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Some Properties of Inflated Modified Power Series Distributions

Pewne własności zmodyfikowanych rozkładów szeregowo-potęgowych ze zniekształceniem

Некоторые свойства модефицированных распределений степенных рядов с искажением

1. Introduction. The interest concerning the inflated power series distributions has increased lately. These distributions include a vast class of discrete inflated distributions which constitute adequate descriptions of some natural phenomena.

Gupta [2] has defined the so-called modified power series distribution from which the power series distribution follows as a special case. It seems to be quite natural to introduce a more general distribution, namely an inflated modified power series distribution. That generalization gives a possibility of an investigation of a quite large class of discrete distributions; in particular, the inflated generalized negative binomial distributions. The aim of this note is to give the distribution of the sum of random variables having inflated modified power series distributions. The result which has been obtained in [6] is a particular case of our results.

Let $\mathbf{N}$ be the set of positive integers and let $T_1 = \{x_0, x_1, x_2, \ldots\}$ be a given subset of $\mathbf{N} \cup \{0\}$.

The random variable X is said to have the inflated modified power series distribution with parameters α, λ, if

$$P[X=x] = \begin{cases} 1-\lambda+\lambda \dfrac{a(x)[h(\alpha)]^x}{f(\alpha)} & \text{for } x = x_0, \\[2ex] \lambda \dfrac{a(x)[h(\alpha)]^x}{f(\alpha)} & \text{for } x \in T_1 \text{ and } x \neq x_0, \end{cases} \tag{1}$$

where $0 < \lambda \leqslant 1$, $f(\alpha) = \sum_{x \in \mathbf{N} \cup \{0\}} a(x)[h(\alpha)]^x$, $a(x) > 0$ for $x \in T_1$, $a(x) = 0$ for $x \in \mathbf{N} \cup \{0\} \setminus T_1$.

2. Distribution of the sum. It is well known that the sum of independent and *identically distributed random* variables having the power series distribution is also power series distributed. In the case of the sum of independent random variables having inflated modified power series distribution the situation is different.

For $m > 1$, define

$$T_m = \left\{x \in \mathrm{N} : x = x_{i_1} + x_{i_2} + \ldots + x_{i_m},\ x_{i_j} \in T_1,\ j = 1, 2, \ldots, m\right\}$$

Theorem 1. *If $X_1, X_2, \ldots, X_m$ are independent random variables having the same inflated modified power series distribution* (1), *and if $Z = X_1 + X_2 + \ldots + X_m$, then*

$$P[Z = z] = \begin{cases} \sum\limits_{i=0}^{m} \binom{m}{i} \gamma^{m-i} \lambda^i \dfrac{a_i(ix_0)\,[h(\alpha)]^{ix_0}}{f_i(\alpha)}, & z = mx_0, \\ \sum\limits_{i=1}^{m} \binom{m}{i} \gamma^{m-i} \lambda^i \dfrac{a_i(z-(m-i)x_0)\,[h(\alpha)]^{z-(m-i)x_0}}{f_i(\alpha)} & \end{cases} \tag{2}$$

$z \in T_m$ and $z \neq mx_0$,

where $\lambda + \gamma = 1$, $a_0(0) = 1$, $f_0(0) = 1$, $f_i(\alpha) = f^i(\alpha)$ and $a_i(x)$ is the coefficient of $[h(\alpha)]^x$ in the expansion of $f_i(\alpha)$.

Proof. We apply the mathematical induction. If $m = 2$, then for $z = 2x_0$ we have

$$P[Z = 2x_0] = P[X_1 = x_0]\,P[X_2 = x_0] = \sum_{i=0}^{2} \binom{2}{i} \gamma^{2-i} \lambda^i \frac{a_i(ix_0)[h(\alpha)]^{ix_0}}{f_i(\alpha)},$$

for $z \in T_2$ and $z \neq 2x_0$,

$$P[Z = z] = \sum_{\{x \in T_1 \,:\, z - x \in T_1\}} P[X_1 = x]\,P[X_2 = z - x] =$$

$$= \sum_{x \in \langle x_0,\, z - x_0\rangle} P[X_1 = x]\,P[X_2 = z - x] =$$

$$= \sum_{i=1}^{2} \binom{2}{i} \gamma^{2-i} \lambda^i \frac{a_i(z-(2-i)x_0)\,[h(\alpha)]^{z-(2-i)x_0}}{f_i(\alpha)}$$

what proves (2) in the considered case.

Suppose that (2) holds for $m \geqslant 2$. Then for $z = (m+1)x_0$,

$$P[Z = (m+1)x_0] = P[X_{m+1} = x_0]\,P[X_1 + X_2 + \ldots + X_m = mx_0] =$$

$$= [\gamma + \lambda \frac{a(x_0)[h(\alpha)]^{x_0}}{f(\alpha)}] \sum_{i=0}^{m} \binom{m}{i} \gamma^{m-i} \lambda^i \frac{a_i(ix_0)[h(\alpha)]^{ix_0}}{f_i(\alpha)} =$$

$$= \binom{m+1}{0} \gamma^{m+1} + \sum_{i=1}^{m} [\binom{m}{i-1} + \binom{m}{i}] \gamma^{m+1-i} \lambda^i \frac{a_i(ix_0)[h(\alpha)]^{ix_0}}{f_i(\alpha)} +$$

$$+ \binom{m+1}{m+1} \lambda^{m+1} \frac{a_{m+1}((m+1)x_0)[h(\alpha)]^{(m+1)x_0}}{f_{m+1}(\alpha)} =$$

$$= \sum_{i=0}^{m+1} \binom{m+1}{i} \gamma^{m+1-i} \lambda^i \frac{a_i(ix_0)[h(\alpha)]^{ix_0}}{f_i(\alpha)};$$

for $z \in T_{m+1}$ and $z \neq (m+1)x_0$,

$$P[Z=z] = \sum_{\{x \in T_1 : z-x \in T_m\}} P[X_{m+1}=x] P[X_1 + X_2 + \dots + X_m = z-x] =$$

$$= \sum_{x \in \langle x_0, z-mx_0 \rangle} P[X_{m+1}=x] P[X_1 + X_2 + \dots + X_m = z-x] =$$

$$= \sum_{i=1}^{m} \binom{m}{i} \gamma^{m+1-i} \lambda^i \frac{a_i(z-(m+1-i)x_0)[h(\alpha)]^{z-(m+1-i)x_0}}{f_i(\alpha)} +$$

$$+ \gamma^m \lambda \frac{a_1(z-mx_0)[h(\alpha)]^{z-mx_0}}{f(\alpha)} +$$

$$+ \sum_{i=1}^{m} \binom{m}{i} \gamma^{m-i} \lambda^{i+1} \frac{[h(\alpha)]^{z-(m-i)x_0}}{f_{i+1}(\alpha)} \sum_{x \in \langle x_0, z-mx_0 \rangle} a_1(x) a_i(z-(m-i)x_0 - x) =$$

$$= \sum_{i=1}^{m+1} \binom{m+1}{i} \gamma^{m+1-i} \lambda^i \frac{a_i(z-(m+1-i)x_0)[h(\alpha)]^{z-(m+1-i)x_0}}{f_i(\alpha)}.$$

Thus (2) is proved.

In the particular case when $f(\alpha) = (1-\alpha)^{-n}$, $h(\alpha) = \alpha(1-\alpha)^{\beta-1}$, $T_1 = N \cup \{0\}$, we have the so-called inflated generalized negative binomial distribution

$$P[X=x] = \begin{cases} 1 - \lambda + \lambda(1-\alpha)^n \text{ for } x = 0, \\ \lambda \frac{n}{n+\beta x} \binom{n+\beta x}{x} \alpha^x (1-\alpha)^{n+\beta x - x} \text{ for } x = 1, 2, \dots \end{cases} \tag{3}$$

where $0<\alpha<1, \beta\in\langle 1,+\infty)\cup\{0\}$, $|\alpha\beta|<1$ and $n>0$.

If $\lambda=1$, then we have the generalized negative binomial distribution, introduced in [3], [1], i.e.

$$P[X=x]=\frac{n}{n+\beta x}\binom{n+\beta x}{x}\alpha^x(1-\alpha)^{n+\beta x-x},\ x=0,1,2,\ldots \tag{4}$$

where $0<\alpha<1, \beta\in\langle 1,+\infty)\cup\{0\}$, $|\alpha\beta|<1$ and $n>0$.

It is known that the classical negative binomial distribution with parameters α and n is a special case of the generalized distribution (4) with parameters α, n, and β and is obtained for $\beta=1$. The binomial distribution is also a particular case of the generalized negative binomial distribution and is obtained when $\beta=0$. It can be seen that (3) includes the inflated binomial distribution and the inflated negative binomial distribution [7], [5].

Corollary 1. *If $Y=X_1+X_2+\ldots+X_m$ where X_i, $i=1,2,\ldots,m$, are independent random variables having the same distribution* (3), *then according to Theorem 1 and applying Lagrange's formula* [4] *we have*

$$P[Y=y]=\begin{cases}[1-\lambda+\lambda(1-\alpha)^n]^m & \text{for } y=0,\\ \sum\limits_{i=1}^{m}\frac{in}{in+\beta y}\binom{m}{i}\binom{in+\beta y}{y}\gamma^{m-i}\lambda^i(1-\alpha)^{ni+(\beta-1)y} & \text{for } y=1,2,\ldots\end{cases} \tag{5}$$

For $\beta=0$ the distribution given by (5) is identically the same as that of (2) in [5], i. e.

$$P[Y=y]=\begin{cases}[1-\lambda+\lambda(1-\alpha)^n]^m & \text{for } y=0,\\ \sum\limits_{i=1}^{m}\binom{m}{i}\binom{in}{y}\gamma^{m-i}\lambda^i\alpha^y(1-\alpha)^{ni-y} & \text{for } y=1,2,\ldots\end{cases}$$

Now, we are going to discuss the distribution of the sum of sums of generalized inflated negative binomial variates, which are truncated at the point 0.

Theorem 2. *Let $Z_1, Z_2, \ldots, Z_l$ be independent and identically distributed random variables having probability function*

$$P[Z=z]=\{1-[\gamma+\lambda(1-\alpha)^n]^m\}^{-1}\sum_{i=1}^{m}\binom{m}{i}\binom{in+\beta x}{z}\frac{in}{in+\beta z}\cdot$$
$$\cdot\lambda^i\gamma^{m-i}\alpha^z(1-\alpha)^{ni+(\beta-1)z},\ \text{for } z=1,2,\ldots, \tag{6}$$

where $0<\lambda\leqslant 1$, $\lambda+\beta=1$, $0<\alpha<1$, $\beta\in\langle 1,+\infty)\cup\{0\}$, $|\alpha\beta|<1$ and $n>0$.

If $Y=Z_1+Z_2+\ldots+Z_l$, then the probability function of the random variable Y is given by

$$P[Y=y]=\{1-[\gamma+\lambda(1-\alpha)^n]^m\}^{-l}\sum_{r=0}^{l}\sum_{s=0}^{rm}(-1)^{l-r}\binom{l}{r}\binom{mr}{s}\binom{ns+\beta y}{y}\cdot$$

$$\cdot \frac{ns}{ns+\beta y} \lambda^s \gamma^{mr-s} [1-\lambda(1-\alpha)^n]^{ml-r} \alpha^y (1-\alpha)^{ns+\beta y-y},$$

for $y = l+1, l+2, \ldots$ *and* 0 *otherwise.*

Proof. Let $T_Z(x)$ be a generalized probability generating function, i.e.

$$T_Z(x) = \sum_z P[Z=z][f(x)]^z .$$

Of course, the generalized probability generating function of the sum of independent and identically distributed random variables is equal to the product of generalized probability generating functions of respective random variables. Putting $f(\alpha, x) = x\left(\frac{1-\alpha x}{1-\alpha}\right)^{\beta-1}$, where $0 < \alpha < 1$ and $|\alpha\beta| < 1$, we will compute the generalized probability generating function of the distribution (6).

Namely, we have

$$T_Z(x) = \left\{1-[\gamma+\lambda(1-\alpha)^n]^m\right\}^{-1} \sum_{i=1}^{m} \binom{m}{i} [\lambda(1-\alpha)^n]^i \gamma^{n-i} \cdot$$

$$\cdot \sum_{z=1}^{\infty} \frac{in}{in+\beta z} \binom{in+\beta z}{z} \left(\frac{\alpha x}{1-\alpha x}\right)^z \left(1+\frac{\alpha x}{1-\alpha x}\right)^{-\beta z}.$$

Using the expression (2.2) of [3], i.e.

$$(1+z)^n = \sum_{x=1}^{\infty} \frac{n}{n+x} \binom{n+x}{x} z^x (1+z)^{-\beta x}, \quad \left|\frac{\beta z}{1+z}\right| < 1 \tag{8}$$

we get

$$T_Z(x) = \frac{\left[\gamma+\lambda \frac{1-\alpha}{1-\alpha x}\right]^{nm} - [\gamma+\lambda(1-\alpha)^n]^m}{1-[\gamma+\lambda(1-\alpha)^n]^m}.$$

Now, we will find the generalized probability generating function of Y.

$$T_Y(x) = \frac{\left\{\left[\gamma+\lambda \frac{1-\alpha}{1-\alpha x}\right]^{nm} - [\gamma+\lambda(1-\alpha)^n]^m\right\}^l}{\left\{1-[\gamma+\lambda(1-\alpha)^n]^m\right\}^l},$$

which can be written as follows:

$$T_Y(x) = \left\{1-[\gamma+\lambda(1-\alpha)^n]^m\right\}^{-l} \sum_{r=0}^{l} (-1)^{l-r} \binom{l}{r} \binom{mr}{s} \lambda^s \gamma^{mr-s} \cdot$$

$$\cdot [\gamma + \lambda (1-\alpha)^n]^{m (l-r)} \left(\frac{1-\alpha}{1-\alpha x} \right)^{ns}.$$

In order to finish the proof of the theorem we expend the function $\left(\frac{1}{1-\alpha x} \right)^{ns}$ according to Lagrange's formula [4].

Putting in (8) $z = \frac{\alpha x}{1-\alpha x}$, we get

$$\frac{1}{(1-\alpha x)^{ns}} = \sum_{y=0}^{\infty} \frac{ns}{ns+\beta y} \binom{ns+\beta y}{y} \alpha^y (1-\alpha)^{(\beta-1)y} \left[x \left(\frac{1-\alpha x}{1-\alpha} \right)^{\beta-1} \right]^y$$

Equating coefficents of $\left[x \left(\frac{1-\alpha x}{1-\alpha} \right)^{\beta-1} \right]^y$ we complete the proof. For $\beta = 0$ the distribution given by (7) is identically the same as that of (3) in [5].

3. The formula for a reccurence relation between k-th moments of the inflated modified power series distributions.

Theorem 3. *If X is a random variable with the distribution* (1), *then the central moments of X satisfied the relation: $M_0 = 1, M_1 = 0$, and for $k \geqslant 2$*

$$M_{k+1} = \frac{h(\alpha)}{h'(\alpha)} \left[\frac{dM_k}{d\alpha} + k \frac{dEX}{d\alpha} M_{k-1} \right] + \frac{\gamma}{\lambda} (x_0 - EX) M_k + \frac{\gamma}{\lambda} (x_0 - EX)^{k+1} \tag{9}$$

where $\lambda + \gamma = 1$ and $EX = \gamma x_0 + \lambda \frac{h(\alpha) f'(\alpha)}{h'(\alpha) f(\alpha)}$.

Proof. For $k \geqslant 2$ we have

$$M_k = \gamma (x_0 - EX)^k + \lambda \sum_{x \in T_1} (x - EX)^k \frac{a(x) [h(\alpha)]^x}{f(\alpha)}.$$

Differentiating it with respect to α, we obtain

$$\frac{dM_k}{d\alpha} = -k \frac{dEX}{d\alpha} \left[\gamma (x_0 - EX)^{k-1} + \lambda \sum_{x \in T_1} (x - EX)^{k-1} \frac{a(x) [h(\alpha)]^x}{f(\alpha)} \right] +$$

$$+ \lambda h'(\alpha) \sum_{x \in T_1} x (x - EX)^k \frac{a(x) [h(\alpha)]^{x-1}}{f(\alpha)} - \lambda \frac{f'(\alpha)}{f(\alpha)} \sum_{x \in T_1} (x - EX)^k \frac{a(x) [h(\alpha)]^x}{f(\alpha)}.$$

Multiplying both sides of the last expresion by $h(\alpha)$, we get:

$$h(\alpha)\frac{dM_k}{d\alpha}=-kh(\alpha)\frac{dEX}{d\alpha}M_{k-1}+\lambda h'(\alpha)\sum_{x\in T_1}(x-EX)^{k+1}\frac{a(x)[h(\alpha)]^x}{f(\alpha)}+$$

$$+\lambda\sum_{x\in T_1}(x-EX)^k\frac{a(x)[h(\alpha)]^x}{f(\alpha)}\left[h'(\alpha)EX-h(\alpha)\frac{f'(\alpha)}{f(\alpha)}\right].$$

Using the formula for EX, we have

$$h(\alpha)EX-h(\alpha)\frac{f'(\alpha)}{f(\alpha)}=\frac{\gamma}{\lambda}h'(\alpha)(x_0-EX)$$

proving (9).

Putting in (9) $h(\alpha)=\alpha(1-\alpha)^{\beta-1}$, $f(\alpha)=(1-\alpha)^{-n}$ and $T_1=N\cup\{0\}$, we get the reccurence relation between k-th central moments of the generalized inftaled negative binomial distribution, i.e.

$$M_{k+1}=\frac{\alpha(1-\alpha)}{1-\alpha\beta}\left[\frac{dM_k}{d\alpha}+\lambda\frac{nk}{(1-\alpha\beta)^2}M_{k-1}\right]+\frac{n\alpha}{1-\alpha\beta}M_k+\gamma\lambda^k\frac{n^k}{\alpha\beta-1}$$

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STRESZCZENIE

W pracy wprowadzono definicję tak zwanego zmodyfikowanego rozkładu szeregowo-potęgowego ze zniekształceniem, a także wyznaczono rozkłady sum niezależnych zmiennych losowych o rozkładach wyżej wymienionego typu. Wyprowadzono również wzory rekurencyjne na momenty centralne dla zmiennej losowej o zmodyfikowanym rozkładzie szeregowo-potęgowym ze zniekształceniem.

РЕЗЮМЕ

В работе введено определение так называемых модифицированных распределений степенных рядов с искажением, а также исследовано распределение сумм независимых случайных величин, имеющих распределение данного типа. Выведено рекурретные формулы на центральные моменты для случайной величины, которая имеет модифицированное распределение степенных рядов с исскажением.