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On A Functional Central Limit Theorem of a Function of the Average of Independent Random Variables

O funkcjonałowym centralnym twierdzeniu granicznym dla funkcji średnich arytmetycznych niezależnych zmiennych losowych

Об функциональной центральной предельной теореме для функций арифметических средних независимых случайных величин

1. Introduction. Let $\{X_k, k \geq 1\}$ be a sequence of random variables and put $S_n = \Sigma_{k=1}^n X_k$. In statistical inference it is sometimes necessary to investigate the asymptotic behavior of $\{g(S_n/n), n \geq 1\}$, where g is a real function. In [5], p. 259, one can find, among other things, the theorem which states that in the case when X_k, $k \geq 1$, are independent and identically distributed with $EX_1 = \mu$, $0 < \sigma^2 X_1 = \sigma^2 < \infty$, and g is differentiable at μ with $g'(\mu) \neq 0$,

$$\frac{\sqrt{n}}{\sigma g'(\mu)} (g(S_n/n) - g(\mu)) \xrightarrow{D} \mathcal{N}_{0,1}, \; n \to \infty, \tag{1}$$

where $\mathcal{N}_{a,b}$ stands for a normal random variable with mean a and standard deviation b, and D denotes weak convergence. Some further consideration on the asymptotic behaviour of $\{g(S_n/n), n \geq 1\}$ can be found in [2] and [3].

The aim of this to note give the functional version of (1).

2. Results. Let $\{X_k, k \geq 1\}$ be a sequence of independent identically distributed random variables with $EX_1 = \mu$, $0 < \sigma^2 X_1 = \sigma^2 < \infty$. Suppose that g is a real differentiable function with continuous derivative such that $g'(\mu t) \neq 0$ for $t \in [0,1]$. Define

$$Y_n(0) = 0, n = 1, 2, \dots$$

$$Y_n(t) = \frac{\sqrt{n}}{g'(\mu t)\,\sigma}\left(g\left(\frac{S_{[nt]}}{n}\right) - g(\mu t)\right) + \tag{2}$$

$$+ \frac{\sqrt{n}\,(nt - [nt])}{g'(\mu t)\,\sigma}\left(g\left(\frac{S_{[nt]+1}}{n}\right) - g\left(\frac{S_{[nt]}}{n}\right)\right),$$

$0 < t \leqslant 1, n = 1, 2 \dots$

Under the above assumptions we prove

Theorem 1. *The random functions defined by* (2) *are weakly convergent to the standard Wiener process on* [0,1], *i.e.*

$$Y_n \xrightarrow{D} W, n \to \infty,$$

where $W = \left\{ W(t), 0 \leqslant t \leqslant 1 \right\}$ *denoted the standard Wiener process on* $(C_{[0,1]}, \mathcal{C})$.

Proof. Note that

$$Y_n(t) = \frac{\sqrt{n}}{g'(\mu t)\,\sigma}\left(\frac{S_{[nt]}}{n} - \mu t\right) g'\left(\frac{S_{[nt]}}{n} + \theta_1(t)\left(\frac{S_{[nt]}}{n} - \mu t\right)\right) +$$

$$+ \frac{(nt - [nt])\,X_{[nt]+1}}{g'(\mu t)\,\sigma\sqrt{n}}\, g'\left(\frac{S_{[nt]}}{n} + \theta_2(t)\,\frac{X_{[nt]+1}}{n}\right) =$$

$$= \frac{S_{[nt]} - n\mu t}{\sigma\sqrt{n}} + \frac{S_{[nt]} - n\mu t}{g'(\mu t)\,\sigma\sqrt{n}}\left(g'\left(\frac{S_{[nt]}}{n} + \theta_1(t)\left(\frac{S_{[nt]}}{n} - \mu t\right)\right)\right.$$

$$\left. - g'(\mu t)\right) + \frac{(nt - [nt])\,X_{[nt]+1}}{g'(\mu t)\,\sigma\sqrt{n}}\, g'\left(\frac{S_{[nt]}}{n} + \theta_2(t)\,\frac{X_{[nt]+1}}{n}\right),$$

where $\theta_1(\cdot)$ and $\theta_2(\cdot)$ are random functions (depending on n) taking values in $(-1,0)$.

Since for every $t \in [0,1]$

$$\frac{S_{[nt]} - n\mu t}{\sigma\sqrt{n}} \xrightarrow{D} \mathcal{N}_{0,\sqrt{t}}, n \to \infty,$$

$$\frac{S_{[nt]}}{n} \xrightarrow{P} \mu t, \quad \frac{X_{[nt]+1}}{\sqrt{n}} \xrightarrow{P} 0, n \to \infty$$

(P. in probability), and g' is continuous, then

$$\frac{S_{[nt]}-n\mu t}{g'(\mu t)\,\sigma\sqrt{n}}\left(g'\left(\frac{S_{[nt]}}{n}+\theta_1(t)\left(\frac{S_{[nt]}}{n}-\mu t\right)\right)-g'(\mu t)\right)\xrightarrow{P.}0,\ n\to\infty,$$

and

$$\frac{(nt-[nt])\,X_{[nt]+1}}{g'(\mu t)\,\sigma\sqrt{n}}\,g'\left(\frac{S_{[nt]}}{n}+\theta_2(t)\,\frac{X_{[nt]+1}}{n}\right)\xrightarrow{P.}0,\ n\to\infty$$

whence

$$Y_n(t)\xrightarrow{D}\mathcal{N}_{0,\sqrt{t}},\ n\to\infty,\ 0\leqslant t\leqslant 1\,. \tag{3}$$

Let $s<t$. Note that

$$Y_n(t)-Y_n(s)=\frac{S_{[nt]}-n\mu t}{\sigma\sqrt{n}}-\frac{S_{[ns]}-n\mu s}{\sigma\sqrt{n}}+\frac{S_{[nt]}-n\mu t}{g'(\mu t)\sigma\sqrt{n}}\left(g'\left(\frac{S_{[nt]}}{n}+\right.\right.$$

$$\left.\left.+\,\theta_1(t)\left(\frac{S_{[nt]}}{n}-\mu t\right)\right)-g'(\mu t)\right)-\frac{S_{[ns]}-n\mu s}{g'(\mu s)\,\sigma\sqrt{n}}\left(g'\left(\frac{S_{[ns]}}{n}+\theta_1(s)\left(\frac{S_{[ns]}}{n}-\right.\right.\right.$$

$$\left.\left.\left.-\,\mu s\right)\right)-g'(\mu s)\right)+\frac{(nt-[nt])\,X_{[nt]+1}}{g'(\mu t)\,\sigma\sqrt{n}}\,g'\left(\frac{S_{[nt]}}{n}+\theta_2(t)\,\frac{X_{[nt]+1}}{n}\right)- \tag{4}$$

$$-\,\frac{(ns-[ns])\,X_{[ns]+1}}{g'(\mu s)\,\sigma\sqrt{n}}\,g'\left(\frac{S_{[ns]}}{n}+\theta_2(s)\,\frac{X_{[ns]+1}}{n}\right).$$

Taking into account that

$$\frac{S_{[nt]}-n\mu t}{\sigma\sqrt{n}}-\frac{S_{[ns]}-n\mu s}{\sigma\sqrt{n}}\xrightarrow{D}\mathcal{N}_{0,\sqrt{t-s}},\ n\to\infty\ ([1],\text{ p. }69)$$

and the left terms in (4) converge in probability to zero, we conclude that

$$Y_n(t)-Y_n(s)\xrightarrow{D}\mathcal{N}_{0,\sqrt{t-s}},\ n\to\infty. \tag{5}$$

Moreover, it is known that

$$\left(\frac{S_{[ns]}-n\mu s}{\sigma\sqrt{n}},\ \frac{S_{[nt]}-nt\mu}{\sigma\sqrt{n}}-\frac{S_{[ns]}-n\mu s}{\sigma\sqrt{n}}\right)\xrightarrow{D}(W(s),\,W(t)-W(s)). \tag{6}$$

$n \to \infty$ ([1], p. 69). Put now

$$U_n(s,t) = \left(\frac{S_{[ns]} - n\mu s}{\sigma\sqrt{n}}, \frac{S_{[nt]} - n\mu t}{\sigma\sqrt{n}} - \frac{S_{[ns]} - n\mu s}{\sigma\sqrt{n}}\right);$$

$$V_n(s,t) = (Y_n(s), Y_n(t) - Y_n(s)).$$

By (6), $U_n(s,t) \xrightarrow{D} U(s,t) = (W(s), W(t) - W(s)), n \to \infty$, and as in the derivation of (3) and (5) $\rho(U_n(s,t), V_n(s,t)) \to 0$, $n \to \infty$, where ρ denotes the Euclidean distance. Hence by Theorem 4.1 ([1], p. 25), we get

$$(Y_n(s), Y_n(t) - Y_n(s)) \xrightarrow{D} (W(s), W(t) - W(s)), n \to \infty,$$

and whence by Corollary 1 of Theorem 5.1 ([1], p. 31), we conclude that for all s, t; $0 \leq s, t \leq 1$,

$$(Y_n(s), Y_n(t)) \xrightarrow{D} (W(s), W(t)), n \to \infty,$$

and also that the finite-dimensional distributions of $\{Y_n(t), 0 \leq t \leq 1\}$ converge weakly to the finite-dimensional distributions of the Wiener process.

We now prove the tightness of $\{Y_n(t), 0 \leq t \leq 1\}$, By Theorem 8.3 ([1], p. 56) it is enough to prove that for any given $\epsilon > 0$ and $\eta > 0$ there exists δ, $0 < \delta < 1$, and a positive integer n_0 such that

$$P[\sup_{t \leq s \leq t+\delta} |Y_n(s) - Y_n(t)| \geq \epsilon] \leq \eta\delta \tag{7}$$

for $n \geq n_0$ and $0 \leq t \leq 1$.

In what follows we will consider (7) with δ replaced by $1/2\delta$, ϵ by 8ϵ and similar changes convenient in evaluations.

Note that

$$P[\sup_{t \leq s \leq t+\delta/2} |Y_n(s) - Y_n(t)| \geq 8\epsilon] \leq$$

$$\leq P[\sup_{t \leq s \leq t+\delta/2} |\frac{S_{[ns]} - n\mu s}{\sigma\sqrt{n}} - \frac{S_{[nt]} - n\mu t}{\sigma\sqrt{n}}| \geq 4\epsilon] +$$

$$+ P[\sup_{t \leq s \leq t+\delta/2} |\frac{S_{[ns]} - n\mu s}{g'(\mu s)\sigma\sqrt{n}}| \, |g'(\frac{S_{[ns]}}{n} +$$

$$+ \theta_1(s)(\frac{S_{[ns]}}{n} - \mu s)) - g'(\mu s)| \geq \epsilon] +$$

$$+P\Big[\sup_{t \le s \le t+\delta/2} \frac{(ns-[ns])\,|X_{[ns]+1}|}{g'(\mu s)\,\sigma\sqrt{n}}\,\Big|g'\Big(\frac{S_{[ns]}}{n}+\theta_2(s)\frac{X_{[ns]+1}}{n}\Big)\Big| \ge \epsilon\Big] +$$

$$+P\Big[\Big|\frac{S_{[nt]}-n\mu t}{g'(\mu t)\,\sigma\sqrt{n}}\Big|\,\Big|g'\Big(\frac{S_{[nt]}}{n}+\theta_1(t)\Big(\frac{S_{[nt]}}{n}-\mu t\Big)\Big)-g'(\mu t)\Big| \ge \epsilon\Big] + \tag{8}$$

$$+P\Big[\frac{(nt-[nt])\,|X_{[nt]+1}|}{g'(\mu t)\,\sigma\sqrt{n}}\,\Big|g'\Big(\frac{S_{[nt]}}{n}+\theta_2(t)\frac{X_{[nt]+1}}{n}\Big)\Big| \ge \epsilon\Big].$$

We need now to recall the definition of the modulus of continuity. The modulus of continuity is defined for g, $\delta \ge 0$, by

$$\omega(g;\delta) = \sup_x \sup_{|h| \le \delta} |g(x+h)-g(x)|,$$

having the properties that $\omega(g;\delta)$ is a decreasing functions of δ and $\omega(g;\lambda\delta) \le \le (1+\lambda)\,\omega(g;\delta)$ for each $\lambda > 0$. Put also $A = \sup_{0 \le s \le 1} |1/g'(\mu s)|$.

We note that

$$\sup_{t \le s \le t+\delta/2} \Big|\frac{S_{[ns]}-n\mu s}{g'(\mu s)\,\sigma\sqrt{n}}\Big|\,\Big|g'\Big(\frac{S_{[ns]}}{n}+\theta_1(s)\Big(\frac{S_{[ns]}}{n}-\mu s\Big)\Big)-g'(\mu s)\Big| \le$$

$$\le A \sup_{0 \le s \le 1} \Big|\frac{S_{[ns]}-n\mu s}{\sigma\sqrt{n}}\Big| \sup_{0 \le s \le 1} \Big|g'\Big(\frac{S_{[ns]}}{n}+\theta_1(s)\Big(\frac{S_{[ns]}}{n}-\mu s\Big)\Big)-g'(\mu s)\Big| \le$$

$$\le A\Big(\sup_{0 \le s \le 1}\Big|\frac{S_{[ns]}-[ns]\mu}{\sigma\sqrt{n}}\Big| + \frac{|\mu|}{\sigma\sqrt{n}}\Big) \sup_{0 \le s \le 1} \omega\Big(g';(1+\theta_1(s))\Big|\frac{S_{[ns]}}{n}-\mu s\Big|\Big) \le$$

$$\le A\Big(\sup_{k \le n}\frac{|S_k-k\mu|}{\sigma\sqrt{n}} + \frac{|\mu|}{\sigma\sqrt{n}}\Big)\,\omega\Big(g';\ \sup_{0 \le s \le 1}\Big|\frac{S_{[ns]}}{n}-\mu s\Big|\Big) \le$$

$$\le 2A\Big(\sup_{k \le n}\frac{|S_k-k\mu|}{\sigma\sqrt{n}} + \frac{|\mu|}{\sigma\sqrt{n}}\Big)\,\omega\Big(g';\ \sup_{k \le n}\frac{|S_k-k\mu|}{n} + \frac{|\mu|}{n}\Big)$$

$$\xrightarrow{P.} 0,\ n \to \infty, \text{ for all } t,\ 0 \le t \le 1,$$

since $\sup_{k \le n} |S_k - k\mu| / \sigma\sqrt{n}$ converges in distribution, $(S_n - n\mu)/n \to 0$, a.s., $n \to \infty$.

Analogously we prove that

$$\sup_{t < s < t+\delta/2} \frac{(ns-[ns])\,|X_{[ns]+1}|}{\sigma\sqrt{n}}\,|g'(\frac{S_{[ns]}}{n}+\theta_2(s)\frac{X_{[ns]+1}}{n})|\xrightarrow{P.} 0,$$

$n \to \infty$ uniformly with respect t, $0 \leqslant t \leqslant 1$. Taking into account the estimate for

$$P\left[\sup_{t < s < t+\delta/2} \left|\frac{S_{[ns]}-n\mu s}{\sigma\sqrt{n}} - \frac{S_{[nt]}-n\mu t}{\sigma\sqrt{n}}\right| \geqslant 4\epsilon\right]$$

following from the considerations of [1], p. 60, we conclude, by the facts given above, and (8), that (7) is satisfied, which completes the proof of Theorem 1.

Following the considerations of [4], p. 472 and of [1], p. 147 one can extended, by standard arguments, Theorem 1 to random indexed sums.

Theorem 2. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent and identically distributed random variables with* $EX_1 = \mu$, $0 < \sigma^2 X_1 = \sigma^2 < \infty$. *Suppose that g is a real differentiable function with continuous derivative such that* $g'(\mu t) \neq 0$, $0 \leqslant t \leqslant 1$.

If $\{N_n, n \geqslant 1\}$ *is a sequence of positive integer-valued random variables such that* $\{X_n, n \geqslant 1\}$ *and* $\{N_n \geqslant 1\}$ *are independent, and* $N_n \xrightarrow{P.} \infty$, $n \to \infty$, *then*

$$Y_{N_n} \xrightarrow{D} W, n \to \infty. \tag{9}$$

Theorem 3. *If under the assumptions of Theorem 2 we do not assume the independence of* $\{X_n, n \geqslant 1\}$ *and* $\{N_n, n \geqslant 1\}$, *then (9) holds if*

$$N_n / n \xrightarrow{P.} \lambda, n \to \infty,$$

where λ *is a positive random variable.*

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STRESZCZENIE

W pracy rozszerzono do funkcjonalnej wersji centralne twierdzenie graniczne dla funkcji średnich arytmetycznych niezależnych zmiennych losowych ([5], str. 259).

РЕЗЮМЕ

В работе расширено к функциональной версии центральную предельную теорему для функции арифметических средних независимых случайных величин ([5], от 259).