 \end{vmatrix} \geqslant 0, \quad \begin{vmatrix} 1 & b_1 & b_2 & b_3 \\ \bar{b}_1 & 1 & b_1 & b_2 \\ \bar{b}_2 & \bar{b}_1 & 1 & b_1 \\ \bar{b}_3 & \bar{b}_2 & \bar{b}_1 & 1 \end{vmatrix} \geqslant 0, \ldots \tag{12}$$

hold (e.g. [8]).

By using (11) and (12) we obtain the following form of the first three inequalities (12) within the class Ω:

$$\begin{aligned} &|c_1| \leqslant 1 \\ &|c_2| \leqslant 1 - |c_1|^2 \\ &|c_3(1-|c_1|^2) + \bar{c}_1 c_2^2| \leqslant (1-|c_1|^2)^2 - |c_2|^2\,. \end{aligned} \tag{13}$$

Without loss of generality we may assume $c_1 \geqslant 0$ and remark that if $c_1 = 1$, then $\omega(z) = z$ and $\Psi(\omega) = |\nu|$.

In the case $\nu \leqslant 0$ the sharp estimate for (6) is given in [10], however of $\nu \geqslant 0$ in order to get the sharp estimate of (6) it is necessary to argue in a different way.

Let us observe next that if $\omega(z) \in \Omega$ then $-\omega(-z) \in \Omega$ which implies that we can restrict our considerations to the case $\mu \geqslant 0, \nu \geqslant 0$.

Now we are going to find sup $\Psi(\omega)$, where c_1, c_2, c_3 satisfy (13) and $\mu \geqslant 0, \nu \geqslant 0$, are arbitrary fixed numbers.

Let us assume that c_1 and c_2 are fixed as in (13). Then by straightforward calculations may be checked that sup $\Psi(\omega)$ in the disc

$$\left| c_3 + \frac{c_1 c_2^2}{1-c_1^2} \right| \leqslant 1 - c_1^2 - \frac{|c_2|^2}{1-c_1^2}, \quad 0 \leqslant c_1 < 1,$$

is attained on its boundary, i.e. when

$$c_3 = -\frac{c_1 c_2^2}{1-c_1^2} + \left(1 - c_1^2 - \frac{|c_2|^2}{1-c_1^2}\right) e^{i\theta}, 0 \leqslant \theta < 2\pi .$$

So for $0 \leqslant c_1 < 1$, $|c_2| \leqslant 1 - c_1^2$, we have

$$\Psi(\omega) \leqslant \left| -\frac{c_1 c_2^2}{1-c_1^2} + \left(1 - c_1^2 - \frac{|c_2|^2}{1-c_1^2}\right) e^{i\theta} + \mu c_1 c_2 + \nu c_1^3 \right| .$$

Putting $c_1 = x$, $c_2 = ye^{i\phi}$, $0 \leqslant x < 1, 0 \leqslant y \leqslant 1 - x^2, 0 \leqslant \phi < 2\pi$, we should find the supremum of the expression

$$\left| \left(1 - x^2 - \frac{y^2}{1-x^2}\right) e^{i\theta} + x \left(\nu x^2 + \mu y e^{i\phi} - \frac{y^2 e^{2i\phi}}{1-x^2}\right) \right| .$$

For fixed x, y we have

$$\left| \left(1 - x^2 - \frac{y^2}{1-x^2}\right) e^{i\theta} + x \left(\nu x^2 + \mu y e^{i\phi} - \frac{y^2 e^{2i\phi}}{1-x^2}\right) \right| \leqslant$$

$$\leqslant \left(1 - x^2 - \frac{y^2}{1-x^2}\right) + x \left| \nu x^2 + \mu y e^{i\phi} - \frac{y^2 e^{2i\phi}}{1-x^2} \right|$$

and the sign of equality in (14) holds if and only if

$$\theta = \arg \left(\nu x^2 + \mu y e^{i\phi} - \frac{y^2 e^{2i\phi}}{1-x^2} \right) .$$

We find that

$$\left|\nu x^2+\mu y e^{i\phi}-\frac{y^2 e^{2i\phi}}{1-x^2}\right|^2=\frac{2y}{1-x^2}\left[-2\nu x^2 y t^2+\mu(\nu x^2(1-x^2)-y^2)t+\nu x^2 y\right]+$$
$$+\left(\mu y^2+\nu^2 x^4+\frac{y^4}{(1-x^2)^2}\right),$$

$t=\cos\phi$.

If we denote

$$g(t)=-2\nu x^2 y t^2+\mu(\nu x^2(1-x^2)-y^2)t+\nu x^2 y$$

then we see that for fixed x and y $\max\limits_{-1\leqslant t\leqslant 1} g(t)=g(t_0)$, where:

(a) for $\mu\leqslant 4$

$$t_0=\begin{cases}-1 & \text{if } 0\leqslant x\leqslant x'',\ y_2(x)\leqslant y\leqslant 1-x^2\\ \dfrac{\mu[\nu x^2(1-x^2)-y^2]}{4\nu x^2 y} & \text{if } 0\leqslant x\leqslant x'',\ y_1(x)\leqslant y\leqslant y_2(x)\\ & \text{or } x''\leqslant x<1,\ y_1(x)\leqslant y\leqslant 1-x^2\\ 1 & \text{if } 0\leqslant x<1,\ 0\leqslant y\leqslant y_1(x)\end{cases}\qquad(15)$$

(b) for $\mu\geqslant 4$

$$t_0=\begin{cases}-1 & \text{if } 0\leqslant x\leqslant x'',\ y_2(x)\leqslant y\leqslant 1-x^2\\ \dfrac{\mu[\nu x^2(1-x^2)-y^2]}{4\nu x^2 y} & \text{if } 0\leqslant x\leqslant x'',\ y_1(x)\leqslant y\leqslant y_2(x)\\ & \text{or } x''\leqslant x\leqslant x',\ y_1(x)\leqslant y\leqslant 1-x^2\\ 1 & \text{if } 0\leqslant x\leqslant x',\ 0\leqslant y\leqslant y_1(x)\\ & \text{or } x'\leqslant x<1,\ 0\leqslant y\leqslant 1-x^2,\end{cases}\qquad(16)$$

where

$$x''=\left(\frac{\mu}{\mu\nu+\mu+4\nu}\right)^{1/2},\ x'=\left(\frac{\mu}{\mu\nu+\mu-4\nu}\right)^{1/2},$$

$$y_1(x)=\frac{x}{\mu}\left(\sqrt{4\nu^2x^2+\mu^2\nu(1-x^2)}-2\nu x\right),$$

$$y_2(x)=\frac{x}{\mu}\left(\sqrt{4\nu^2x^2+\mu^2\nu(1-x^2)}+2\nu x\right).$$

Now we are in position to determine $\sup_{x,y} h(x, y)$, where

$$h(x, y) = \left(1 - x^2 - \frac{y^2}{1-x^2}\right) + x\left|\nu x^2 + \mu y e^{i\phi} - \frac{y^2 e^{2i\phi}}{1-x^2}\right|$$

and $\cos\phi = t_0$ is given by (15) or (16) respectively. We will distinguish two cases: (A): $\nu \leqslant \frac{1}{4}\mu^2$, (B): $\nu > \frac{1}{4}\mu^2$.

Now the case (A): $\nu \leqslant \frac{1}{4}\mu^2$. We show in fact that $\sup_{x,y} h(x, y)$ will be attained for $y = 0$ or $y = 1 - x^2$. We have

1. if $y = 0$, then $h(x, 0) = 1 - x^2 + \nu x^3 \leqslant \max(1, \nu)$;
2. if $t_0 = -1$, then according to (15) and (16) we have

$$h(x, y) = h_{-1}(x, y) = 1 - x^2 + x\left|\nu x^2 - \mu y - \frac{y^2}{1-x^2}\right| - \frac{y^2}{1-x^2} = -\frac{1}{1+x} y^2 +$$

$$+ \mu x y + (1 - x^2 - \nu x^3), \text{ for } 0 \leqslant x \leqslant x'', y_2(x) \leqslant y \leqslant 1 - x^2.$$

The function h_{-1} attains maximum at $y_0(x) = \frac{1}{2}\mu x(1 + x)$ if $0 \leqslant x \leqslant \frac{2}{2+\mu}$ or at $y = 1 - x^2$ if $\frac{2}{2+\mu} \leqslant x \leqslant x''$. Moreover, we have

$$h_{-1}(x, y_0(x)) = 1 + \left(\frac{1}{4}\mu^2 - 1\right)x^2 + \left(\frac{1}{4}\mu^2 - \nu\right)x^3, \quad 0 \leqslant x \leqslant \frac{2}{2+\mu}. \tag{18}$$

From (18) we get at once that

$$\max_{0 \leqslant x \leqslant \frac{2}{2+\mu}} h_{-1}(x, y_0(x)) = \max\left[h_{-1}(0, y_0(0)), h_{-1}\left(\frac{2}{2+\mu}, y_0\left(\frac{2}{2+\mu}\right)\right)\right],$$

which implies

$$\sup_{\substack{0 \leqslant x \leqslant x'' \\ y_2(x) \leqslant y \leqslant 1-x^2}} h_{-1}(x, y) = \max\left[h_{-1}(0, 0), \sup_{0 \leqslant x \leqslant x''} h_{-1}(x, 1 - x^2)\right]; \tag{19}$$

3. if $t_0 = \dfrac{\mu[\nu x^2(1 - x^2) - y^2]}{4\nu x^2 y}$, then we have

$$h(x, y) = h_0(x, y) =$$

$$= 1 - x^2 + \frac{\nu x^2}{2} \sqrt{\frac{\mu^2 - (\mu^2 - 4\nu) x^2}{\nu}} + \frac{y^2}{2(1-x^2)} \left(\sqrt{\frac{\mu^2 - (\mu^2 - 4\nu) x^2}{\nu}} - 2 \right).$$

Because for $\nu \leqslant \frac{1}{4}\mu^2$ the inequality $\sqrt{\frac{\mu^2 - (\mu^2 - 4\nu) x^2}{\nu}} \geqslant 2$ holds, then taking into account (15) and (16) we obtain

$$\max_y h_0(x, y) = \begin{cases} h_0(x, y_2(x)) & \text{if} \quad 0 \leqslant x \leqslant x'', \mu \geqslant 0, \\ h_0(x, 1 - x^2) & \text{if} \quad x'' \leqslant x < 1, 0 \leqslant \mu \leqslant 4, \\ & \text{or} \quad x'' \leqslant x \leqslant x', \mu \geqslant 0. \end{cases} \tag{20}$$

But $h_0(x, y_2(x)) = h_{-1}(x, y_2(x))$ and this case was already discussed above (2);

4. if $t_0 = 1$, then according to (15) and (16) we have

$$h(x, y) = h_1(x, y) = 1 - x^2 - \frac{y^2}{1-x^2} + x \left| \nu x^2 + \mu y - \frac{y^2}{1-x^2} \right| =$$

$$= -\frac{1}{1+x} y^2 + \mu x y + (1 - x^2 + \nu x^3).$$

When $0 \leqslant \mu \leqslant 4$ we obtain $\max_y h_1(x, y) = h_1(x, y_1(x))$, whereas if $\mu \geqslant 4$, then

$$\max_y h_1(x, y) = \begin{cases} h_1(x, y_1(x)) & \text{if } 0 \leqslant x \leqslant x' \\ h_1(x, 1 - x^2) & \text{if } x' \leqslant x \leqslant 1. \end{cases}$$

The equality $h_1(x, y_1(x)) = h_0(x, y_1(x))$ together with (20) and (19) imply the final relation

$$\sup_{\substack{0 < x < 1 \\ 0 < y < 1 - x^2}} h(x, y) = \max \left[h(0, 0), \sup_{0 < x < 1} h(x, 1 - x^2) \right],$$

where the function h is given by the formula (17).

Now we determine (17). We have $h(0, 0) = 1$ and in order to find $\sup_{0 < x < 1} h(x, 1 - x^2)$ we should calculate the maximum of the following function ($0 \leqslant \nu \leqslant \frac{1}{4}\mu^2$):

$$h(x)=\begin{cases} h_{-1}(x)=(\mu+1)x-(\mu+1+\nu)x^3 & \text{if } 0\leqslant x\leqslant x'',\ \mu\geqslant 0\\ h_0(x)=\dfrac{1}{2}[(\nu-1)x^2+1]\sqrt{\dfrac{\mu^2-(\mu^2-4\nu)x^2}{\nu}} & \text{if } x''\leqslant x\leqslant 1 \text{ when } 0\leqslant\mu\leqslant 4 \\ & \text{or } x''\leqslant x\leqslant x' \text{ when } \mu\geqslant 4\\ h_1(x)=(\mu-1)x-(\mu-1-\nu)x^3 & \text{if } x''\leqslant x\leqslant 1,\ \mu\geqslant 4\,. \end{cases} \tag{21}$$

One can check that $h'(x)$ has at most one zero in the interval $[0, 1]$ in which h attains its maximum.

Namely:

(i) h has the maximum at the point x_{-1} in the interval $[0, x'']$ if and only if

$$x_{-1}^2=\frac{1}{3}\,\frac{\mu+1}{\mu+1+\nu}\leqslant x''^2,$$

which is equivalent to the inequality

$$\nu\leqslant\frac{2\mu(\mu+1)}{\mu^2+2\mu+4}\,.$$

(ii) h has the maximum at the point x_0 in the interval $[x'', 1]$ when $0\leqslant\mu\leqslant 4$ or in the interval $[x'', x']$ when $\mu\geqslant 4$ if and only if

$$x_0^2=\frac{3\mu^2-2(\mu^2+2)\nu}{3(\nu-1)(4\nu-\mu^2)}\in\begin{cases}[x''^2, 1] & \text{when } 0\leqslant\mu\leqslant 4\\ [x''^2, x'^2] & \text{when } \mu\geqslant 4,\end{cases}$$

($\nu\neq 1, \nu\neq\frac{1}{4}\mu^2$), which is equivalent to the inequalites $\mu\geqslant 2$ and

$$\frac{2\mu(\mu+1)}{\mu^2+2\mu+4}\leqslant\nu\leqslant\frac{1}{12}(\mu^2+8)\quad\text{when } 2\leqslant\mu\leqslant 4$$

$$\frac{2\mu(\mu+1)}{\mu^2+2\mu+4}\leqslant\nu\leqslant\frac{2\mu(\mu-1)}{\mu^2-2\mu+4}\quad\text{when } \mu\geqslant 4\,.$$

(iii) h has the maximum at the point x_1 in the interval $[x'', 1]$ when $\mu\geqslant 4$ if and only if

$$x_1^2=\frac{1}{3}\,\frac{\mu-1}{\mu-1-\nu}\in[x''^2, 1],$$

which is equivalent to the inequalities

$$\frac{2\mu(\mu-1)}{\mu^2-2\mu+4} \leqslant \nu \leqslant \frac{2}{3}(\mu-1),\ \mu \geqslant 4 .$$

(iv) h has the maximum at the point $x = 1$ if and only if $h'(1) \geqslant 0$, which is equivalent to the inequalities

$$\nu \geqslant \frac{1}{12}(\mu^2+8) \quad \text{when } 2 \leqslant \mu \leqslant 4 ,$$

$$\nu \geqslant \frac{2}{3}(\mu-1) \quad \text{when } \mu \geqslant 4 .$$

In order to finish the proof in the case $\nu \leqslant \frac{1}{4}\mu^2$ we should compare the values $h(0, 0)$, $h_{-1}(x_{-1})$, $h_0(x_0)$, and $h_1(x_1)$ in the appropriate sets, which leads to the inequality $1 \geqslant h_{-1}(x_{-1})$ which is equivalent to

$$\frac{1}{4}\mu^2 \geqslant \nu \geqslant \frac{4}{27}(\mu+1)^3-(\mu+1) .$$

Now we consider the case (B): $\nu \geqslant \frac{1}{4}\mu^2$. First of all let us observe that for the point $\mu = 2, \nu = 1$ we obtain for $x \in [0, 1]$ $h_{-1}(x_{-1}) = h_0(x) = 1$, which implies that for these values of parameters the functions $\omega(z) = z$ and $\omega(z) = z^3$ are extremal for $\Psi(\omega)$.

Further on we remark that the set G of values (μ, ν), $\nu \geqslant 0$, in the plane for which the function $\omega(z) = z$ is extremal appears to be a convex set.

Indeed if $(\mu_k, \nu_k) \in G$, $k = 1, 2$, then for any $\lambda \in [0, 1]$

$$|c_3 + (\lambda\mu_1 + (1-\lambda)\mu_2)c_1c_2 + (\lambda\nu_1 + (1-\lambda)\nu_2)c_1^3| \leqslant$$
$$\leqslant \lambda\, |c_3 + \mu_1c_1c_2 + \nu_1c_1^3| + (1-\lambda)\, |c_3 + \mu_2c_1c_2 + \nu_2c_1^3| \leqslant \lambda\nu_1 + (1-\lambda)\nu_2$$

and the sign of equality holds for the function $\omega(z) = z$.

The same property has the set H of values (μ, ν) in the plane for which the function $\omega(z) = z^3$ is extremal.

In [10] it was proved that the function $\omega(z) = z$ is extremal w.r.t. $\Psi(\omega)$ in the set

$$G_1 = \left\{(\mu, \nu): 0 \leqslant \mu \leqslant \frac{1}{2}, \nu \geqslant 1\right\} \cup \left\{(\mu, \nu): \mu \geqslant \frac{1}{2}, \nu \geqslant \frac{2}{3}(\mu+1)\right\}.$$

We have shown above that $\omega(z) = z$ is also extremal in the set

$$G_2 = \left\{(\mu, \nu): 2 \leqslant \mu \leqslant 4, \frac{1}{12}(\mu^2+8) \leqslant \nu \leqslant \frac{1}{4}\mu^2\right\} \cup \left\{(\mu, \nu): \mu \geqslant 4, \frac{2}{3}(\mu-1) \leqslant \nu \leqslant \frac{1}{4}\mu^2\right\}.$$

Taking into account the convexity of the set G we conclude that $G = \operatorname{conv}(G_1 \cup G_2)$.

In the similar way we find the set H. Namely in [10] it was proved that $\omega(z) = z^3$ is extremal w.r.t. $\Psi(\omega)$ in the set

$$H_1 = \left\{(\mu, \nu): 0 \leqslant \mu \leqslant \frac{1}{2}, 0 \leqslant \nu \leqslant \frac{1}{2}\right\} \cup$$

$$\cup \left\{(\mu, \nu): \frac{1}{2} \leqslant \mu \leqslant \frac{3\sqrt{3}}{2} - 1, 0 \leqslant \nu \leqslant (\mu + 1) - \frac{4}{27}(\mu + 1)^3\right\}.$$

On the other hand we have shown above that $\omega(z) = z^3$ is also extremal w.r.t. $\Psi(\omega)$ in the set

$$H_2 = \left\{(\mu, \nu): 0 \leqslant \mu \leqslant \frac{3\sqrt{3}}{2} - 1, 0 \leqslant \nu \leqslant \frac{1}{4}\mu^2\right\} \cup$$

$$\cup \left\{(\mu, \nu): \frac{3\sqrt{3}}{2} - 1 \leqslant \mu \leqslant 2, \frac{4}{27}(\mu + 1)^3 - (\mu + 1) \leqslant \nu \leqslant \frac{1}{4}\mu^2\right\}.$$

From the convexity of H it follows that $H = \text{conv}\,(H_1 \cup H_2)$.

Now if we take simultineously the results which we proved above in the case $\nu \geqslant 0$ with the results from [10] in the case $\nu \leqslant 0$ we obtain (10).

The extremal values of $\Psi(\omega)$ are equal to $h(0, 0)$, $h(1, 0)$, $h_{-1}(x_{-1})$, $h_0(x_0)$, $h_1(x_1)$ respectively.

The form of the extremal functions w.r.t. $\Psi(\omega)$ (up to the rotation) depends on the values (μ, ν). We have:

I. if $(\mu, \nu) \in D_1 \cup D_2 \cup \{(2, 1)\}$ then the extremal function has the form $\omega(z) = z^3$;

II. if $(\mu, \nu) \in \bigcup_{k=3}^{7} D_k \cup \{(2, 1)\}$ then the extremal function has the from $\omega(z) = z$;

III. if $(\mu, \nu) \in D_8 \cup D_9$, then the extremal function has the form

$$\omega_{-1}(z) = c_1^{(-1)} z + c_2^{(-1)} z^2 + c_3^{(-1)} z^3 + \ldots, \tag{22}$$

where

$$c_1^{(-1)} = \left\{\frac{1}{3}\,\frac{\mu + 1}{\mu + 1 + \nu}\right\}^{1/2}, c_2^{(-1)} = -(1 - c_1^{(-1)2}), c_3^{(-1)} = c_1^{(-1)} c_2^{(-1)}.$$

IV. if $(\mu, \nu) \in D_{10} \cup D_{11} - \{(2, 1)\}$ then the extremal function has the form

$$\omega_0(z) = c_1^{(0)} z + c_2^{(0)} z^2 + c_3^{(0)} z^3 + \ldots, \tag{23}$$

where

$$c_1^{(0)} = \left[\frac{3\mu^2 - 2(\mu^2 + 2)\nu}{3(\nu - 1)(4\nu - \mu^2)}\right]^{1/2}, c_2^{(0)} = (1 - c_1^{(0)2}) e^{i\phi_0}, c_3^{(0)} = -c_1^{(0)} c_2^{(0)} e^{i\phi_0},$$

and

$$\phi_0 = \pm \arccos \frac{\mu[2(\mu^2 + 2) - (\mu^2 + 8)\nu]}{2[3\mu^2 - 2(\mu^2 + 2)\nu]}.$$

V. if $(\mu, \nu) \in D_{12}$, then the extremal function has the form

$$\omega_1(z) = c_1^{(1)} z + c_2^{(1)} z^2 + c_3^{(1)} z^3 + \dots, \tag{24}$$

where

$$c_1^{(1)} = \left(\frac{1}{3}\,\frac{\mu-1}{\mu-1-\nu}\right)^{1/2},\ c_2^{(1)} = (1 - c_1^{(1)2}),\ c_3^{(1)} = -c_1^{(1)} c_2^{(1)}.$$

Remark 1. The explicit formulae for extremal functions (22)–(24) may be found from the relation (11) where p is the function with positive real part in D and has the form

$$p(z) = \lambda\,\frac{1+\epsilon_1 z}{1-\epsilon_1 z} + (1-\lambda)\frac{1+\epsilon_2 z}{1-\epsilon_2 z},\ 0 \leqslant \lambda \leqslant 1,\ |\epsilon_1| = |\epsilon_2| = 1,$$

with

$$\lambda = \frac{(c_1-\epsilon_2)^2}{(\epsilon_2-c_1)^2 + c_2},\ \epsilon_1 = \frac{c_1\epsilon_2 - c_2 - c_1^2}{\epsilon_2 - c_1},\ \arg((\epsilon_2-c_1)^2 + c_2) = 2\arg(c_1-\epsilon_2),$$

where c_1, c_2 are given by (22)–(24) respectively.

Remark 2. Obviously an analogous result like Lemma 2 is also true for the Carathéodory class of functions with positive real part.

3. Now we are in a position to prove

Theorem 1. *If $f \in M(\alpha, \beta)$ then for any real numbers s, u the following sharp estimate*

$$|a_4 + s a_2 a_3 + u a_2^3| \leqslant \frac{2(1-\beta)}{3(1+3\alpha)}\,\Phi(\mu,\nu) \tag{25}$$

holds. The function Φ is given by (10) *with*

$$\mu = 2 + 3\frac{(1+3\alpha)s + (1+5\alpha)}{(1+\alpha)(1+2\alpha)}(1-\beta) \tag{26}$$

$$\nu = 1 + \frac{3(1+3\alpha)\{4(1+2\alpha)(1-\beta)u + [2(1+3\alpha)(1-\beta) + (1+\alpha)^2]s\}(1-\beta)}{(1+\alpha)^3(1+2\alpha)} + \tag{27}$$

$$+\frac{2(17\alpha^2+6\alpha+1)(1-\beta)^2 + 3(1+\alpha)^2(1+5\alpha)(1-\beta)}{(1+\alpha)^3(1+2\alpha)}.$$

Proof. Let $f \in M(\alpha, \beta)$. In order to get (25) we should find the connection between the coefficients of functions from the classes $M(\alpha,\beta)$ and Ω. From the definition (2) we have the equality

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=\frac{1+(1-2\beta)\,\omega(z)}{1-\omega(z)}, \tag{28}$$

where ω is an arbitrary function from Ω with the expansion (7).

Comparing the coefficients in (28) we obtain the relations

$$c_1=\frac{1+\alpha}{2(1-\beta)}\,a_2,$$

$$c_2=\frac{1+2\alpha}{1-\beta}\,a_3-\frac{[(1+\alpha)^2+2(1+3\alpha)(1-\beta)]}{4(1-\beta)^2}\,a_2^2,$$

$$c_3=\frac{3}{2}\,\frac{1+3\alpha}{1-\beta}\,a_4-\frac{1}{2(1-\beta)^2}\,[3(1+5\alpha)(1-\beta)+2(1+2\alpha)(1+\alpha)]\,a_2a_3+$$

$$+\frac{1}{8(1-\beta)^3}\,[4(1+7\alpha)(1-\beta)^2+4(1+\alpha)(1+3\alpha)(1-\beta)+(1+\alpha)^3]\,a_2^3.$$

Now if we apply Lemma 2, then we obtain (25) with μ and ν as in (26) and (27). From Lemma 1 and Theorem 1 follows

Corollary 1. *If $f\in M(\alpha,\beta)$ then the following sharp estimates*

$$|a_2|\leqslant\frac{2(1-\beta)}{1+\alpha},$$

$$|a_3|\leqslant\frac{(1-\beta)\,[\alpha^2+(8-6\beta)\,\alpha+3-2\beta]}{(1+\alpha)^2\,(1+2\alpha)},$$

$$|a_4|\leqslant$$

$$\leqslant\frac{4(1-\beta)[2(17\alpha^2+6\alpha+1)(1-\beta)^2+3(1+\alpha)^2(1+5\alpha)(1-\beta)+(1+\alpha)^3(1+2\alpha)]}{3(1+\alpha)^2(1+2\alpha)(1+3\alpha)}$$

hold. The extremal function in all three cases is the "Koebe-type" function (3).

Let now introduce some denotations

$$A=A(\alpha)=\frac{(31\alpha^2+33\alpha+8)(1+2\alpha)}{9(2+5\alpha)^2(1+\alpha)}; \tag{29}$$

$$B=B(\alpha)=\frac{2+5\alpha}{(1+\alpha)(1+2\alpha)}; \tag{30}$$

$$\beta_0(\alpha)=\frac{-\alpha^2+3\alpha+2}{3+5\alpha},\ \beta_0=\beta_0(0)=\frac{2}{3};$$

$$\beta_1(\alpha)=\frac{3B(12A-1)-4}{3B(12A-1)},\ \beta_1=\beta_1(0)=\frac{3}{5};$$

$$\beta_2(\alpha)=\frac{12AB-(2A+1)-\sqrt{1-12A^2}}{12AB},\ \beta_2=\beta_2(0)=\frac{35-\sqrt{33}}{43}=0.609\ldots;$$

$\beta_3(\alpha)$ is the unique root in the interval (0, 1) of the equation $(6B(1-\tau)-1)^3 = 216AB^2(1-\tau)^2$; $\beta_3=\beta_3(0)$ is the unique root in the interval (0, 1) of the equation

$$1728\tau^3-4656\tau^2+4164\tau-1235=0,\ 0.77<\beta_3<0.78;$$

$$\widehat{\beta}(\alpha)=1-\frac{3(1+\alpha)^2(2+5\alpha)}{2(31\alpha^2+33\alpha+8)};\tag{31}$$

α' is the unique root of the equation

$$23\tau^3-17\tau^2-20\tau-4=0,\ 1.4<\alpha'<1.5;\tag{32}$$

α'' is the unique root of the equation

$$15\tau^3-26\tau^2-39\tau-10=0,\ 2.7<\alpha''<2.8.\tag{33}$$

Theorem 2. *If $F\in\widehat{M}(\alpha,\beta)$, then the following sharp estimates*

$$|A_2|\leqslant\frac{2(1-\beta)}{1+\alpha}\tag{34}$$

$$|A_3|\leqslant\begin{cases}\dfrac{1-\beta}{1+2\alpha}\left[\dfrac{2(3+5\alpha)(1-\beta)}{(1+\alpha)^2}-1\right] & \text{if } \alpha\in[0,\alpha_0],\ \beta\in[0,\beta_0(\alpha)]\\[2ex] \dfrac{1-\beta}{1+2\alpha} & \text{if } \alpha\in[\alpha_0,+\infty),\ \beta\in[\beta_0(\alpha),1)\end{cases}\tag{35}$$

$$|A_4| \leqslant \begin{cases} \dfrac{2(1-\beta)}{3(1+3\alpha)}\nu & \text{if } \alpha \in [0,\alpha'], \beta \in [0,\beta_1(\alpha)] \\ & \text{or } \alpha \in [\alpha',\alpha''], \beta \in [0,\beta(\alpha)] \\ \dfrac{2(1-\beta)(\mu^2-4)}{9(1+3\alpha)(\mu^2-4\nu)}\left[\dfrac{\mu^2-4}{3(\nu-1)}\right]^{1/2}\cdot\nu & \text{if } \alpha \in [0,\alpha'], \beta \in [\beta_1(\alpha),\beta_2(\alpha)] \\ \dfrac{4(1-\beta)}{9(1+3\alpha)}(1-\mu)\left[\dfrac{1-\mu}{3(1-\mu+\nu)}\right]^{1/2} & \text{if } \alpha \in [0,\alpha'], \beta \in [\beta_2(\alpha),\beta_3(\alpha)] \\ \dfrac{2(1-\beta)}{3(1+3\alpha)} & \text{if } \alpha \in [0,\alpha'], \beta \in [\beta_3(\alpha),1) \\ & \text{or } \alpha \in [\alpha',\alpha''], \beta \in [\beta(\alpha),1) \\ & \text{or } \alpha \in [\alpha'',+\infty), \beta \in [0,1) \end{cases} \tag{36}$$

hold.

Proof. If $F \in \widehat{M}(\alpha, \beta)$, then the relations (4) hold. The inequality (34) follows from Corollary 1 and the inequality (35) follows from (8) if we put in it $\sigma = 2$. In order to get (36) we put $s = -5$, $u = 5$ in Theorem 1. We obtain

$$\mu = 2 - 6B(1-\beta),\ \nu = \mu - 1 + A(\mu-2)^2, \tag{37}$$

where A and B are given by (29), (30).

For fixed α the equation

$$\nu = A(\mu-2)^2 + \mu - 1 \tag{38}$$

is the equation of a parabola in the (μ, ν) plane. It may be checked that $A'_\alpha > 0$ for every $\alpha \geqslant 0$. Taking into account that $2 - 6B \leqslant \mu \leqslant 2$ we obtain the arc of parabola (38) which, according to the Lemma 2, may intersect the curves

$$\nu = \frac{1}{12}(\mu^2+8),\ \nu = \frac{2\mu(\mu-1)}{\mu^2-2\mu+4},\ \nu = \frac{4}{27}(1-\mu)^3 - (1-\mu), \nu = 1.$$

We observe that all these curves have the common end point $\mu = -2$, $\nu = 1$, which lies on the arc of parabola (38) with $A = A(\alpha')$, α' is the unique positive root of the equation (32).

Now we find the values of parameters α and β for which the function (3) will be extremal. It corresponds to the fact that in the class Ω the extremal function w.r.t. $\Psi(\omega)$ is the function $\omega(z) = z$. This will be equivalent to the inequalities

$$A(\mu-2)^2 + \mu - 1 \geqslant \frac{1}{12}(\mu^2+8),\ 2 - 6B \leqslant \mu \leqslant -2,$$

or

$$A(\mu-2)^2 + \mu - 1 \geqslant 1, -2 \leqslant \mu \leqslant 2.$$

According to (37) these conditions are equivalent to the inequalities $\beta \leqslant \beta_1(\alpha)$, $0 \leqslant \alpha \leqslant \alpha'$, or $\beta \leqslant \hat{\beta}(\alpha)$, $\alpha \geqslant \alpha'$. The last condition has sense if $\hat{\beta}(\alpha) \geqslant 0$ and leads to the inequality $15\alpha^3 - 26\alpha^2 - 39\alpha - 10 \leqslant 0$, satisfied for $\alpha \in [\alpha', \alpha'']$.

Now we find the values of α and β for which the are of parabola (38) lies between the curves

$$\nu = \frac{1}{12}(\mu^2 + 8) \text{ and } \nu = \frac{-2\mu(-\mu+1)}{\mu^2 - 2\mu + 4}, \quad -4 \leqslant \mu \leqslant -2.$$

According to (37) it occurs if $\beta_1(\alpha) \leqslant \beta \leqslant \beta_2(\alpha)$, $0 \leqslant \alpha \leqslant \alpha'$.

In the same way, according to (37) we check that the arc of parabola (38) lies between the curves

$$\nu = \frac{-2\mu(-\mu+1)}{\mu^2 - 2\mu + 4} \text{ and } \nu = \frac{4}{27}(1-\mu)^3 - (1-\mu)$$

if $\beta_2(\alpha) \leqslant \beta \leqslant \beta_3(\alpha)$, $0 \leqslant \alpha \leqslant \alpha'$.

Taking into account all facts mentioned above we see that:

(a) if $\alpha \in [0, \alpha']$, $\beta \in [0, \beta_1(\alpha)]$ or $\alpha \in [\alpha', \alpha'']$, $\beta \in [0, \hat{\beta}(\alpha)]$ then the extremal function w.r.t. $\Psi(\omega)$ is $\omega(z) = z$;

(b) if $\alpha \in [0, \alpha']$, $\beta \in [\beta_1(\alpha), \beta_2(\alpha)]$ then the extremal function w.r.t. $\Psi(\omega)$ is $\omega(z) = \omega_0(z)$ given by (23) with the substitution $-\mu$ instead of μ;

(c) if $\alpha \in [0, \alpha']$, $\beta \in [\beta_2(\alpha), \beta_3(\alpha)]$ then the extremal function w.r.t. $\Psi(\omega)$ is $\omega(z) = \omega_{-1}(z)$ given by (22) with the substitution $-\mu$ instead of μ;

(d) if $\alpha \in [0, \alpha']$, $\beta \in [\beta_3(\alpha), 1)$ or $\alpha \in [\alpha', \alpha'']$, $\beta \in [\hat{\beta}(\alpha), 1)$ or $\alpha \in [\alpha'', +\infty)$, $\beta \in [0, 1)$ then the extremal function is $\omega(z) = z^3$.

The function f corresponding to $F = f^{-1}$ for which $|A_4|$ is maximal in (36) may be found from the relation (28). In this way the proof of Theorem 2 is complete.

As corollaries we have:

Theorem 2'. *If $F \in \widehat{M}(\alpha, 0)$ then the following sharp estimates*

$$|A_2| \leqslant \frac{2}{1+\alpha}$$

$$|A_3| \leqslant \begin{cases} \dfrac{-\alpha^2 + 8\alpha + 5}{(1+2\alpha)(1+\alpha)^2} & \text{if } \alpha \in \left[0, \dfrac{3+\sqrt{17}}{2}\right] \\ \dfrac{1}{1+2\alpha} & \text{if } \alpha \in \left[\dfrac{3+\sqrt{17}}{2}, +\infty\right) \end{cases}$$

$$|A_4| \leqslant \begin{cases} \dfrac{2}{3(1+3\alpha)}\nu_\alpha & \text{if } \alpha \in [0, \alpha''] \\ \dfrac{2}{3(1+\alpha)} & \text{if } \alpha \in [\alpha'', +\infty) \end{cases}$$

hold where α'' *is given by* (33) *and* v_α *is given by* (27) *with* $\beta = 0$, $u = +5$, $s = -5$. *The extremal functions have the form:* $m_{(\alpha,0)}(z), [m_{(\alpha,0)}(z^2)]^{1/2}, [m_{(\alpha,0)}(z^3)]^{1/3}$.

Theorem 2″ *If* $F \in \hat{M}(0,\beta) = \hat{S}^*_\beta$, *then the following sharp estimates hold*

$$|A_2| \leqslant 2(1-\beta)$$

$$|A_3| \leqslant \begin{cases} (1-\beta)(5-6\beta) & \text{if } \beta \in [0, \frac{2}{3}] \\ (1-\beta) & \text{if } \beta \in [\frac{2}{3}, 1) \end{cases}$$

$$|A_4| \leqslant \begin{cases} \frac{2}{3}(1-\beta)(3-4\beta)(7-8\beta) & \text{if } \beta \in [0, \frac{3}{5}] \\ \frac{4}{3}(2-3\beta)(3-4\beta)(7-8\beta)\left(\frac{2-3\beta}{5-8\beta}\right)^{1/2} & \text{if } \beta \in \left[\frac{3}{5}, \frac{35-\sqrt{33}}{48}\right] \\ \frac{2}{3}\left(\frac{11-12\beta}{6}\right)^{3/2} & \text{if } \beta \in \left[\frac{35-\sqrt{33}}{48}, \beta_3\right] \\ \frac{2}{3}(1-\beta) & \text{if } \beta \in [\beta_3, 1) \end{cases}$$

where β_3 *is the unique root in the interval* (0, 1) *of the equation* (31).

Proof. We have for $\alpha = 0$: $\mu = 12\beta - 10$ and $\nu = 32\beta^2 - 52\beta + 21$ in Theorem 2 and hence the result follows immediately.

Corollary 2. *If* $F \in \hat{M}(1,0) = \hat{K}_0$, *then* $|A_k| \leqslant 1$, $k = 2, 3, 4$, *and the result is sharp.* In the case $k = 4$ this result improves the result of Kirwan and Schober [2].

REFERENCES

[1] Aleksandrov, I. A., *Parametric Extensions in the Theory of Univalent Functions* (Russian), Izd. "Nauka", Moscow, 1976.

[2] Kirwan, W. E., Schober, G., *Inverse Coefficients for Functions of Bounded Boundary Rotation*, Tech. Report, University of Maryland, August 1979, 79–42.

[3] Krzyż, J. G., Libera, R. J., Złotkiewicz, E. J., *Coefficients of Inverses of Regular Starlike Functions*, Ann. Univ. M. Curie-Skłodowska, Sect. A, 33 (1979), 103–110.

[4] Loewner, K., *Untersuchungen uber schlichte konforme Abbildungen des Einheitskreises I*, Math. Ann., 89 (1923), 103–121.

[5] Ławrynowicz, J., Tammi, O., *On Estimating of a Fourth Order Functional for Bounded Univalent Functions*, Ann. Acad. Sci. Fenn., Ser. A I, 490 (1971), 18 pp.

[6] Mocanu, P. T., *Une propriété de convexité généralisée dans la theorie de la representation conforme*, Mathematica (Cluj), 11, 34 (1969), 127–133.

[7] Netanyahu, E., *Extremal Problems for Schlicht Functions in the Exterior of the Unit Circle*, Canad, J. Math., 17 (1965), 335–341.
[8] Pommerenke, Ch., *Univalent Functions*, Vanderhoeck and Ruprecht, Gottingen, 1975.
[9] Szynal, J., *Some Remarks on Coefficients Inequality for α-convex Functions*, Bull. Acad. Sci., Sér. Math., Astronim. Phys., 20, 11 (1972), 917–919.
[10] Szynal, J., Wajler, S., *On the Fourth Coefficient for α-convex Functions*, Rev. Roumane Math. Poures Appl., 19, 9 (1974), 1153–1157.

STRESZCZENIE

Praca zawiera dokładne oszacowanie funkcjonału

$$|a_4 + sa_2a_3 + ua_2^3|, \; s, u \in R$$

dla funkcji holomorficznych

$$f(z) = z + a_2z^2 + \dots, \; |z| < 1,$$

spełniających warunek

$$z^{-1}f(z)f'(z) \neq 0, \; |z| < 1 \quad \text{i}$$

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta, \; |z| < 1, \alpha \geq 0, \; 0 \leq \beta < 1. \qquad (*)$$

Jako zastosowanie otrzymano dokładne oszacowanie dla

$$|a_k|, \; |A_k|, \; k = 2, 3, 4 \; (F = f^{-1}, \; F(w) = w + A_2w^2 + \dots)$$

dla funkcji f, spełniających warunek (*).

Podstawową nierównością (mającą również inne zastosowanie) pozwalającą otrzymać wynik jest dokładne oszacowanie dla funkcjonału

$$|c_3 + pc_1c_2 + qc_1^3|, \; p, q \in R$$

w klasie funkcji holomorficznych

$$\omega(z) = c_1z + \dots, \; |\omega(z)| < 1, \; |z| < 1.$$

РЕЗЮМЕ

В работе подана точная оценка функционала

$$|a_4 + sa_2a_3 + ua_2^3|, \; s, u \in R$$

для функций голоморфных

$$f(z) = z + a_2z^2 + \dots, \; |z| < 1,$$

удовлетворяющих условию

$$z^{-1} f(z) f'(z) \neq 0, \ |z| < 1 \quad \text{и}$$

$$\operatorname{Re}\left\{(1-\alpha)\ \frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} < \beta, \ |z| < 1, \alpha > 0, \ 0 < \beta < 1. \qquad (*)$$

Как применение получены точные оценки для

$$|a_k|, \ |A_k|, \ k = 2, 3, 4 \ (F = f^{-1}, F(w) = w + A_2 w^2 + \dots)$$

для функций f удовлетворяющих условию (*).

Основное неравенство (имеет также и другие применения), из которого вытекает резуль-таь работы есть точная оценка функционала

$$|c_3 + pc_1c_2 + qc_1^3|, \ p, q \in R$$

в классе голоморфных функций

$$\omega(z) = c_1 z + \dots, \ |\omega(z)| < 1, \ |z| < 1.$$