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Statistics Departament
University of Adelaide, South Australia
Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

Kerwin W. MORRIS, Dominik SZYNAL

Some Asymtotic Results for Binomial Models

Pewne graniczne własności w modelach dwumianowych

Некоторые предельные свойства в биномяльных моделях

1. Introduction. Let $\pi(u)$ be a given strictly increasing distribution function. We assume that the probability P that an individual 'responds' depends on the values of l determining variables $t_1, \dots, t_l$, and is given by $P = \pi(\underset{\sim}{t}'\underset{\sim}{\theta}^*)$ for some unknown parameter values $\underset{\sim}{\theta}^* \in R^l$.

For $p = 1, 2, \dots, k > l$ a sample of n_p individuals are examined with $\underset{\sim}{t} = \underset{\sim}{t}_p$, of which f_p are observed to respond. We write $T = (\underset{\sim}{t}_1, \dots, \underset{\sim}{t}_k) = (t_{ij})$, and assume that T has rank l. We also write $\bar{y}_p = f_p/n_p$, $N = \operatorname{diag}(n_1, \dots, n_k)$

$$d\pi/du = g(u),\ d^2\pi/du^2 = h(u),\ d^3\pi/du^3 = k(u),$$

$$\pi_p = \pi_p(\theta) = \pi(t_p'\theta),\ g_p = g(t_p'\theta),\ h_p = h(t_p'\theta),$$

$$k_p = k(t_p'\theta), \text{ and } \alpha_p = 1/\pi_p(1-\pi_p).$$

We denote the values of $\pi_p, g_p, \dots$ evaluated at the true value $\underset{\sim}{\theta}^*$ by $\pi_p^*, g_p^*, \dots$, and write

$$G^* = \operatorname{diag}(g_1^*, \dots, g_k^*),\ \Delta^* = (\alpha_1^*, \dots, \alpha_k^*).$$

Then $\partial\pi_p/\partial\theta_i = g_p t_{ip}$, $\partial g_p/\partial\theta_i = h_p t_{ip}$, $\partial h_p/\partial\theta_i = k_p t_{ip}$ and $\partial\alpha_p/\partial\theta_i = \alpha_p^2(2\pi_p - 1)g_p t_{ip}$.

Since $f_p \sim B(n_p, \pi_p^*)$ and $f_1, \dots, f_k$ are independent, then the log-likelyhood function for $\underset{\sim}{\theta}$ is

$$L(\theta) = \text{const} + \sum_{p=1}^{k} n_p(\bar{y}_p \ln \pi_p + (1-\bar{y}_p)\ln(1-\pi_p)),\ \theta \in R^l$$

and the score-vector has components

$$\partial L/\partial\theta_i = \sum_p n_p (\bar{y}_p - \pi_p)\alpha_p g_p t_{ip},\ i = 1, \ldots, l\,.$$

The MLE $\hat{\underset{\sim}{\theta}}_N = \hat{\underset{\sim}{\theta}}_N(\bar{\underset{\sim}{y}}_N)$ is obtained by solving $\partial L/\partial\theta_1 = \ldots = \partial L/\partial\theta_l = 0$. Since

$$\partial^2 L/\partial\theta_i\partial\theta_j = \sum_p n_p (y_p - \pi_p)\alpha_p (h_p + \alpha_p g_p^2 (2\pi_p - 1))\, t_{ip}t_{jp} - \sum_p n_p \alpha_p g_p^2 t_{ip} t_{jp}$$

then the information matrix at $\underset{\sim}{\theta}^*$ is $w_N^* = (w_{ij}^*)$, where

$$w_{ij}^* = w_{ji}^* = -E(\partial^2 L/\partial\theta_i\partial\theta_j)_{\underset{\sim}{\theta} = \underset{\sim}{\theta}^*} = \sum_p n_p\, \alpha_p^* g_p^{*2} t_{ip} t_{jp}\,.$$

Thus

$$w_N^* = TN\Delta^* G^{*2} T' > 0 \tag{1}$$

Denoting by C_N a positive definite symmetric matrix such that

$$C_N\, w_N^{*-1}\, C_N = I_l \tag{2}$$

we show that $C_N(\hat{\underset{\sim}{\theta}}_N - \underset{\sim}{\theta}^*) \xrightarrow{D} \underset{\sim}{U} \sim N(\underset{\sim}{0}, I_l)$, in the multiplyindexed sense of [1]; equivalently, that $\hat{\underset{\sim}{\theta}}_N \dot{\sim} N(\underset{\sim}{\theta}^*, w_N^{*-1})$ when all the sample sizes are large.

2. **The Taylor expansions.** For each N, we consider the parameters $\phi_N^* = C_N\theta^*$. Since by the invariance property of MLE $\hat{\underset{\sim}{\phi}}_N = C_N\hat{\underset{\sim}{\theta}}_N$, the stated results can be written equivalently as $\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}_N^* \xrightarrow{D} U$.

Writing

$$A_N = C_N^{-1} = (\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_l) = (a_{ij}) \tag{3}$$

then $A_N = A_N'$, and the log-likelihood function $L_1(\underset{\sim}{\phi})$ for $\underset{\sim}{\phi}$ is given by $L_1(\underset{\sim}{\phi}) = L(A_N\underset{\sim}{\phi})$, and $\hat{\underset{\sim}{\phi}}_N$ is obtained by solving $\partial L_1/\partial\phi_1 = \ldots = \partial L_1/\partial\phi_l = 0$. Defining now

$$\Gamma_j = \Gamma_{Nj}(\underset{\sim}{\theta}, \underset{\sim}{\kappa}) = \sum_p n_p \kappa_p g_p (\kappa_p - \pi_p)\, t_{jp},\ \theta \in R^l,\ 0 \leqslant \kappa_p \leqslant 1\ \forall_p$$

and $G_j = G_{Nj}(\underset{\sim}{\phi}, \underset{\sim}{\kappa}) = \sum_{i=1}^{l} \Gamma_{Ni}(A_N\underset{\sim}{\phi}, \underset{\sim}{\kappa})\, a_{ij}$, $j = 1, \ldots, l$, the equations $G_1 = \ldots = G_l = 0$ define $\underset{\sim}{\phi}$ implicitly as functions of $\underset{\sim}{\kappa}$, which we denote by $\underset{\sim}{\Phi}_N(\underset{\sim}{\kappa})$, and then

$$\hat{\underset{\sim}{\phi}}_N = \underset{\sim}{\Phi}_N(\bar{\underset{\sim}{y}}_N)\,. \tag{4}$$

Since from [1], Theorem 3,

$$\underset{\sim}{Z}_N = (N\Delta^*)^{1/2}(\underset{\sim}{\bar{y}}_N - \pi^*) \xrightarrow{D} \underset{\sim}{Z} \sim N(\underset{\sim}{0}, I_k) \tag{5}$$

we consider first the Taylor expansions of $G_1, .., G_l$ about the point $M(\phi_N^*, \pi^*)$. For G_i we have

$$G_{Ni}(\underset{\sim}{\phi}, \underset{\sim}{\kappa}) = G_{Ni}(\underset{\sim}{\phi}_N^*, \underset{\sim}{\pi}^*) + \begin{pmatrix} \underset{\sim}{\phi} - \underset{\sim}{\phi}_N^* \\ \underset{\sim}{\kappa} - \underset{\sim}{\pi}^* \end{pmatrix}' \begin{pmatrix} \underset{\sim}{G}_\phi^{i*} \\ \underset{\sim}{G}_\kappa^{i*} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \underset{\sim}{\phi} - \underset{\sim}{\phi}_N^* \\ \underset{\sim}{\kappa} - \underset{\sim}{\pi}^* \end{pmatrix}' \begin{pmatrix} \bar{G}_{\phi\phi}^i & \bar{G}_{\phi\kappa}^i \\ \bar{G}_{\phi\kappa}^i & \bar{G}_{\kappa\kappa}^i \end{pmatrix} \begin{pmatrix} \underset{\sim}{\phi} - \underset{\sim}{\phi}_N^* \\ \underset{\sim}{\kappa} - \underset{\sim}{\pi}^* \end{pmatrix},$$

where $\underset{\sim}{G}_\phi^{i*}, \underset{\sim}{G}_\kappa^{i*}$ are the vectors $(\partial G_i/\partial\phi_j), (\partial G_i/\partial\kappa_p)$ evaluated at M, and $\bar{G}_{\phi\phi}^i, \bar{G}_{\phi\kappa}^i, \bar{G}_{\kappa\kappa}^i$ are the matrices $(\partial^2 G_i/\partial\phi_j\partial\phi_k)$, $(\partial^2 G_i/\partial\phi_j\partial\kappa_p)$, $(\partial^2 G_i/\partial\kappa_p\partial\kappa_q)$ evaluated at some point $\bar{M}_i(\underset{\sim}{\bar{\phi}}_{Ni}, \underset{\sim}{\bar{\kappa}}_{Ni})$ between $(\underset{\sim}{\phi}, \underset{\sim}{\kappa})$ and M, i.e. $\underset{\sim}{\bar{\phi}}_{Ni} = \underset{\sim}{\phi}_N^* + \lambda_i(\underset{\sim}{\phi} - \underset{\sim}{\phi}_N^*)$, $\underset{\sim}{\bar{\kappa}}_{Ni} = \underset{\sim}{\pi}^* + \lambda_i(\underset{\sim}{\kappa} - \underset{\sim}{\pi}^*)$ for some

$$\lambda_i = \lambda_{Ni}(\underset{\sim}{\phi}, \underset{\sim}{\kappa}) \text{ between 0 and 1.} \tag{6}$$

Since $\Gamma_{Nj}(\underset{\sim}{\theta}^*, \underset{\sim}{\pi}^*) = 0$, $\forall N, j$, then $G_{Nj}(\phi_N^*, \pi^*) = 0$ $\forall N, j$. Furthermore, routine calculations show that at M

$$\partial G_i/\partial\phi_j = -(A_N w_N^* A_N)_{ij} = -\delta_{ij} \text{ (using (2) and (3))}$$

and $\partial G_i/\partial\kappa_p = (B_N \Delta^* G^* N)_{ip}$, where

$$B_N = A_N T = (\underset{\sim}{b}_1, .., \underset{\sim}{b}_k) = (b_{ip}), \tag{7}$$

and that $\partial^2 G_i/\partial\phi_j\partial\phi_k = \sum_p n_p b_{ip} b_{jp} b_{kp} D_{1p}$,

$$\partial^2 G_i/\partial\phi_j\partial\kappa_p = n_p b_{ip} b_{jp} D_{2p},$$

$$\partial^2 G_i/\partial\kappa_p\partial\kappa_l = 0,$$

where

$$\begin{aligned} D_{1p} &= D_{1p}(\underset{\sim}{\theta}, \kappa_p) = (\kappa_p - \pi_p) A_p - B_p, \\ A_p &= A_p(\underset{\sim}{\theta}) = g_p \alpha_p^2 (3h_p(2\pi_p - 1) + 2g_p^2 \alpha_p (2\pi_p - 1)^2 + 2g_p^2) + \alpha_p k_p \\ B_p &= B_p(\underset{\sim}{\theta}) = g_p \alpha_p (2\alpha_p g_p^2 (2\pi_p - 1) + 3h_p), \\ D_{2p} &= D_{2p}(\underset{\sim}{\theta}) = \alpha_p^2 g_p^2 (2\pi_p - 1) + \alpha_p h_p \text{ and } \underset{\sim}{\theta} = A_N \underset{\sim}{\phi}. \end{aligned} \tag{8}$$

Writing the l Taylor expansions as a single equation gives

$$\underset{\sim}{G}_N = \underset{\sim}{G}_N(\underset{\sim}{\phi}, \underset{\sim}{\kappa}) = (E_N - I)(\underset{\sim}{\phi} - \underset{\sim}{\phi}_N^*) + (B_N \Delta^* G^* N + F_N)(\underset{\sim}{\kappa} - \underset{\sim}{\pi}^*)$$

where

$$(E_N)_{ij} = \sum_p n_p b_{ip} b_{jp} (\underset{\sim}{b}_p' (\underset{\sim}{\phi} - \underset{\sim}{\phi}_N^*)) \bar{D}_{1pi} \tag{9}$$

$$(F_N)_{ip} = n_p b_{ip} (\underset{\sim}{b}_p' (\underset{\sim}{\phi} - \underset{\sim}{\phi}_N^*)) \bar{D}_{2ip} \text{ and } \bar{D}_{1pi} = D_{1p} (\underset{\sim}{\bar{\theta}}_{Ni}, \bar{\kappa}_{Nip}),$$

$$\bar{D}_{2pi} = D_{2p} (\underset{\sim}{\bar{\theta}}_{Ni}), \quad \underset{\sim}{\bar{\theta}}_{Ni} = A_N \underset{\sim}{\bar{\phi}}_{Ni}$$

3. The functions $\underset{\sim}{\Phi}_N$. From 2, the functions $\underset{\sim}{\Phi}_N(\underset{\sim}{\kappa})$ satisfy the equations

$$(E_N - I)(\underset{\sim}{\Phi}_N - \underset{\sim}{\phi}_N^*) + (B_N \Delta^* G^* N + F_N)(\underset{\sim}{\kappa} - \underset{\sim}{\pi}^*) = 0$$

where now in E_N and F_N,

$$\underset{\sim}{\phi} = \underset{\sim}{\Phi}_N, \underset{\sim}{\bar{\phi}}_{Ni} = \underset{\sim}{\phi}_N^* + \lambda_i (\underset{\sim}{\Phi}_N - \underset{\sim}{\phi}_N^*) \text{ and } \lambda_i = \lambda_{Ni} (\underset{\sim}{\Phi}_N, \underset{\sim}{\kappa}) \tag{10}$$

Changing variables for each N from $\underset{\sim}{\kappa}$ to $\underset{\sim}{\zeta}_N = (N\Delta^*)^{1/2} (\underset{\sim}{\kappa} - \underset{\sim}{\pi}^*)$, $\underset{\sim}{\Phi}_N$, regarded now as a function of $\underset{\sim}{\zeta}_N$, satisfies

$$(E_N - I)(\underset{\sim}{\Phi}_N - \underset{\sim}{\phi}_N^*) + (H_N + F_N (N\Delta^*)^{-1/2}) \underset{\sim}{\zeta}_N = \underset{\sim}{0} \tag{11}$$

where $H_N = B_N G^* (N\Delta^*)^{1/2}$, and from (1) and (7),

$$H_N H_N' = A_N w_N^* A_N = I_l \tag{12}$$

and $\underset{\sim}{\kappa}$ is replaced by $\pi^* + (N\Delta^*)^{-1/2} \underset{\sim}{\zeta}_N$, so that e.g. in (6)

$$\underset{\sim}{\bar{\kappa}}_{Ni} = \underset{\sim}{\pi}^* + \lambda_i (N\Delta^*)^{-1/2} \underset{\sim}{\zeta}_N .$$

To discuss the solution $\underset{\sim}{\Phi}_N (\underset{\sim}{\zeta}_N)$ of (11) we use the following

Lemma. *$\exists$ C^* such that $\forall$ N the elements of B_N in (7) satisfy $|b_{ip}| < C^* / \sqrt{n_p}$.*

Proof. Since $\tau_N = Tr (B_N N B_N') = \sum_p \sum_i n_p b_{ip}^2$, it is enough to show that $\exists C^*$ such that $\tau_N < C^{*2}$ $\forall$ N. From (3) and (7),

$$\tau_N = Tr ((T' A_N)' N (T' A_N)) = \sum_{i=1}^{l} \underset{\sim}{C}_i' N \underset{\sim}{C}_i, \text{ where } \underset{\sim}{C}_i = T' \underset{\sim}{a}_i.$$

Also, since $A_N w_N^* A_N = I_l$ then from (1),

$$\underset{\sim}{C}_i' N \Delta^* G^{*2} \underset{\sim}{C}_i = 1 \ \forall\ i.$$

Writing $\min_j (\alpha_j^* g_j^{*2}) = p^* > 0$, it follows that

$$p^* \underset{\sim}{C}_i' N \underset{\sim}{C}_i \leq \underset{\sim}{C}_i' N \Delta^* G^{*2} \underset{\sim}{C}_i = 1 \ \forall i,$$

and hence that $\tau_N \leq l / p^* = C^{*2}$, as required.

It follows that $\lim_{N \to \infty} B_N = 0$, and hence that $\lim_{N \to \infty} A_N = 0$, since from (7), $A_N = B_N T'(TT')^{-1}$.

For given $a > 0$ write $\mathscr{A}_l = \{ \underset{\sim}{\kappa}; \underset{\sim}{\kappa}' \underset{\sim}{\kappa} < a^2, \underset{\sim}{\kappa} \in R^l \}$, and consider values of $\underset{\sim}{\Phi}_N(\underset{\sim}{\zeta}_N)$ defined by (11) and such that $\underset{\sim}{\Phi}_N - \underset{\sim}{\phi}_N^* \in \mathscr{A}_l$. We show that given $\epsilon_0 > 0$, $\exists n_0(\epsilon_0, a)$ such that $\forall N > n_0 l$ $\underset{\sim}{\Phi}_N - \underset{\sim}{\phi}_N^*$ has the form

$$\underset{\sim}{\Phi}_N - \underset{\sim}{\phi}_N^* = H_N \underset{\sim}{\zeta}_N + X_N \underset{\sim}{\zeta}_N \text{ where } |(X_N)_{ip}| < \epsilon_0 \ \forall i, p \tag{13}$$

and the matrix X_N is defined below.

From (9) and the Lemma,

$$|(E_N)_{ij}| < a l C^{*3} \sum_p |D_{1pi}| / \sqrt{n_p}$$

Since $|\bar{\kappa}_{Nip} - \pi_p(\underset{\sim}{\bar{\theta}}_{Ni})| < 1$, it follows from (9) that

$$|\bar{D}_{1pi}| < |A_p(\underset{\sim}{\bar{\theta}}_{Ni})| + |B_p(\underset{\sim}{\bar{\theta}}_{Ni})|$$

where $\underset{\sim}{\bar{\theta}}_{Ni} = A_N \underset{\sim}{\bar{\phi}}_{Ni} = \underset{\sim}{\theta}^* + \lambda_i A_N(\underset{\sim}{\Phi}_N - \underset{\sim}{\phi}_N^*)$ and $|\lambda_i| < 1$. Since $\lim_{N \to \infty} A_N = 0$ and $|\underset{\sim}{\Phi}_N - \underset{\sim}{\phi}_N^*| < a$, then $\lim_{N \to \infty} \bar{\theta}_{Ni} = \underset{\sim}{\theta}^*$, $\lim_{N \to \infty} A_p(\underset{\sim}{\bar{\theta}}_{Ni}) = A_p^*$ and $\lim_{N \to \infty} B_p(\underset{\sim}{\bar{\theta}}_{Ni}) = B_p^*$ $\forall i$,

and so the elements of E_N are all arbitrarily small $\forall N$ sufficiently large. The same is then true of $\bar{E}_N$, where $(I - E_N)^{-1} = I + \bar{E}_N$.

Similarly the elements of F_N are uniformly bounded $\forall$ i, p and $\forall N$ sufficiently large, and so the elements of $F_N(N\Delta^*)^{-1/2}$ are also all arbitrarily small $\forall N$ sufficiently large. Finally, from (12), $|(H_N)_{ip}| \leq 1$ $\forall$ i, p, N, and so, from (11)

$$\underset{\sim}{\Phi}_N - \underset{\sim}{\phi}_N^* = (I + \bar{E}_N)(H_N + F_N(N\Delta^*)^{-1/2}) \underset{\sim}{\zeta}_N = H_N \underset{\sim}{\zeta}_N + X_N \underset{\sim}{\zeta}_N,$$

where $X_N = \bar{E}_N H_N + \bar{E}_N F_N (N\Delta^*)^{-1/2} + F_N (N\Delta^*)^{-1/2}$ has the stated property.

4. Convergence in distribution of $\underset{\sim}{\hat{\phi}}_N - \underset{\sim}{\phi}_N^* = C_N(\underset{\sim}{\hat{\theta}}_N - \underset{\sim}{\theta}^*)$. From Theorem 1 of [1] it is sufficient to prove that $\lim_{N \to \infty} P(\underset{\sim}{\hat{\phi}}_N - \underset{\sim}{\phi}_N^* \in \mathscr{A}) = P(\underset{\sim}{U} \in \mathscr{A})$ for every bounded open 'rectangle' $\mathscr{A} = \{ \underset{\sim}{\kappa}; a_{1i} < \kappa_i < a_{2i}, i = 1, 2, .., l \} \subset R^l$.

Since from (4) $\underset{\sim}{\hat{\phi}}_N = \underset{\sim}{\Phi}_N(\underset{\sim}{y}_N)$, then, choosing in (13) $a = \sup_{\underset{\sim}{\kappa} \in \mathscr{A}} |\underset{\sim}{\kappa}|$ it follows that

$\forall N > n_0 l$

$$P(\hat{\phi}_N - \phi_N^* \in \mathcal{A}) = P(H_N Z_N + X_N Z_N \in \mathcal{A})$$

where, from (5), $Z_N \xrightarrow{D} Z \sim N(0, I_k)$, and the absolute values of the elements of the random matrix $X_N = X_N(Z_N)$ are all $< \epsilon_0$.

Define now

$$\mathcal{B}_N = \{\zeta \in R^k; \kappa = H_N\zeta + X_N\zeta, \kappa \in \mathcal{A}\}$$

$$\mathfrak{b}_N = \{\zeta \in R^k; \kappa = H_N\zeta, \kappa \in \mathcal{A}\} \tag{14}$$

$$\mathfrak{F}_r = \{\zeta \in R^k; \zeta'\zeta < r^2\}$$

Then $P(\hat{\phi}_N - \phi_N^* \in \mathcal{A}) = P(Z_N \in \mathcal{B}_N) = P(Z_N \in \mathfrak{b}_N) + P(Z_N \in \mathcal{B}_N \cap \bar{\mathfrak{b}}_N) - P(Z_N \in \mathfrak{b}_N \cap \bar{\mathcal{B}}_N)$. Since $P(Z_N \in \mathfrak{b}_N) = P(H_N Z_N \in \mathcal{A})$, then

$$|P(\hat{\phi}_N - \phi_N^* \in \mathcal{A}) - P(U \in \mathcal{A})| \leqslant$$
$$\leqslant |P(H_N Z_N \in \mathcal{A}) - P(U \in \mathcal{A})| + P(Z_N \in \xi_N) + P(Z_N \in \xi_N') + P(Z_N \in \bar{\mathfrak{F}}_r), \tag{15}$$

where

$$\xi_N = \mathcal{B}_N \cap \bar{\mathfrak{b}}_N \cap \mathfrak{F}_r \text{ and } \xi_N' = \mathfrak{b}_N \cap \bar{\mathcal{B}}_N \cap \mathfrak{F}_r. \tag{16}$$

Consider now the terms in (15)

(a) From (12) for each N there exists a $k \times k$ orthogonal matrix $R_N = \begin{matrix} H_N \\ K_N \end{matrix}$. We define new variates

$$Z_N^* = \begin{pmatrix} U_N \\ V_N \end{pmatrix} = R_N Z_N = \begin{pmatrix} H_N Z_N \\ K_N Z_N \end{pmatrix}. \tag{17}$$

From (5) and Theorem 1 of [1], the c.f. $\chi_N(\nu)$ of Z_N has the form

$$\chi_N(\nu) = E[\exp(i\nu' Z_N)] = \exp(-\tfrac{1}{2}\nu'\nu) + f_N(\nu)$$

where

$$\lim_{N \to \infty} f_N(\nu) = 0 \text{ uniformly } \forall \nu \text{ in any bounded domain } \mathcal{D} \subset R^k. \tag{18}$$

The c.f. $\xi_N(\nu_1)$ of Z_N^* is then

$$\xi_N(\nu_1) = \chi_N(\nu_N), \text{ where } \nu_N' = \nu_1' R_N.$$

Since R_N is orthogonal, then $\nu_N'\nu_N = \nu_1'\nu_1 \quad \forall N$, whence

$$\xi_N(\nu_1) = \exp(-\tfrac{1}{2}\nu_1'\nu_1) + f_N(\nu_N).$$

For fixed $\underset{\sim}{\nu}_1$, choose in (18) $\mathfrak{D} = \{ \underset{\sim}{\kappa}; \underset{\sim}{\kappa}'\underset{\sim}{\kappa} \leqslant \underset{\sim}{\nu}_1' \underset{\sim}{\nu}_1 \}$. Then $\forall N \, \nu_N \in \mathfrak{D}$, $\lim_{N \to \infty} \xi_N(\nu_1) = \exp(-1/2 \, \underset{\sim}{\nu}_1' \underset{\sim}{\nu}_1)$, and so, from Theorem 1 of [1] $\underset{\sim}{Z}_N^* \xrightarrow{D} \underset{\sim}{Z}^* \sim N(\underset{\sim}{0}, I_k)$, and, in particular

$$\underset{\sim}{U}_N \xrightarrow{D} \underset{\sim}{U} \sim N(\underset{\sim}{0}, I_l) \text{ and } \underset{\sim}{V}_N \xrightarrow{D} \underset{\sim}{V} \sim N(\underset{\sim}{0}, I_{k-l}). \tag{19}$$

It follows that, for given $\epsilon > 0$, $\exists \, n_1$ such that

$$|P(H_N \underset{\sim}{Z}_N \in \mathcal{A}) - P(\underset{\sim}{U} \in \mathcal{A})| < \epsilon/5 \quad \forall N > n_1 I.$$

(b) Since $\underset{\sim}{Z} \sim N(\underset{\sim}{0}, I_k)$, then, for given $\epsilon > 0$, $r = r(\epsilon)$ such that $P(\underset{\sim}{Z} \in \bar{\mathfrak{F}}_r) < \epsilon/10$, and since $\underset{\sim}{Z}_N \xrightarrow{D} \underset{\sim}{Z}$, $\exists \, n_2 = n_2(\epsilon)$ such that in (15) $P(\underset{\sim}{Z}_N \in \bar{\mathfrak{F}}_r) < \epsilon/5 \quad \forall N > n_2 I$.

(c) To discuss $P(\underset{\sim}{Z}_N \in \xi_N)$, consider a point $\underset{\sim}{\zeta}_1 \in \xi_N$. From (14),

$$\underset{\sim}{\kappa}_1 = H_N \underset{\sim}{\zeta}_1 + X_N \underset{\sim}{\zeta}_1 \in \mathcal{A}, \; H_N \underset{\sim}{\zeta}_1 \notin \mathcal{A} \text{ and } \underset{\sim}{\zeta}_1' \underset{\sim}{\zeta}_1 < r^2 \tag{20}$$

Since the solutions $\underset{\sim}{\zeta}$ of $H_N \underset{\sim}{\zeta} = \underset{\sim}{\kappa}_1$ are all $\in \mathfrak{b}_N$, then the distance d_N between $\underset{\sim}{\zeta}_1$ and $_N$ satisfies

$$d_N \leqslant \inf_{\underset{\sim}{\zeta} : H_N \underset{\sim}{\zeta} = \underset{\sim}{\kappa}_1} (|\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_1|).$$

Consider now the solution of $H_N \underset{\sim}{\zeta} = \underset{\sim}{\kappa}_0$. The general solution is of the from $\underset{\sim}{\zeta} = K_N' t + H_N' \underset{\sim}{\kappa}_0$, $\underset{\sim}{t} \in R^{k-l}$ and $\underset{\sim}{\zeta}'\underset{\sim}{\zeta} = \underset{\sim}{t}'\underset{\sim}{t} + \underset{\sim}{\kappa}_0' \underset{\sim}{\kappa}_0$. From (20), $H_N(\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_1) = X_N \underset{\sim}{\zeta}_1$, whence $d_N^2 \leqslant \underset{\sim}{\zeta}_1' X_N' X_N \underset{\sim}{\zeta}_1$. Further from (13) $\forall N > n_0 I$ each component of $X_N \underset{\sim}{\zeta}_1$ is less then $\epsilon_0 \sum_p |\zeta_{1p}|$, whence $\underset{\sim}{\zeta}_1' X_N' X_N \underset{\sim}{\zeta}_1 < kl \, \epsilon_0^2 \, \underset{\sim}{\zeta}_1' \underset{\sim}{\zeta}_1$, and so

$$d_N < \epsilon_0 r \sqrt{kl} \qquad \forall N > n_0 I$$

Consider now a 'face' $\{\kappa; \kappa_i = a_{mi}, a_{1j} < \kappa_j < a_{2j}, j \neq i\}$ of $\mathcal{A}$. Writing $H_N' = (\underset{\sim}{h}_1, \ldots, \underset{\sim}{h}_l)$, the corresponding 'face' of $\mathfrak{b}_N$ is

$$\{\underset{\sim}{\zeta}; \underset{\sim}{\zeta} = K_N' \underset{\sim}{t} + a_{mi} \underset{\sim}{h}_i + \sum_{j \neq i} \kappa_j \underset{\sim}{h}_j; \underset{\sim}{t} \in R^{k-l}, a_{1j} < \kappa_j < a_{2j}, j \neq i\}$$

and since $|\underset{\sim}{h}_i| = 1$, the parallel 'face' $\subset \mathfrak{b}_N$ at a distance d_N has the term $a_{mi} \underset{\sim}{h}_i$ replaced by $(a_{mi} + d_N) \underset{\sim}{h}_i$, assuming that $a_{mi} > 0$ (if not, the modyfication is trivial). It follows that $\forall N > n_0 I$

$$P(\underset{\sim}{Z}_N \in \xi_N) \leqslant P(\underset{\sim}{Z}_N \in \bigcup_{m,i} D_{Nmi}),$$

where $D_{Nmi} = \{\underset{\sim}{\zeta}; \; \underset{\sim}{\zeta} = K_N' \underset{\sim}{t} + H_N' \underset{\sim}{\kappa}; \; \underset{\sim}{t} \in R^{k-l}; \; a_{mi} < \kappa_i < a_{mi} + \epsilon_0 r \sqrt{kl}; \; a_{1j} < \kappa_j < a_{2j}, j \neq i; \; \underset{\sim}{\zeta}'\underset{\sim}{\zeta} < r^2\}$. From (17), $P(\underset{\sim}{Z}_N \in D_{Nmi}) = P(\underset{\sim}{Z}_N^* \in D_{mi}')$, where $D_{mi}' =$

$= \left\{ \zeta^* = (\kappa / \underset{\sim}{t});\ \underset{\sim}{t} \in R^{k-l}, a_{mi} < \kappa_i < a_{mi} + \epsilon_0 r \sqrt{kl};\ a_{1j} < \kappa_j < a_{2j}, j \neq i;\ \underset{\sim}{\kappa}'\underset{\sim}{\kappa} + \underset{\sim}{t}'\underset{\sim}{t} < r^2 \right\}$, and since $D'_{mi} \subset D_{mi} = \left\{ \zeta^*; a_{mi} < \kappa_i < a_{mi} + \epsilon_0 r \sqrt{kl} \right\}$, then $P(\underset{\sim}{Z}_N \in \xi_N) \leqslant P(\underset{\sim}{Z}^*_N \in D)$, where $D = \bigcup_{m,i} D_{mi}$. Now $\underset{\sim}{Z}^*_N \xrightarrow{D} \underset{\sim}{Z}^* \sim N(\underset{\sim}{0}, I_k)$, so that $\exists n_3$ such that

$$|P(\underset{\sim}{Z}^*_N \in D) - P(\underset{\sim}{Z}^* \in D)| < \epsilon/5 \ \ \forall N > n_3 I.$$

Furthemore, $P(\underset{\sim}{Z}^* \in D) < 2l\, \epsilon_0 r \sqrt{kl}\ \ \forall N > n_0 I$, whence $P(\underset{\sim}{Z}_N \in \xi_N) < \epsilon/5 + 2l\, \epsilon_0 r \sqrt{kl}$ $\forall N > n_4 I$, $n_4 = \max(n_0, n_3)$.

We find similarly that $\exists n_5$ such that

$$P(\underset{\sim}{Z}_N \in \xi'_N) < \epsilon/5 + 2l\, \epsilon_0 r \sqrt{kl} \ \ \forall N > n_5 I$$

(d) Choosing now $\epsilon_0 = \epsilon/20\, lr(\epsilon) \sqrt{kl}$, it follows from (a), (b), (c) and (15) that $|P(\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N \in \mathscr{A}) - P(\underset{\sim}{U} \in \mathscr{A})| < \epsilon \ \ \forall N$ sufficently large, which completes the proof.

We note that the convergence in distribution of $\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N$ implies that $\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N$ and $H_N \underset{\sim}{Z}_N$ asymptotically equivalent, in the sense that

$$\underset{\sim}{w}_N = H_N \underset{\sim}{Z}_N - (\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N) \xrightarrow{D} \underset{\sim}{0} \tag{21}$$

To estabilish this, it is sufficient to show that given $\epsilon > 0$, $\eta > 0$, $\exists n$ such that $P(\underset{\sim}{w}'_N \underset{\sim}{w}_N > \epsilon) < \eta \ \ \forall N > nI$.

For given $a > 0$,

$$P(\underset{\sim}{w}'_N \underset{\sim}{w}_N > \epsilon) \leqslant P((\underset{\sim}{w}'_N \underset{\sim}{w}_N > \epsilon) \cap (|\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N| < a)) + P(|\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N| > a).$$

Since $\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N \xrightarrow{D} U$, $\exists a = a(\eta), n_1 = n_1(\eta)$ such that $P(|\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N| > a) < \eta/2 \ \ N > n_1 I$. Also, from (13), and 4 (c) given ϵ_0, $n_0 = n_0(\epsilon_0, a)$ such that $\forall N > n_0 I$

$$P((\underset{\sim}{w}'_N \underset{\sim}{w}_N > \epsilon) \cap (|\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N| < a)) \leqslant P((\underset{\sim}{Z}'_N X'_N X_N \underset{\sim}{Z}_N > \epsilon) \cap (|\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N| < a)) \leqslant$$

$$\leqslant P((\underset{\sim}{Z}'_N \underset{\sim}{Z}_N > \epsilon/kl\epsilon_0^2) \cap (|\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}^*_N| < a)) \leqslant P(\underset{\sim}{Z}'_N \underset{\sim}{Z}_N > \epsilon/kl\epsilon_0^2).$$

Finally, since from (5) $\underset{\sim}{Z}'_N \underset{\sim}{Z}_N \xrightarrow{D} Z^2 \sim \chi^2_k$, $\exists r = r(\eta)$, $n_2 = n_2(\eta)$ such that $P(\underset{\sim}{Z}'_N \underset{\sim}{Z}_N > > r) < \eta/2 \ \ \forall N > n_2 I$.

The result then follows by choosing $\epsilon_0^2 = \epsilon/klr$.

5. A goodness-of-fit test of the model. A standard test of goodness of fit the model uses the statistic $X_N^2 = \sum_p (f_p - n_p \hat{\pi}_p)^2 / n_p \hat{\pi}_p (1 - \hat{\pi}_p)$ where $\hat{\pi}_p = \pi_p(\hat{\underset{\sim}{\theta}}_N)$. We show that $X_N^2 \xrightarrow{D} \chi^2_{k-l}$.

Consider first the Taylor expansion of $\pi_p(\underset{\sim}{\kappa})$ about $\underset{\sim}{\kappa} = \underset{\sim}{\theta}^*$, namely

$$\pi_p(\underset{\sim}{\kappa}) - \pi_p^* = g_p^* \underset{\sim}{t}'_p (\underset{\sim}{\kappa} - \underset{\sim}{\theta}^*) + \frac{1}{2} h_p(\underset{\sim}{\tilde{\kappa}}) (\underset{\sim}{t}'_p(\underset{\sim}{\kappa} - \underset{\sim}{\theta}^*))^2$$

for some $\bar{\kappa}$ between $\underset{\sim}{\kappa}$ and $\underset{\sim}{\theta}^*$. Then

$$(n_p \alpha_p^*)^{1/2} (\hat{\pi}_p - \pi_p^*) = (n_p \alpha_p^*)^{1/2} g_p^* \underset{\sim}{t}_p' (\hat{\underset{\sim}{\theta}}_N - \underset{\sim}{\theta}^*) + \frac{1}{2} (n_p \alpha_p^*)^{1/2} h_p (\bar{\underset{\sim}{\theta}}_N) (\underset{\sim}{t}_p' (\hat{\underset{\sim}{\theta}}_N - \underset{\sim}{\theta}^*))^2 \qquad (22)$$

for some $\bar{\underset{\sim}{\theta}}_N$ between $\hat{\underset{\sim}{\theta}}_N$ and $\underset{\sim}{\theta}^*$. Now from (12)

$$I = A_N w_N^* A_N = \sum_p n_p \alpha_p^* g_p^{*2} A_N \underset{\sim}{t}_p \underset{\sim}{t}_p' A_N ,$$

whence $\sqrt{n_p}\ \ \alpha_p^* g_p^{*2} (\underset{\sim}{\kappa}' A_N \underset{\sim}{t}_p)^2 \leqslant \underset{\sim}{\kappa}' \underset{\sim}{\kappa} / \sqrt{n_p}\ \forall \underset{\sim}{\kappa}, p$. Thus $\sqrt{n_p} (\underset{\sim}{t}_p' (\hat{\underset{\sim}{\theta}}_N - \underset{\sim}{\theta}^*))^2 =$ $= \sqrt{n_p} ((\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}_N^*)' A_N \underset{\sim}{t}_p)^2 \leqslant (\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}_N^*)' (\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}_N^*) / \sqrt{n_p} \alpha_p^* g_p^{*2}$, and, since $h_p (\bar{\underset{\sim}{\theta}}_N) \xrightarrow{D} h_p^*$ and $(\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}_N^*)' (\underset{\sim}{\phi}_N - \underset{\sim}{\phi}_N^*) \xrightarrow{D} \underset{\sim}{U}' \underset{\sim}{U} \sim \chi_l^2$ it follows that the second term in (22) converges in distribution to zero and hence that

$$(N\Delta^*)^{1/2} (\hat{\underset{\sim}{\pi}}_N - \underset{\sim}{\pi}^*) = (N\Delta^*)^{1/2} G^* T \quad (\hat{\underset{\sim}{\theta}}_N - \theta^*) + \underset{\sim}{e}_{1N} = H_N' (\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}_N^*) + \underset{\sim}{e}_{1N}, \text{ from (12)}$$

where $\underset{\sim}{e}_{1N} \xrightarrow{D} 0$. Using now (5), (17) and (21),

$$(N\Delta^*)^{1\ 2} (\bar{\underset{\sim}{y}}_N - \hat{\underset{\sim}{\pi}}_N) = \underset{\sim}{Z}_N - H_N' (\hat{\underset{\sim}{\phi}}_N - \underset{\sim}{\phi}_N^*) - \underset{\sim}{e}_{1N} = K_N' \underset{\sim}{V}_N + H_N' \underset{\sim}{w}_N - \underset{\sim}{e}_{1N} = K_N' \underset{\sim}{V}_N + \underset{\sim}{e}_{2N}$$

where $\underset{\sim}{e}_{2N} \xrightarrow{D} 0$. Writ/ing $\hat{\Delta}_N = \operatorname{diag}(\hat{\alpha}_1, .., \hat{\alpha}_k)$, we have

$$(N\hat{\Delta}_N)^{1/2} (\bar{\underset{\sim}{y}}_N - \hat{\underset{\sim}{\pi}}_N) = (\hat{\Delta}_N \Delta^{*-1})^{1/2} K_N' \underset{\sim}{V}_N + \underset{\sim}{e}_{3N} ,$$

where $\underset{\sim}{e}_{3N} \xrightarrow{D} 0$ since $\hat{\Delta}_N \xrightarrow{D} \Delta^*$. Finally

$$X_N^2 = (\bar{\underset{\sim}{y}}_N - \hat{\underset{\sim}{\pi}}_N)' (N\hat{\Delta}_N) (\bar{\underset{\sim}{y}}_N - \hat{\underset{\sim}{\pi}}_N) = \underset{\sim}{V}_N' \underset{\sim}{V}_N + \underset{\sim}{V}_N' K_N (I - \Delta^{*-1} \hat{\Delta}_N) K_N' \underset{\sim}{V}_N +$$
$$+ 2 \underset{\sim}{e}_{3N}' (\Delta^{*-1} \hat{\Delta}_N)^{1/2} K_N' \underset{\sim}{V}_N + \underset{\sim}{e}_{3N}' \underset{\sim}{e}_{3N} \xrightarrow{D} \underset{\sim}{V}' \underset{\sim}{V} \sim \chi_{k-l}^2$$

since $\underset{\sim}{V}_N \xrightarrow{D} \underset{\sim}{V} \sim N(\underset{\sim}{0}, I_{k-l})$, $\Delta^{*-1} \hat{\Delta}_N \xrightarrow{D} I$, $\underset{\sim}{e}_{3N} \xrightarrow{D} \underset{\sim}{0}$ and the elements of K_N' are bounded in modul/s by 1.

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STRESZCZENIE

W pracy bada się graniczne własności (w sensie określonym w [1]) estymatorów największej wiarygodności parametrów w uogólnionych liniowych modelach typu binomialnego (opisanych w [2]) oraz graniczne własności testu zgodności.

РЕЗЮМЕ

В работе исследуются предельные свойства (в смысле определенном в [1]) оценек максимального правдоподобия параметров в обобщенных линейных моделях биномялного типа, а также предельные свойства критерия согласия.